FRAMED MAPPING CLASS GROUPS AND THE MONODROMY OF STRATA OF ABELIAN DIFFERENTIALS

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Abstract. This paper investigates the relationship between strata of abelian differentials and various mapping class groups afforded by means of the topological monodromy representation. Building off of prior work of the authors, we show that the fundamental group of a stratum surjects onto the subgroup of the mapping class group which preserves a fixed framing of the underlying Riemann surface, thereby giving a complete characterization of the monodromy group. In the course of our proof we also show that these “framed mapping class groups” are finitely generated (even though they are of infinite index) and give explicit generating sets.

1. Introduction

The moduli space $\Omega M_g$ of holomorphic 1–forms (abelian differentials) of genus $g$ is a complex $g$–dimensional vector bundle over the moduli space $M_g$. The complement of its zero section is naturally partitioned into strata, sub-orbifolds with fixed number and degree of zeros. Fixing a partition $\kappa := (\kappa_1, \ldots, \kappa_n)$ of $2g - 2$, we let $\Omega M_g(\kappa)$ denote the stratum consisting of those pairs $(X, \omega)$ where $\omega$ is an abelian differential on $X \in M_g$ with zeros of orders $\kappa$.

As strata are quasi-projective varieties their (orbifold) fundamental groups are finitely presented. Kontsevich and Zorich famously conjectured that strata should be $K(G, 1)$’s for “some sort of mapping class group” [KZ97], but little progress has been made in this direction. This paper continues the work begun in [Cal19] and [CS19], where the authors investigate these orbifold fundamental groups by means of a “topological monodromy representation.”

Any (homotopy class of) loop in $\Omega M_g(\kappa)$ based at $(X, \omega)$ gives rise to a(n isotopy class of) self–homeomorphism $X \to X$ which preserves $Z = \text{Zeros}(\omega)$. This gives rise to the topological monodromy representation

$$\rho : \pi_1^{\text{orb}}(\mathcal{H}) \to \text{Mod}_g^n$$

where $\mathcal{H}$ is the component of $\Omega M_g(\kappa)$ containing $(X, \omega)$ and $\text{Mod}_g^n$ is the mapping class group of $X$ relative to $Z$.

The fundamental invariant: framings. The horizontal vector field $1/\omega$ of any $(X, \omega) \in \Omega M_g(\kappa)$ defines a trivialization $\tilde{\phi}$ of the tangent bundle of $X \setminus Z$, or an “absolute framing” (see §2.1 and 6.2; the terminology reflects the finer notion of a “relative framing” to be discussed below). The mapping class group $\text{Mod}_g^n$ generally does not preserve (the isotopy class of) this absolute framing, and its stabilizer $\text{Mod}_g^n[\tilde{\phi}]$ is of infinite index. On the other hand, the canonical nature of $1/\omega$ means that the image of $\rho$ does leave some such absolute framing fixed. Our first main theorem identifies the image of the monodromy representation as the stabilizer of an absolute framing.
Theorem A. Suppose that $g \geq 5$ and $\kappa$ is a partition of $2g - 2$. Let $\mathcal{H}$ be a non-hyperelliptic component of $\Omega \mathcal{M}_g(\kappa)$; then

$$\rho \left( \pi_1^{\text{orb}}(\mathcal{H}) \right) \cong \text{Mod}_g^\kappa [\bar{\phi}]$$

where $\bar{\phi}$ is the absolute framing induced by the horizontal vector field of any surface in $\Omega \mathcal{M}_g(\kappa)$.

For an explanation as to why we restrict to the non-hyperelliptic components of strata, see the discussion after Theorem 7.2. The bound $g \geq 5$ is an artifact of the method of proof and can probably be relaxed to $g \geq 3$. We invoke $g \geq 5$ in Proposition 3.10, Proposition 4.1, and Lemma 5.5; it is not needed elsewhere.

Equivalently, the universal property of $\mathcal{M}_{g,n}$ implies there is a map

$$\ell : \quad \mathcal{H} \to \mathcal{M}_{g,\kappa} \quad (X, \omega) \mapsto (X, \text{Zeros}(\omega)).$$

where $\mathcal{M}_{g,\kappa}$ denotes the moduli space of Riemann surfaces with $n$ marked points labeled by $(\kappa_1, \ldots, \kappa_n)$, in which one may only permute marked points if they have the same label. In this language, Theorem A characterizes the image of $\ell$ at the level of (orbifold) fundamental groups.

Remark 1.1. In a recent preprint [Ham18], Hamenstädt has also analyzed the image of the monodromy representation $\rho(\pi_1^{\text{orb}}(\mathcal{H}))$, although she does not give a characterization of this image as the stabilizer of a framing as in Theorem A.

Remark 1.2. Via the mapping class group’s action on relative homology, Theorem A also implicitly characterizes the image of the homological monodromy representation

$$\rho_H : \pi_1^{\text{orb}}(\mathcal{H}) \to \text{Aut}(H_1(X, Z; \mathbb{Z})).$$

In a companion paper [CS20], we give an explicit description of the image of $\rho_H$ as the kernel of a certain crossed homomorphism on $\text{Aut}(H_1(X, Z; \mathbb{Z}))$.

Application: realizing curves and arcs geometrically. Using the identification of Theorem A, we can apply a framed version of the “change of coordinates” principle (see Proposition 2.14) to deduce the following characterization of which curves can be realized as the core curves of embedded cylinders. To formulate this, we observe that the data of an absolute framing $\phi$ gives rise to a “winding number function” (also denoted $\phi$) that sends an oriented simple closed curve to the $\mathbb{Z}$–valued holonomy of its forward–pointing tangent vector relative to the framing (c.f. §2.1).

Corollary 1.3 (c.f. Corollary 1.1 of [CS19]). Fix $g \geq 5$ and a partition $\kappa$ of $2g - 2$. Pick some $(X, \omega)$ in a non-hyperelliptic component $\mathcal{H}$ of $\Omega \mathcal{M}_g(\kappa)$ and let $\bar{\phi}$ denote the induced (absolute) framing. Pick a nonseparating simple closed curve $c \subset X \setminus Z$.

Then there is a path $\gamma : [0, 1] \to \mathcal{H}$ with $\gamma(0) = (X, \omega)$ and such that the parallel transport of $c$ along $\gamma$ is a cylinder on $\gamma(1)$ if and only if the winding number of $c$ with respect to $\bar{\phi}$ is 0.

\[\text{1Of course, } \mathcal{M}_{g,\kappa} \text{ is a finite cover of } \mathcal{M}_{g,n}, \text{ corresponding to the group } \text{Sym}(\kappa) \text{ of label–preserving permutations.}\]
Proof. The condition that \( \bar{\phi}(c) = 0 \) is necessary, as the core curve of a cylinder has constant slope with respect to \( \bar{\phi} \).

To see that it is sufficient, we note that there is some cylinder on \( X \) with core curve \( d \) and the winding number of \( d \) with respect to \( \bar{\phi} \) is 0. Therefore, by the framed change–of–coordinates principle (Proposition 2.14), there is some element \( g \in \Mod[n][\bar{\phi}] \) taking \( d \) to \( c \). By Theorem A, \( g \) also lies in the monodromy group, so there is some loop \( \gamma \in \pi_{\text{orb}}^1(\mathcal{H}) \) whose monodromy is \( g \). This \( \gamma \) is the desired path. \( \square \)

We can also deduce a complementary result for arcs using the same principle. Recall that a saddle connection on an abelian differential is a nonsingular geodesic segment connecting two zeros.

**Corollary 1.4.** Fix \( g \geq 5 \) and a partition \( \kappa \) of \( 2g - 2 \). Pick some \( (X, \omega) \) in a non-hyperelliptic component \( \mathcal{H} \) of \( \Omega \mathcal{M}_g(\kappa) \) and fix a nonseparating arc connecting distinct zeros of \( Z \).

Then there is a path \( \gamma : [0, 1] \to \mathcal{H} \) with \( \gamma(0) = (X, \omega) \) and such that the parallel transport of \( a \) along \( \gamma \) is realized as a saddle connection on \( \gamma(1) \).

The proof of this corollary uses machinery developed throughout the paper and is therefore deferred to Section 8.3. In Section 8 we collect other corollaries that we can obtain by the methods of the paper: we also give a classification of components of strata with marking data (Section 8.1) and we show that for \( \mathcal{H} \) a sufficiently general stratum–component, the monodromy image \( \rho(\pi_{\text{orb}}^1(\mathcal{H})) \leq \PMod[n][\bar{\phi}] \) is not generated by shears about cylinders (Section 8.2).

**Other monodromy groups.** Theorem A is a consequence of our characterization of the images of certain other monodromy representations: in Theorem 7.13, we compute the monodromy of a “prong–marked” stratum into the mapping class group \( \Mod[g,n] \) of a surface with boundary. Theorem 7.14 computes the monodromy of a stratum into the “pronged mapping class group,” denoted \( \Mod^*[g,n] \), a refinement which captures the combinatorics of the zeros of the differential. In both Theorems 7.13 and 7.14 we find that the monodromy group is the stabilizer \( \Mod[g,n][\bar{\phi}] \), respectively \( \Mod^*[g,n][\bar{\phi}] \), of an appropriate “relative framing” \( \bar{\phi} \).

A relative framing is an isotopy class of framing of \( \Sigma_{g,n} \) where the isotopies are required to be trivial on \( \partial \Sigma_{g,n} \) (see §2). To promote an absolute framing \( \bar{\phi} \) into a relative framing \( \bar{\phi} \), we “blow up” the zeros of a differential into boundary components (see §7.2). Under this transformation, a zero of order \( k \) becomes a boundary component with winding number \( -1 - k \), and so an element of \( \Omega \mathcal{M}_g(\kappa) \) induces a relative framing \( \phi \) on its blow-up with “signature” \( -1 - \kappa := (-1 - \kappa_1, \ldots, -1 - \kappa_n) \) (see §2.1). Thus each boundary component has negative winding number; a framing with this property is said to be of holomorphic type.

**Generating the framed mapping class group.** The monodromy computations in Theorems A 7.13 and 7.14 rest on a development in the theory of stabilizers of relative framings as subgroups of \( \Mod[g,n] \); we determine simple explicit finite generating sets. For the results of this paper we only need to consider the situation where each boundary component has winding number at most \(-2\); see Remark 1.6 below for a brief discussion of how to formulate Theorem B for general framings of holomorphic type.
Theorem B. Fix \( g \geq 5 \) and a partition \( \kappa = (\kappa_1, \ldots, \kappa_n) \) of \( 2g - 2 \). Let \( \phi \) be a relative framing of \( \Sigma_{g,n} \) of signature \(-1 - \kappa\). Let

\[
C = \{a_0, \ldots, a_{2g-1}\} \cup \left\{ b_i : i = \sum_{j=1}^{\ell} \kappa_j \text{ for } \ell = 1, \ldots, n \right\}
\]

be a configuration of curves on \( \Sigma_{g,n} \) such that \( \phi(c) = 0 \) for all \( c \in C \), and such that \( C \) has either of the two intersection patterns shown in Figure 1 (type 1 or type 2). Then \( \text{Mod}_{g,n}[\phi] \) is generated by \( \{T_c \mid c \in C\} \).

![Figure 1](image_url)

**Figure 1.** Configurations of types 1 and 2 determining generating sets for \( \text{Mod}_{g,n}[\phi] \). We label the boundary components \( \Delta_i \) for \( i = 0, \ldots, 2g - 3 \), with \( \Delta_i \) positioned between \( b_i \) and \( b_{i+1} \) (for clarity most of the labels have been omitted). Given a partition \( \kappa = (\kappa_1, \ldots, \kappa_n) \) of \( 2g - 2 \), only the \( n \) boundary components \( \Delta_i \) for \( i_\ell \) of the form \( \sum_{j=1}^{\ell} \kappa_j \) (interpreted mod \( 2g - 2 \)) are included, and \( \Delta_i \) is assigned the signature \(-1 - \kappa_\ell\). Under this scheme, each complementary region determined by \( C \) contains exactly one boundary component, and the signatures are arranged so that each curve in \( C \) is admissible.
Remark 1.5. As shown in Section 2.3 (see Proposition 2.14 in particular), for any $\phi$ of signature $-1 - \kappa$, there is a configuration $C$ satisfying the hypotheses of Theorem B. If $\text{gcd}(\kappa)$ is odd, then configurations of both types exist. If $\text{gcd}(\kappa)$ is even and

$$\text{Arf}(\phi) = \begin{cases} 1 & g \equiv 0, 3 \pmod{4} \\ 0 & g \equiv 1, 2 \pmod{4}, \end{cases}$$

then only the type 1 configuration exists, and in the remaining case where $\text{gcd}(\kappa)$ is even and

$$\text{Arf}(\phi) = \begin{cases} 1 & g \equiv 1, 2 \pmod{4} \\ 0 & g \equiv 0, 3 \pmod{4}, \end{cases}$$

then only the type 2 configuration exists. See §2.2 for the definition of $\text{Arf}(\phi)$.

Remark 1.6. Theorem B has an extension to general framings of holomorphic type. A general such framing has $m \geq 0$ boundary components of winding number $-1$ in addition to the boundary components $\Delta_1, \ldots, \Delta_n$ discussed in Theorem B. These can be placed on the surface by “splitting” any collection of the curves $b_i$ into multiple curves $b_{i,1}, \ldots, b_{i,k_i+1}$ separated by $k_i$ boundary components of winding number $-1$. See Figure 2. The arguments of Section 3 (in particular, Lemma 3.8) apply equally well to this more general class of configurations of curves; the remaining steps in the proof make no reference to the specific configuration and thus require no modification whatsoever.

![Figure 2](image.png)

**Figure 2.** “Splitting” a curve $b_i$ into parallel copies $b_{i,1}, \ldots, b_{i,k_i+1}$ separated by $k_i$ boundary components of winding number $-1$. Here $k_i = 3$.

Theorem B in turn implies a finite generation result for stabilizers of absolute framings.

**Corollary 1.7.** Let $g, \kappa$ and $\phi$ be as above. Let $\bar{\phi}$ be the absolute framing on $\Sigma_g^n$ obtained by shrinking the boundary components of $\Sigma_{g,n}$ to punctures; then $\text{Mod}_g^n[\bar{\phi}]$ is generated by finitely many Dehn twists.

An explicit finite generating set for $\text{Mod}_g^n[\bar{\phi}]$ is given in Corollary 6.10. In general, the set of Dehn twists described in Theorem B only generates a finite–index subgroup of $\text{Mod}_g^n[\bar{\phi}]$ (see Proposition 6.11).

**Remark 1.8.** Both $\text{Mod}_{g,n}[\phi]$ and $\text{Mod}_g^n[\bar{\phi}]$ are of infinite index in their respective ambient mapping class groups, and so a priori could be infinitely generated. To the best of the authors’ knowledge, Theorem B and Corollary 1.7 are the first proofs that these groups are finitely generated. This is
another instance of a surprising and poorly-understood theme in the study of mapping class groups: *stabilizers of geometric structures often have unexpectedly strong finiteness properties*. The most famous instance of this principle is Johnson’s proof that the Torelli group is finitely generated for all \( g \geq 3 \) [Joh83]; this was recently and remarkably improved by Ershov–He and Church–Ershov–Putman to establish finite generation for each term in the Johnson filtration [EH18, CEP17].

**Remark 1.9.** In Theorem [B] we impose the requirement that \( \phi \) have holomorphic type. Not all framings of \( \Sigma_{g,n} \) are of holomorphic type, although, as the terminology suggests, the framings induced from holomorphic differentials always are. The hypothesis that \( \phi \) be of holomorphic type is used only in Section [5]. We do not know if the results there can be improved to include arbitrary framings, although it seems that non-holomorphic framings may behave in genuinely different ways. Are their stabilizers still finitely generated? By Dehn twists?

**Remark 1.10.** In forthcoming work [PCS20], the second author and P. Portilla Cuadrado apply Theorem [B] to give a complete description of the set of vanishing cycles for Morsifications of arbitrary plane curve singularities.

**Relation with previous work.** As mentioned above, this paper serves as a sequel to [CS19]. The main result of that work considers a weaker version of the monodromy representation attached to a stratum of abelian differentials. In [CS19], we study the monodromy representation valued in the *closed* mapping class group \( \text{Mod}_g \); here we enrich our monodromy representation so as to track the location of the zeroes. There, we find that an object called an “*r*-spin structure” (c.f. Section [2.1]) governs the behavior of the monodromy representation. Here, the added structure of the locations of the zeroes allows us to refine these *r*-spin structures to the more familiar notion of a globally invariant framing of the fibers. Where the technical core of [CS19] is an analysis of the group theory of the stabilizer in \( \text{Mod}_g \) of an *r*-spin structure, here the corresponding work is to understand these “framed mapping class groups” and to work out their basic theory, including the surprising fact that these infinite-index subgroups admit the remarkably simple finite generating sets described in Theorem [B].

1.1. **Structure of the paper.** This paper is roughly divided into two parts: the first deals exclusively with relative framings on surfaces with boundary and their associated framed mapping class groups, while the second deals with variations on framed mapping class groups and their relationship with strata of abelian differentials. Readers interested only in Theorem [B] can read Sections [2–5] independently, while readers interested only in Theorem [A] need only read the introductory Section [2] together with Sections [6–8] (provided they are willing to accept Theorem [B] as a black box).

**Outline of Theorem [B]** The proof of Theorem [B] has two steps; these roughly parallel the steps of [CS19, Theorem B]. For the first step, we show in Proposition [3.1] that the Dehn twists specified in the Theorem generate the “admissible subgroup” \( T_{\phi} \leq \text{Mod}_{g,n}[\phi] \) (see Section [2.3]). The proof of this step is an *ad hoc* extension of arguments appearing in [CS19, §5] and [Sal19, §§8, 9], and occupies the majority of Section [3].

The second step is to show that the admissible subgroup is the *entire* stabilizer of the relative framing; the proof of this step spans both Sections [4] and [5] in [CS19] and [Sal19]. The analogous step
is accomplished using the “Johnson filtration” of the mapping class group, a strategy which does not work for surfaces with multiple boundary components. Instead, we prove that $T_\phi = \text{Mod}_{g,n}[\phi]$ by induction on the number of boundary components of $\Sigma_{g,n}$.

The base case of the induction (when there is a single boundary component of winding number $1 - 2g$) takes place in Section 4. Its proof relies heavily on the analysis of $\text{CS19}$ and the relationship between framings and “$r$–spin structures,” their analogues on closed surfaces (see the end of §2.1). Using a version of the Birman exact sequence adapted to framed mapping class groups (Lemma 4.6), we show that the equality $T_\phi = \text{Mod}_{g,n}[\phi]$ is equivalent to the statement that $T_\phi$ contains “enough separating twists.” We directly exhibit these twists in Proposition 4.1, refining $\text{CS19}$, Lemma 6.4 and its counterpart in $\text{Sal19}$; the reader is encouraged to think of Proposition 4.1 as the “canonical version” of this statement.

The inductive step of the proof that $T_\phi = \text{Mod}_{g,n}[\phi]$ is contained in Section 5. The overall strategy is to introduce a connected graph $A^s$ on which $\text{Mod}_{g,n}[\phi]$ acts vertex– and edge–transitively (see Sections 5.1 through 5.4). The heart of the argument is thus to establish these transitivity properties (Lemma 5.3) and the connectedness of $A^s$ (Lemma 5.4); these both require a certain amount of care, and the arguments are lengthy. Standard techniques then imply $\text{Mod}_{g,n}[\phi]$ is generated by the stabilizer of a vertex (which we can identify with $\text{Mod}_{g,n-1}[\phi']$ for some $\phi'$) together with an element that moves along an edge (Lemma 5.10). Applying the inductive hypothesis and explicitly understanding the action of certain Dehn twists on $A^s$ together yield that $T_\phi = \text{Mod}_{g,n}[\phi]$, completing the proof of Theorem B.

Variations on framed mapping class groups. Section 6 is an interlude into the theory of other framed mapping class groups. In Section 6.1 we introduce the theory of pronged surfaces, surfaces with extra tangential data which mimic the zero structure of an abelian differential. After discussing the relationship between the mapping class groups $\text{Mod}^*_{g,n}$ of pronged surfaces and surfaces with boundaries or marked points, we introduce a theory of relative framings of pronged surfaces and hence a notion of framed, pronged mapping class group $\text{Mod}^*_{g,n}[\phi]$. The main result of this subsection is Proposition 6.6, which exhibits $\text{Mod}^*_{g,n}[\phi]$ as a certain finite extension of $\text{Mod}_{g,n}[\phi]$.

We then proceed in Section 6.2 to a discussion of absolute framings of pointed surfaces, as in the beginning of this Introduction. When a surface has marked points instead of boundary components, framings can only be considered up to absolute isotopy. Therefore, the applicable notion is not a relative but an absolute framing $\bar{\phi}$. In this section we prove Theorem 6.8 which states that the (pronged) relative framing stabilizer $\text{Mod}^*_{g,n}[\phi]$ surjects onto the (pointed) absolute framing stabilizer $\text{PMod}_{g,n}[\bar{\phi}]$. Combining this theorem with work of the previous subsection also gives explicit generating sets for $\text{PMod}_{g,n}[\bar{\phi}]$ (see Corollary 6.10).

Outline of Theorem A. The proof of Theorem A is accomplished in Section 7. After recalling background material on abelian differentials (§7.1) and exploring the different sorts of framings a differential induces (§7.2), we record the definitions of certain marked strata, first introduced in $\text{BSW16}$ (§7.3). These spaces fit together in a tower of coverings §14 which evinces the structure of the pronged
mapping class group, as discussed in Section 6.1. By a standard continuity argument, the monodromy of each covering must stabilize a framing (see Lemma 7.10 and Corollaries 7.11 and 7.12).

Using these marked strata, we can upgrade the $\text{Mod}_{g}^{\ast}$--valued monodromy of $H$ into a $\text{Mod}_{g,n}^{\ast}$--valued homomorphism, and passing to a certain finite cover of the stratum therefore results in a space $H^{\text{pr}}$ whose monodromy lies in $\text{Mod}_{g,n}$. By realizing the generating set of Theorem B as cylinders on a prototype surface in $H^{\text{pr}}$, we can explicitly construct deformations whose monodromy is a Dehn twist, hence proving that the Mod_{g,n}^{\ast}$--valued monodromy group of $H^{\text{pr}}$ is the entire stabilizer of the appropriate framing (Theorem 7.13).

To deduce Theorem A from the monodromy result for $H^{\text{pr}}$ requires an understanding of the interactions between all three types of framed mapping class groups. Using the diagram of coverings (14) together with the structural results of Section 6, we conclude that the Mod_{g,n}^{\ast}$--valued monodromy of $H$ is exactly the framing stabilizer $\text{Mod}_{g,n}^{\ast}[\phi]$ (Theorem 7.14). An application of Theorem 6.8 together with a discussion of the permutation action of $\text{Mod}_{g}^{\ast}[\phi]$ on $Z$ finishes the proof of Theorem A.

The concluding Section 8 contains applications of our analysis to the classification of components of certain covers of strata (Corollaries 8.1 and 8.2) as well as to the relationship between cylinders and the fundamental groups of strata (§8.2). This section also contains the proof of Corollary 1.4.

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2. Framings and framed mapping class groups

2.1. Framings. We begin by recalling the basics of framed surfaces. Our conventions ultimately follow those of Randal-Williams [RW13 Sections 1.1, 2.3], but we have made some convenient cosmetic alterations and use language compatible with our previous papers [Sal19,CS19]. See Remark 2.1 below for an explanation of how to reconcile these two presentations.

**Framings, (relative) isotopy.** Let $\Sigma_{g,n}$ denote a compact oriented surface of genus $g$ with $n \geq 1$ boundary components $\Delta_{1}, \ldots, \Delta_{n}$. Through Section 5 we will work exclusively with boundary components, but in Section 6 we will also consider surfaces equipped with marked points. We formulate our discussion here for surfaces with boundary components; we will briefly comment on the changes necessary to work with marked points in Section 6.

Throughout this section we fix an orientation $\theta$ and a Riemannian metric $\mu$ of $\Sigma_{g,n}$, affording a reduction of the structure group of the tangent bundle $T\Sigma_{g,n}$ to $SO(2)$. A framing of $\Sigma_{g,n}$ is an
isomorphism of $\text{SO}(2)$-bundles

\[ \phi : T \Sigma_{g,n} \to \Sigma_{g,n} \times \mathbb{R}^2. \]

With $\theta, \mu$ fixed, framings are in one-to-one correspondence with nowhere-vanishing vector fields $\xi$; in the sequel we will largely take this point of view. In this language, we say that two framings $\phi, \psi$ are isotopic if the associated vector fields $\xi_\phi$ and $\xi_\psi$ are homotopic through nowhere-vanishing vector fields.

Suppose that $\phi$ and $\psi$ restrict to the same framing $\delta$ of $\partial \Sigma_{g,n}$. In this case, we say that $\phi$ and $\psi$ are relatively isotopic if they are isotopic through framings restricting to $\delta$ on $\partial \Sigma_{g,n}$. With a choice of $\delta$ fixed, we say that $\phi$ is a relative framing if $\phi$ is a framing restricting to $\delta$ on $\partial \Sigma_{g,n}$.  

(Relative) winding number functions. Let $(\Sigma_{g,n}, \phi)$ be a (relatively) framed surface. We explain here how the data of the (relative) isotopy class of $\phi$ can be encoded in a topological structure known as a (relative) winding number function. Let $c : S^1 \to \Sigma_{g,n}$ be a $C^1$ immersion. Given two vectors $v, w \in T_x \Sigma_{g,n}$, we denote the angle (relative to the metric $\mu$) between $v, w$ by $\angle(v, w)$. We define the winding number $\phi(c)$ of $c$ as the degree of the “Gauss map” restricted to $c$:

\[ \phi(c) = \int_{S^1} d\angle(c'(t), \xi\phi(c(t))). \]

The winding number $\phi(c)$ is clearly an invariant of the isotopy class of $\phi$, and is furthermore an invariant of the isotopy class of $c$ as an immersed curve in $\Sigma_{g,n}$.  

Possibly after altering $\phi$ by an isotopy, we can assume that each component $\Delta_i$ of $\partial \Sigma_{g,n}$ contains a point $p_i$ such that $\xi_\phi(p_i)$ is orthogonally inward-pointing. We call such a point $p_i$ a legal basepoint for $\Delta_i$. We emphasize that even though $\Delta_i$ may contain several legal basepoints, we choose exactly one legal basepoint on each $\Delta_i$, so that all arcs based at $\Delta_i$ are based at the same point.

Let $a : [0, 1] \to \Sigma_{g,n}$ be a $C^1$ immersion with $a(0), a(1)$ equal to distinct legal basepoints $p_i, p_j$; assume further that $a'(0)$ is orthogonally inward-pointing and $a'(1)$ is orthogonally outward pointing. We call such an arc legal. Then the winding number

\[ \phi(a) := \int_0^1 d\angle(a'(t), \xi\phi(a(t))) \]

is necessarily half-integral, and is invariant under the relative isotopy class of $\phi$ and under isotopies of $a$ through legal arcs.

Thus a framing $\phi$ gives rise to an absolute winding number function which we denote by the same symbol. Let $\mathcal{S}$ denote the set of isotopy classes of oriented simple closed curves on $\Sigma_{g,n}$. Then the framing $\phi$ determines the winding number function

\[ \phi : \mathcal{S} \to \mathbb{Z}; \quad x \mapsto \phi(x). \]

Likewise, let $\mathcal{S}^+$ be the set obtained from $\mathcal{S}$ by adding the set of isotopy classes of legal arcs. Then $\phi$ also determines a relative winding number function

\[ \phi : \mathcal{S}^+ \to \frac{1}{2}\mathbb{Z}; \quad x \mapsto \phi(x). \]

\[ \text{It is not, however, an invariant of the homotopy class of the map } c, \text{ since the winding number will change under the addition or removal of small self-intersecting loops. In this paper we will be exclusively concerned with winding numbers of embedded curves and arcs, so we will not comment further on this. See } \text{[HJ89]} \text{ for further discussion.} \]
Signature; holomorphic type. The signature of a framing $\delta$ of $\partial \Sigma_{g,n}$ (or of a framing $\phi$ of $\Sigma_{g,n}$ restricting to $\delta$ on $\partial \Sigma_{g,n}$) is the vector

$$\text{sig}(\delta) := (\delta(\Delta_1), \ldots, \delta(\Delta_n)) \in \mathbb{Z}^n,$$

where each $\Delta_i$ is oriented with $\Sigma_{g,n}$ lying to the left. A relative framing $\phi$ is said to be of holomorphic type if $\text{sig}(\Delta_i) \leq -1$ for all $i$; in Section 7.2 we will see that if $\omega$ is an Abelian differential on a Riemann surface $X$, then the relative framing induced by $\omega$ is indeed of holomorphic type. Given a partition $\kappa = (\kappa_1, \ldots, \kappa_n)$ of $2g-2$, we say that a relative framing $\phi$ has signature $-1 - \kappa$ if the boundary components have signatures $(-1, -1 - \kappa_n)$. 

Remark 2.1. For the convenience of the reader interested in comparing the statements of this section with their counterparts in [RW13], we briefly comment on the places where the two expositions diverge. We have used the term “framing” where Randal-Williams uses “$\theta^r$–structure” with $r = 0$, and we use the term “(relative) winding number function” where Randal-Williams uses an equivalent structure denoted “$q_\xi$.” Randal-Williams also adopts some different normalization conventions. If $x$ is a curve, then $\xi(x) = \phi(x) - 1$, and if $x$ is an arc, $\xi(x) = \phi(x) - 1/2$ (in particular, $\xi$ is integer-valued on arcs).

The lemma below allows us to pass between framings and winding number functions. Its proof is a straightforward exercise in differential topology (c.f. [RW13, Proposition 2.4]).

Lemma 2.2. Let $\Sigma_{g,n}$ be a surface with $n \geq 1$ boundary components and let $\phi$ and $\psi$ be framings. Then $\phi$ and $\psi$ are isotopic as framings if and only if the associated absolute winding number functions are equal. If $\phi |_{\partial \Sigma_{g,n}} = \psi |_{\partial \Sigma_{g,n}}$, then $\phi$ and $\psi$ are relatively isotopic if and only if the associated relative winding number functions are equal.

Moreover, if $\Sigma_{g,n}$ is endowed with the structure of a CW complex for which each 0–cell is a legal basepoint and each 1–cell is either isotopic to a simple closed curve or a legal arc, then $\phi$ and $\psi$ are (relatively) isotopic if and only if the (relative) winding numbers of each 1–cell are equal.

Remark 2.3. Following Lemma 2.2, we will be somewhat lax in our terminology. Often we will use the term “(relative) framing” to refer to the entire (relative) isotopy class, or else conflate the (relative) framing with the associated (relative) winding number function.

Properties of (relative) winding number functions. The terminology of “winding number function” originates with the work of Humphries and Johnson [HJ89] (although we are discussing what they call generalized winding number functions). We recall here some properties of winding number functions which they identified.

Lemma 2.4. Let $\phi$ be a relative winding number function on $\Sigma_{g,n}$ associated to a relative framing of the same name. Then $\phi$ satisfies the following properties.

\footnote{In the non-relative setting.}
(1) **(Twist–linearity)** Let \( a \subset \Sigma_{g,n} \) be a simple closed curve, oriented arbitrarily. Then for any \( x \in S^+ \),
\[
\phi(T_a(x)) = \phi(x) + \langle x, a \rangle \phi(a),
\]
where \( \langle \cdot, \cdot \rangle : H_1(\Sigma_{g,n}, \partial \Sigma_{g,n}; \mathbb{Z}) \times H_1(\Sigma_{g,n}; \mathbb{Z}) \to \mathbb{Z} \) denotes the relative algebraic intersection pairing.

(2) **(Homological coherence)** Let \( S \subset \Sigma_{g,n} \) be a subsurface with boundary components \( c_1, \ldots, c_k \), oriented so that \( S \) lies to the left of each \( c_i \). Then
\[
\sum_{i=1}^{k} \phi(c_i) = \chi(S),
\]
where \( \chi(S) \) denotes the Euler characteristic.

**Functoriality.** In the body of the argument we will have occasion to consider maps between surfaces equipped with framings and related structures. We record here some relatively simple observations about this. Firstly, as we have already implicitly used, if \( S \subset \Sigma_{g,n} \) is a subsurface, any (isotopy class of) framing \( \phi \) of \( \Sigma_{g,n} \) restricts to a (isotopy class of) framing of \( S \). There is a converse as well; the proof is an elementary exercise in differential topology.

**Lemma 2.5.** Let \( S \subset \Sigma_{g,n} \) be a subsurface, and let \( \phi \) be a framing of \( S \). Enumerate the components of \( \Sigma_{g,n} \setminus S \) as \( S_1, \ldots, S_k \). Call such a component relatively closed if \( \partial S_i \subset \partial S \). Then \( \phi \) extends to a framing \( \tilde{\phi} \) of \( \Sigma_{g,n} \) if and only if for each relatively closed component \( S_i \), there is an equality
\[
\sum_{c \text{ a component of } \partial S_i} \phi(c) = \chi(S_i),
\]
with each \( c \) oriented with \( S_i \) to the left.

In particular, suppose that \( S = \Sigma_{g,n} \setminus D \), where \( D \) is an embedded disk, and let \( \phi \) be a framing of \( S \). Then \( \phi \) extends over \( D \) to give a framing of \( \Sigma_{g,n} \) if and only if \( \phi(\partial D) = 1 \) when \( \partial D \) is oriented with \( D \) to the left.

**Closed surfaces; \( r \)–spin structures.** For \( g \geq 2 \), the closed surface \( \Sigma_g \) does not admit any nonvanishing vector fields, but there is a “mod \( r \) analogue” of a framing called an \( r \)–spin structure. As \( r \)–spin structures will play only a passing role in the arguments of this paper (c.f. Section 4.2), we present here only the bare bones of the theory. See [Sal19, Section 3] for a much more complete discussion.

**Definition 2.6.** Let \( \Sigma_g \) be a closed surface, and as above, let \( S \) denote the set of oriented simple closed curves on \( \Sigma_g \). An \( r \)–spin structure is a function
\[
\hat{\phi} : S \to \mathbb{Z}/r\mathbb{Z}
\]
that satisfies the twist–linearity and homological coherence properties of Lemma 2.4.
Above we saw how a nonvanishing vector field $\xi$ on $\Sigma_{g,n}$ gives rise to a winding number function $\phi$ on $\Sigma_{g,n}$. Suppose now that $\xi$ is an arbitrary vector field on $\Sigma_g$ with isolated zeroes $p_1, \ldots, p_k$. For $i = 1, \ldots, k$, let $\gamma_i$ be a small embedded curve encircling $p_i$ (oriented with $p_i$ to the left) and define

$$r = \gcd\{\WN_\xi(\gamma_i) - 1 \mid 1 \leq i \leq k\},$$

where $\WN_\xi(\gamma_i)$ means to take the $\mathbb{Z}$-valued winding number of $\gamma_i$ viewed as a curve on $\Sigma_g \setminus \{p_1, \ldots, p_n\}$ endowed with the framing given by $\xi$. Then it can be shown (c.f. [HJ89, Section 1]) that the map

$$\hat{\phi} : S \to \mathbb{Z}/r\mathbb{Z}; \quad \phi(x) = \WN_\xi(x) \pmod{r}$$

determines an $r$–spin structure.

2.2. The action of the mapping class group. Recall that the mapping class group $\text{Mod}_{g,n}$ of $\Sigma_{g,n}$ is the set of isotopy classes of self–homeomorphisms of $\Sigma_{g,n}$ which restrict to the identity on $\partial \Sigma_{g,n}$. Therefore $\text{Mod}_{g,n}$ acts on the set of relative (isotopy classes of) framings, and hence the set of relative winding number functions, by pullback. As we require $\text{Mod}_{g,n}$ to act from the left, there is the formula

$$(f \cdot \phi)(x) = \phi(f^{-1}(x)).$$

We recall here the basic theory of this action, as developed by Randal-Williams [RW13, Section 2.4]. Throughout this section, we fix a framing $\delta$ of $\partial \Sigma_{g,n}$ and may therefore choose a legal basepoint on each boundary component once and for all.

The (generalized) Arf invariant. The $\text{Mod}_{g,n}$ orbits of relative framings are classified by a generalization of the classical Arf invariant. To define this, we introduce the notion of a distinguished geometric basis for $\Sigma_{g,n}$. For $i = 1, \ldots, n$, let $p_i$ be a legal basepoint on the $i^{th}$ boundary component of $\Sigma_{g,n}$. A distinguished geometric basis is a collection

$$B = \{x_1, y_1, \ldots, x_g, y_g\} \cup \{a_2, \ldots, a_n\}$$

of $2g$ oriented simple closed curves $x_1, \ldots, y_g$ and $n - 1$ legal arcs $a_2, \ldots, a_n$ that satisfy the following intersection properties.

1. $\langle x_i, y_i \rangle = i(x_i, y_i) = 1$ (here $i(\cdot, \cdot)$ denotes the geometric intersection number) and all other pairs of elements of $\{x_1, \ldots, y_g\}$ are disjoint.

2. Each arc $a_i$ is a legal arc running from $p_1$ to $p_i$ and is disjoint from all curves $x_1, \ldots, y_g$.

3. The arcs $a_i, a_j$ are disjoint except at the common endpoint $p_1$.

Remark 2.7. A distinguished geometric basis can easily be used to determine a CW–structure on $\Sigma_{g,n}$ satisfying the hypotheses of Lemma 2.2. In particular, a (relative) winding number function (and hence the associated (relative) isotopy class of framing) is determined by its values on a distinguished geometric basis. Moreover, for any vector $(w_1, \ldots, w_{2g+n-1}) \in \mathbb{Z}^{2g} \times \mathbb{Z}^{n-1}$, there exists a framing $\phi$ of $\Sigma_{g,n}$ realizing the values $(w_1, \ldots, w_{2g+n-1})$ on a chosen distinguished geometric basis (this is a straightforward construction).
Let $B$ be a distinguished geometric basis for the framed surface $(\Sigma_{g,n}, \phi)$. The Arf invariant of $\phi$ relative to $B$ is the element of $\mathbb{Z}/2\mathbb{Z}$ given by
\[
\text{Arf}(\phi, B) = \sum_{i=1}^{g} (\phi(x_i) + 1)(\phi(y_i) + 1) + \sum_{j=2}^{n} (\phi(a_j) + \frac{1}{2})(\phi(\Delta_j) + 1) \pmod{2}
\] (mod 2) (1)
(compare to [RW13, (2.4)]).

**Lemma 2.8** (c.f. Proposition 2.8 of [RW13]). Let $(\Sigma_{g,n}, \phi)$ be a framed surface. Then the Arf invariant $\text{Arf}(\phi, B)$ is independent of the choice of distinguished geometric basis $B$.

**Remark 2.9.** We caution the reader that while the Arf invariant does not depend on the choice of basis, it does depend on the choice of legal basepoints on each boundary component. Since $\text{Mod}_{g,n}$ fixes the boundary pointwise it preserves our choice of legal basepoint on each boundary component.

Following Lemma 2.8, we write $\text{Arf}(\phi)$ to indicate the Arf invariant $\text{Arf}(\phi, B)$ computed on an arbitrary choice of distinguished geometric basis. The next result shows that for $g \geq 2$, the Arf invariant classifies $\text{Mod}_{g,n}$ orbits of relative isotopy classes of framings.

**Proposition 2.10** (c.f. Theorem 2.9 of [RW13]). Let $g \geq 2$ and $n \geq 1$ be given, and let $\phi, \psi$ be two relative framings of $\Sigma_{g,n}$ which agree on $\partial \Sigma_{g,n}$. Then there is an element $f \in \text{Mod}_{g,n}$ such that $f \cdot \psi = \phi$ if and only if $\text{Arf}(\phi) = \text{Arf}(\psi)$.

For surfaces of genus 1, the action is more complicated. In this work we will only need to study the case of one boundary component; this was treated by Kawazumi [Kaw18]. Let $(\Sigma_{1,1}, \phi)$ be a framed surface. Consider the set
\[
a(\phi) = \{ \phi(x) \mid x \subset \Sigma_{1,1} \text{ an oriented s.c.c.} \}.
\]
The twist-linearity formula (Lemma 2.4.1) implies that $a(\phi)$ is in fact an ideal of $\mathbb{Z}$. We define the genus-1 Arf invariant of $\phi$ to be the unique nonnegative integer $\text{Arf}_1(\phi) \in \mathbb{Z}_{\geq 0}$ such that
\[
a(\phi) = \text{Arf}_1(\phi)\mathbb{Z}.
\] (2)

**Remark 2.11.** The normalization conventions for Arf as in [1] and $\text{Arf}_1$ as in [2] are different. In Section 4.1 we will reconcile them, but here we chose to present the “natural” definition of $\text{Arf}_1$.

**Lemma 2.12** (c.f. Theorem 0.3 of [Kaw18]). Let $\phi$ and $\psi$ be relative framings of $\Sigma_{1,1}$ which agree on $\partial \Sigma_{1,1}$. Then there is $f \in \text{Mod}_{1,1}$ such that $f \cdot \psi = \phi$ if and only if $\text{Arf}_1(\phi) = \text{Arf}_1(\psi)$.

**2.3. Framed mapping class groups.** Having studied the orbits of the $\text{Mod}_{g,n}$ action on the set of framings in the previous section, we turn now to the stabilizer of a framing.

**Definition 2.13** (Framed mapping class group). Let $(\Sigma_{g,n}, \phi)$ be a (relatively) framed surface. The framed mapping class group $\text{Mod}_{g,n}[\phi]$ is the stabilizer of the relative isotopy class of $\phi$:
\[
\text{Mod}_{g,n}[\phi] = \{ f \in \text{Mod}_{g,n} \mid f \cdot \phi = \phi \}.
\]
Admissible curves, admissible twists, and the admissible subgroup. In our study of Mod\(g,n[\phi]\), a particularly prominent role will be played by the Dehn twists that preserve \(\phi\). An admissible curve on a framed surface \((\Sigma_{g,n}, \phi)\) is a nonseparating simple closed curve \(a\) such that \(\phi(a) = 0\). It follows from the twist–linearity formula (Lemma 2.4.1) that the associated Dehn twist \(T_a\) preserves \(\phi\). We call the mapping class \(T_a\) an admissible twist. Finally, we define the admissible subgroup to be the group generated by all admissible twists:

\[
T_\phi := \langle T_a \mid a \text{ admissible for } \phi \rangle.
\]

Change–of–coordinates for framed surfaces. The classical “change–of–coordinates principle” for surfaces is a body of techniques for constructing special configurations of curves and subsurfaces on a fixed surface (c.f. [FM11, Section 1.3]). The underlying mechanism is the classification of surfaces, which provides a homeomorphism between a given surface and a “reference surface;” if a desired configuration exists on the reference surface, then the configuration can be pulled back along the classifying homeomorphism.

A similar principle exists for framed surfaces, governing when configurations of curves with prescribed winding numbers exist on framed surfaces. The classification results Proposition 2.10 and Lemma 2.12 assert that the Arf invariant provides the only obstruction to constructing desired configurations of curves in the presence of a framing. We will make extensive and often tacit use of the “framed change–of–coordinates principle” throughout the body of the argument. Here we will illustrate some of the more frequent instances of which we avail ourselves. Recall that a \(k\)–chain is a sequence \(c_1, \ldots, c_k\) of curves such that

\[
i(c_i, c_{i+1}) = 1 \quad \text{for} \quad i = 1, \ldots, k-1, \quad \text{and} \quad i(c_i, c_j) = 0 \quad \text{for} \quad |i - j| \geq 2.
\]

Proposition 2.14 (Framed change–of–coordinates). Let \((\Sigma_{g,n}, \phi)\) be a relatively framed surface with \(g \geq 2\) and \(n \geq 1\). A configuration \(x_1, \ldots, x_k\) of curves and/or arcs with prescribed intersection pattern and winding numbers \(\phi(x_i) = s_i\) exists if and only if

(a) a configuration \(\{x'_1, \ldots, x'_k\}\) of the prescribed topological type exists in the “unframed” setting where the values \(\phi(x'_i)\) are allowed to be arbitrary,

(b) there exists some framing \(\psi\) such that \(\psi(x_i) = s_i\) for all \(i\), and

(c) if \(\text{Arf}(\psi)\) is determined by the constraints of (b), then \(\text{Arf}(\phi) = \text{Arf}(\psi)\).

In particular:

(1) For \(s \in \mathbb{Z}\) arbitrary, there exists a nonseparating curve \(c \subset \Sigma_{g,n}\) with \(\phi(c) = s\).

(2) For \(n = 1\), there exists a \(2g\)–chain of admissible curves on \(\Sigma_{g,1}\) if and only if the pair

\[(g \pmod{4}, \text{Arf}(\phi)) \in (\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})\]

is one of the four listed below:

\[(0,0), (1,1), (2,1), (3,0)\].

(3)

Such a chain is called a maximal chain of admissible curves.

Proof. We will prove (1) and (2), from which it will be clear how the general argument works. We begin with (1). Let \(B = \{x_1, \ldots, y_g, a_2, \ldots, a_n\}\) be a distinguished geometric basis. Following Remark 2.7 a relative framing \(\psi\) of \(\Sigma_{g,n}\) can be constructed by (freely) specifying the values \(\psi(b)\) for each element \(b \in B\). Set \(\psi(x_1) = s\) and let \(\psi(y_1)\) be arbitrary. Since \(g \geq 2\), it is possible to choose the
values of $\psi(x_2), \psi(y_2)$ such that $\text{Arf}(\psi) = \text{Arf}(\phi)$. By Proposition 2.10, there exists a diffeomorphism $f : \Sigma_{g,n} \to \Sigma_{g,n}$ such that $f \cdot \psi = \phi$. We see that $f(x_1)$ is the required curve:

$$\phi(f(x_1)) = (f^{-1} \cdot \phi)(c_1) = \psi(x_1) = s$$

as required.

For (2), consider a maximal chain $a_1, \ldots, a_{2g}$ on $\Sigma_{g,1}$. Define $b_1 := a_1$ and choose curves $b_2, b_3, \ldots, b_g$ each disjoint from all $a_j$ with $j$ odd, such that $B = \{b_1, a_2, b_2, a_4, \ldots, b_g, a_{2g}\}$ is a distinguished geometric basis. We now construct a framing $\psi$ such that each $a_i$ is admissible. By construction, the curves $b_k, a_{2k+1}$ form pairs of pants for each $1 \leq k \leq g - 1$. By the homological coherence property (Lemma 2.4.2), if each $a_i$ is to be admissible, we must have $\psi(b_k) = 1 - k$ for $1 \leq k \leq g$ when $b_k$ is oriented so that the pair of pants cobounded by $b_{k-1}$ and $a_{2k-1}$ lies to the left. $\text{Arf}(\psi)$ is determined by these conditions, and is computed to be

$$\text{Arf}(\psi) = \begin{cases} 0 & g \equiv 0, 3 \pmod{4} \\ 1 & g \equiv 1, 2 \pmod{4}. \end{cases}$$

If the pair $(g, \text{Arf}(\phi))$ is one of those listed in (3), then by Proposition 2.10 there exists $f : \Sigma_{g,1} \to \Sigma_{g,1}$ such that $f \cdot \psi = \phi$. As above, we find that $f(a_1), \ldots, f(a_{2g})$ is the required maximal chain of admissible curves. Conversely, if $(g, \text{Arf}(\phi))$ does not appear in (3), then the Arf invariant of $\phi$ obstructs the existence of a maximal chain of admissible curves.

□

3. Finite generation of the admissible subgroup

Theorem B asserts that the framed mapping class group $\text{Mod}_{g,n} [\phi]$ is generated by the finite collection of Dehn twists shown in Figure 1. In this section, we take the first step to establishing this result. Proposition 3.1 establishes that the finite set of Dehn twists shown in Figure 1 generate the admissible subgroup. In the subsequent sections we will show that there is an equality $\mathcal{T}_\phi = \text{Mod}_{g,n} [\phi]$, establishing Theorem B.

**Proposition 3.1 (Generating the admissible subgroup).** Fix $g \geq 5$ and a partition $\kappa = \{\kappa_1, \ldots, \kappa_n\}$ of $2g - 2$. Let $\phi$ be a framing of $\Sigma_{g,n}$ of signature $-1 - \kappa$. Let $C$ be a configuration of $\phi$-admissible curves of type 1 or 2 (c.f. Figure 2). Define

$$\mathcal{T}_C := \langle T_c \mid c \in C \rangle.$$

Then $\mathcal{T}_C = \mathcal{T}_\phi$.

The proof of Proposition 3.1 closely follows the approach developed in [Sal19]. The heart of the argument (Lemma 3.6) is to show that our finite collection of twists generates a version of a point-pushing subgroup for a subsurface. This will allow us to express all admissible twists supported on this subsurface with our finite set of generators. Having shown this, we can import our method from [Sal19] (appearing below as Proposition 3.10) which allows us to propagate this argument across the set of subsurfaces, proving the result.
3.1. **Framed subsurface push subgroups.** Let $S \subset \Sigma_{g,n}$ be a subsurface and suppose $\Delta \subset \partial S$ is a boundary component. Let $\overline{S}$ denote the surface obtained from $S$ by capping $\Delta$ with a disk, and let $UT\overline{S}$ denote the associated unit tangent bundle. Recall the *disk–pushing homomorphism* $\mathcal{P} : \pi_1(UT\overline{S}) \to \text{Mod}(S)$ [FM11, Section 4.2.5]. The inclusion $S \hookrightarrow \Sigma_{g,n}$ induces a homomorphism $i : \text{Mod}(S) \to \text{Mod}(\Sigma_{g,n})$ which restricts to give a *subsurface push homomorphism* $\mathcal{P} := i \circ \mathcal{P}$:

$$\mathcal{P} : \pi_1(UT\overline{S}) \to \text{Mod}(\Sigma_{g,n}).$$

The *framed subsurface push subgroup* $\Pi(S)$ is the intersection of this image with $\text{Mod}_{g,n}[\phi]$:

$$\Pi(S) := \text{Im}(\mathcal{P}) \cap \text{Mod}_{g,n}[\phi].$$

Note that $\Pi(S)$ is defined relative to the boundary component $\Delta$, suppressed in the notation. In practice, the choice of $\Delta$ will be clear from context.

There is an important special case of the above construction. Let $b \subset \Sigma_{g,n}$ be an oriented nonseparating curve satisfying $\phi(b) = -1$. The subsurface $\Sigma_{g,n} \setminus \{b\}$ has a distinguished boundary component $\Delta$ corresponding to the left-hand side of $b$. For this choice of $(S, \Delta)$, we streamline notation, defining $\phi(\Delta) = -1$ (oriented so that $S$ lies to the left), it follows from Lemma 2.5 that the framing of $S$ can be extended over the capping disk to $\overline{S}$. Such a framing of $\overline{S}$ gives rise to a section $s : \overline{S} \to UT\overline{S}$, and hence a splitting $s_* : \pi_1(\overline{S}) \to \pi_1(UT\overline{S})$.

**Lemma 3.2.** *If $\phi(\Delta) = -1$, there is an equality*

$$\Pi(S) = \mathcal{P}(s_*(\pi_1(\overline{S}))).$$

**Proof.** Let $x_1, \ldots, x_n \in \pi_1(\overline{S})$ be a system of generators such that each $x_i$ is represented by a simple based loop on $\overline{S}$. Under $\mathcal{P} \circ s_*$, each such $x_i$ is sent to a multitwist:

$$\mathcal{P}(s_*(x_i)) = T_{x^L_i}^s T_{x^R_i}^{-1} T_{x^L_i}^{-\phi(x^L_i)} ,$$

where the curves $x^L_i, x^R_i \subset S$ are characterized by the following two conditions:

1. $x^L_i \cup x^R_i \cup \Delta$ form a pair of pants (necessarily lying to the left of $\Delta$),
2. $x^L_i$ (resp. $x^R_i$) lies to the left (resp. right) of $x_i$ as a based oriented curve.

By the twist-linearity formula (Lemma 2.4), $\mathcal{P}(s_*(x_i))$ preserves $\phi$. As the set of $x_i$ generates $\pi_1(\overline{S})$, it follows that $\mathcal{P}(s_*(\pi_1(\overline{S}))) \leq \Pi(S)$.

To establish the opposite containment, we recall that $s_*$ gives a splitting of the sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(UT\overline{S}) \longrightarrow \pi_1(\overline{S}) \longrightarrow 1$$

and so it suffices to show that $\Pi(S) \cap \ker(\mathcal{P} \circ p_*) = \{e\}$. Under $\mathcal{P}$, the generator of $\ker p_*$ is sent to $T_\Delta$. As $\phi(\Delta) = -1$ and $\Delta$ was constructed by cutting along the non-separating curve $b \subset \Sigma_{g,n}$, the twist-linearity formula shows that $\langle T_\Delta \rangle \cap \text{Mod}_{g,n}[\phi] = \{e\}$ and the result follows. $\square$
3.2. Generating framed push subgroups. We want to show that our finitely-generated subgroup $T_C$ contains a framed subsurface push subgroup $\tilde{\Pi}(S)$ for a subsurface $S$ that is "as large as possible". In Lemma 3.3 below, we show that this can be accomplished inductively by successively showing containments $\tilde{\Pi}(S_i) \subseteq T_C$ for an increasing union of subsurfaces $\ldots \subset S_i \subset S_{i+1} \subset \ldots$.

![Figure 3. The configuration discussed in Lemma 3.3.](image)

**Lemma 3.3.** Let $S \subset \Sigma_{g,n}$ be a subsurface and let $\Delta$ be a boundary component of $S$ such that $\phi(\Delta) = -1$, giving rise to the associated framed subsurface push subgroup $\tilde{\Pi}(S)$. Let $c \in \partial S$ be a boundary component distinct from $\Delta$, and let $a \subset \Sigma_{g,n}$ be an admissible curve such that $i(a,c) = 2$ and $a \cap S$ is a single arc that connects $c$ to itself. Let $a' \subset S$ be an admissible curve satisfying $i(a,a') = 1$. Let $S^+$ be the subsurface given by a regular neighborhood of $S \cup a$. Then $\tilde{\Pi}(S^+) \subseteq \langle T_a, T_{a'}, T_C \rangle$.

**Proof.** Let $a'' \subset S^+$ be a curve such that $a \cup a'' \cup \Delta$ forms a pair of pants and such that $a'' \cap S$ is a single arc based at the same boundary component of $S$ as $a \cap S$. By homological coherence (Lemma 2.4.2), $a''$ is admissible. Observe that $T_a T_{a'} T_{a''}^{-1} \in \tilde{\Pi}(S^+)$. Applying $T_a T_{a'}$ takes both $a$ and $a''$ to admissible curves on $S$, and so

$$(T_a T_{a'}) T_a T_{a''}^{-1} (T_a T_{a'})^{-1} \in \tilde{\Pi}(S).$$

Consequently $T_a T_{a''}^{-1} \in \langle T_a, T_{a'}, \tilde{\Pi}(S) \rangle$. Let $x_1, \ldots, x_k \in \tilde{\Pi}(S)$ be a generating set. The inclusion $S \hookrightarrow S^+$ induces an inclusion $\tilde{\Pi}(S) \hookrightarrow \tilde{\Pi}(S^+)$, and $\tilde{\Pi}(S^+)$ is generated by $x_1, \ldots, x_k$ and $T_a T_{a''}^{-1}$. These elements are all contained in the group $\langle T_a, T_{a'}, \tilde{\Pi}(S) \rangle$. $\square$.

**Generation via networks.** The inductive criterion Lemma 3.3 leads to the notion of a *network*, which is a configuration of curves designed such that Lemma 3.3 can be repeatedly applied. Here we discuss the basic theory.

**Definition 3.4 (Networks).** Let $S$ be a surface of finite type. For the purposes of the definition, punctures and boundary components are interchangeable; we convert both into boundary components. A *network* on $S$ is any collection $\mathcal{N} = \{a_1, \ldots, a_n\}$ of simple closed curves (not merely isotopy classes) on $S$ such that $\#(a_i \cap a_j) \leq 1$ for all pairs of curves $a_i, a_j \in \mathcal{N}$, and such that there are no triple intersections.

A network $\mathcal{N}$ has an associated *intersection graph* $\Lambda_N$, whose vertices correspond to curves $x \in \mathcal{N}$, with vertices $x, y$ adjacent if and only if $\#(x \cap y) = 1$. A network is said to be *connected* if $\Lambda_N$ is
connected, and arboreal if $\Lambda_N$ is a tree. A network is filling if

$$S \setminus \bigcup_{a \in N} a$$

is a disjoint union of disks and boundary–parallel annuli.

A network $N$ determines a subgroup $T_N \leq \text{Mod}(\Sigma_{g,n})$ by taking the group generated by the Dehn twists about curves in $N$:

$$T_N := \langle T_a \mid a \in N \rangle.$$  

The following appears in slightly modified form as [Sal19, Lemma 9.4].

**Lemma 3.5.** Let $S \subset \Sigma_{g,n}$ be a subsurface with a boundary component $\Delta$ satisfying $\phi(\Delta) = -1$. Let $N$ be a network of admissible curves on $S$ that is connected, arboreal, and filling, and suppose that there exist $a, a' \in N$ such that $a \cup a' \cup \Delta$ forms a pair of pants. Then $\tilde{\Pi}(S) \leq T_N$.

### 3.3. The key lemma

Proposition 3.10, to be stated below, gives a criterion for a group $H$ to contain the admissible subgroup $T_\phi$. It asserts that containing a framed subsurface push subgroup of the form $\tilde{\Pi}(b)$ is “nearly sufficient.” In preparation for this, we show here that $T_C$ contains such a subgroup. Ideally, we would like to use the network generation criterion (Lemma 3.5), but the configuration $C$ does not satisfy the hypotheses and so more effort is required.

![Figure 4](image-url)

**Figure 4.** The curve $b$ of Lemma 3.6. In Type 1, it is the unique curve bounding a pair of pants with $a_0$ and $a_4$ and disjoint from all elements of $C$ except $a_3$. In Type 2, it is the unique curve bounding a pair of pants with $a_0$ and $a_6$ and disjoint from all elements of $C$ except $a_7$. Here we have illustrated the ordered partition $(1, 2g - 6, 2, 1)$ in Type 1, and in Type 2 we have illustrated $(1, 2, 2g - 6, 1)$. 
Lemma 3.6. Let \( C \) be a configuration of admissible curves of type 1 or 2 (c.f. Figure 4), and let \( b \) be the curve indicated in Figure 4. Then \( \Pi(b) \leq T_C \).

This will be proved in three steps. In the first step, recorded as Lemma 3.7 below, we apply the network generation criterion (Lemma 3.5) on a convenient subsurface \( S_0 \subset \Sigma_{g,n} \setminus b \). In the second step (Lemma 3.8), we use the inductive criterion of Lemma 3.3 to see that \( T_C \) contains \( \Pi(S_{n-1}) \) for \( S_{n-1} \) a certain “large” subsurface of \( \Sigma_{g,n} \setminus b \), and finally in Step 3 we find that \( T_C \) contains one last twist suitable for a final application of Lemma 3.3.

Step 1: Networks on subsurfaces.

![Figure 5](image.png)

**Figure 5.** The surface \( S_0 \) is bounded by \( d \) and the left side of \( b \), and the subsurface \( A \) is bounded by \( c \) and the right side of \( b \). For clarity, we have moved the boundary components from their original locations in Figures 1 and 4; in these coordinates, the curves \( b_i \in C \) would look distorted, so we omit them from the figure. The boundary component \( \Delta_n \) contained in \( A \) is the unique one contained in the component of the complement of \( C \) containing \( b \).

Lemma 3.7. Let \( S_0 \subset \Sigma_{g,n} \) be the subsurface shown in Figure 5. There is a containment \( \Pi(S_0) \leq T_C \).

**Proof.** Let \( C' \) be the network shown in Figure 5 consisting of all the red curves (all depicted curves except for \( b, c, d \)). In both cases, \( C' \) satisfies the hypotheses of Lemma 3.5, so that \( \Pi(S_0) \leq T_{C'} \). In case 1, the elements of \( C' \) are all elements of \( C \), so that the claim is proved in this case.
In case 2, the curves $a_{2g}$ and $a_{2g+1}$ are not elements of $C$. As shown in \cite{CS19} Lemma 5.12, there is a containment:

$$\{T_{a_{2g}}, T_{a_{2g+1}}\} \subset \mathcal{T}_C.$$ 

Consequently, $\mathcal{T}_C \leq \mathcal{T}_C$ in case 2 as well, completing the proof. \hfill \Box

Step 2: Expansion.

**Lemma 3.8.** Let $A \subset \Sigma_{g,n}$ be the annulus shown in Figure 5. There is a containment $\tilde{\Pi}(\Sigma_{g,n} \setminus A) \leq \mathcal{T}_C$.

**Proof.** The configuration $\mathcal{C}$ consists of two types of curves: the $A$–curves $a_0, \ldots, a_{2g-1}$ and the $B$–curves which we enumerate $b_1, \ldots, b_{n-1}$. Define a family of subsurfaces $S_1, \ldots, S_{n-1}$ via

$$S_i = \nu(S_0 \cup b_1 \cup \cdots \cup b_i),$$

where the notation $\nu$ indicates a suitable regular neighborhood. Observe that the subsurfaces $S_{n-1}$ and $\Sigma_{g,n} \setminus A$ are isotopic and so $\tilde{\Pi}(S_{n-1}) = \tilde{\Pi}(\Sigma_{g,n} \setminus A)$.

To prove the claim, we proceed by induction, showing that $\tilde{\Pi}(S_i) \leq \mathcal{T}_C$. The case $i = 0$ was established in Lemma 3.7. Assuming that $\tilde{\Pi}(S_i) \leq \mathcal{T}_C$ for $i \geq 0$, we observe that $S_i$ contains a distinguished boundary component corresponding to the left side of $b$. For $0 \leq i \leq n-2$, the intersection $S_i \cap b_{i+1}$ is a single arc connecting some other boundary component to itself. The curve $b_{i+1}$ intersects some $A$-curve $a_j \in \mathcal{C}$ exactly once. By Lemma 3.3 and the inductive hypothesis, $\tilde{\Pi}(S_{i+1}) \leq \langle T_{b_{i+1}}, T_{a_j}, \tilde{\Pi}(S_i) \rangle \leq \mathcal{T}_C$. \hfill \Box

Step 3: One last twist.

**Lemma 3.9.** There is an admissible curve $a \subset \Sigma_{g,n}$ such that (1) $a \cap \Sigma_{g,n} \setminus A$ is a single arc connecting $c$ to itself, (2) $T_a \in \mathcal{T}_C$, and (3) there is a curve $a' \in \mathcal{C}$ contained in $\Sigma_{g,n} \setminus A$ such that $i(a, a') = 1$.

**Proof.** Figure 6 shows six admissible curves $a_1, \ldots, a_6$ arranged in a configuration of type $E_6$ and the subsurface $A$ bounded by the right side of $b$ and $c$. Observe that in both types 1 and 2, $\mathcal{C}$ contains such a subconfiguration. The $D_5$ relation (c.f. \cite{CS19} Lemma 5.9), as applied to the configuration $a_1, \ldots, a_5$, states that

$$T_{d_1} T_{d_2} \in \mathcal{T}_C,$$

and so $T_a \in \mathcal{T}_C$, where

$$a := T_{d_1} T_{d_2}^3 (a_6).$$

Such $a$ satisfies property (2) of the claim by construction, and satisfies properties (1) and (3) by an inspection of Figure 6. \hfill \Box

\footnote{Although the computations in \cite{CS19} are performed on a closed surface, they all take place in a neighborhood of $\mathcal{C}$ and hence can be imported to this setting.}
Proof of Lemma 3.6. Observe that $\Sigma_{g,n} \setminus b$ is isotopic to a regular neighborhood of $a \cup \Sigma_{g,n} \setminus A$ for $a$ the curve of Lemma 3.9. By Lemma 3.3, $\bar{\Pi}(b) \leq \langle T_a, T_{a'}, \bar{\Pi}(\Sigma_{g,n} \setminus A) \rangle$, where $a'$ is the curve of Lemma 3.9. Combining Lemma 3.9 with Lemma 3.8 it follows that $\langle T_a, T_{a'}, \bar{\Pi}(\Sigma_{g,n} \setminus A) \rangle \leq T_C$, and the result follows. □

3.4. Finite generation of the admissible subgroup. Proposition 3.10 below is taken from [Sal19, Proposition 8.2]. There, it is formulated for $r$–spin structures on closed surfaces of genus $g \geq 5$, but the result and its proof hold mutatis mutandis for framings of $\Sigma_{g,n}$ with $g \geq 5$.

**Proposition 3.10** (C.f. Proposition 8.2 of [Sal19]). Let $\phi$ be a framing of $\Sigma_{g,n}$ for $g \geq 5$. Let $(a_0, a_1, b)$ be an ordered 3–chain of curves with $\phi(a_0) = \phi(a_1) = 0$ and $\phi(b) = -1$. Let $H \leq \text{Mod}(\Sigma_{g,n})$ be a subgroup containing $T_{a_0}, T_{a_1}$, and the framed subsurface push subgroup $\bar{\Pi}(b)$. Then $H$ contains $T_\phi$.

**Proof (of Proposition 3.1).** Since $T_C \leq T_\phi$ by construction, it suffices to apply Proposition 3.10 with the subgroup $H = T_C$. Lemma 3.6 asserts that $\bar{\Pi}(b) \leq T_C$. When $C$ is of type 1 (resp. 2), the chain $(a_4, a_3, b)$ (resp. $(a_8, a_7, b)$) satisfies the hypotheses of Proposition 3.10. The result now follows by Proposition 3.10. □

4. Separating twists and the single boundary case

4.1. Separating twists. We come now to the first of two sections dedicated to showing the equality $T_\phi = \text{Mod}_{g,n}([\phi])$. This will be accomplished by induction on $n$. In this section, we establish the base case $n = 1$, while in the next section we carry out the inductive step.

The base case $n = 1$ is in turn built around a close connection with the theory of $r$–spin structures on closed surfaces (c.f. Definition 2.6) as studied in the prior papers [Sal19, CS19]. We combine
this work with a version of the Birman exact sequence (c.f. [11]) to reduce the problem of showing \( T_\phi = \text{Mod}_{g,1}[\phi] \) to the problem of showing that \( T_\phi \) contains a sufficient supply of Dehn twists about separating curves.

Below and throughout, the group \( K_{g,1} \) is defined to be the group generated by separating Dehn twists:

\[
K_{g,1} := \langle T_c \mid c \subset \Sigma_{g,1} \text{ separating} \rangle.
\]

\( K_{g,1} \) is known as the Johnson kernel. It is a deep theorem of Johnson that \( K_{g,1} \) can be identified with the kernel of a certain “Johnson homomorphism” \([\text{Joh85}]\), but we will not need to pursue this any further here.

**Proposition 4.1.** Fix \( g \geq 5 \), and let \( \phi \) be a framing of \( \Sigma_{g,1} \). Then \( K_{g,1} \subseteq T_\phi \).

A separating curve \( c \subset (\Sigma_{g,1},\phi) \) has two basic invariants. To define these, let \( \text{Int}(c) \) be the component of \( \Sigma_{g,1} \setminus c \) that does not contain the boundary component of \( \Sigma_{g,1} \). The first invariant of \( c \) is the genus \( g(c) \), defined as the genus of \( \text{Int}(c) \). The second invariant of \( c \) is the Arf invariant \( \text{Arf}(c) \). When \( g(c) \geq 2 \), define \( \text{Arf}(c) \) to be the Arf invariant of \( \phi \mid \text{Int}(c) \). As discussed in Section 2.2 the Arf invariant in genus 1 is special. For uniformity of notation later, if \( g(c) = 1 \) and \( \text{Arf}(\text{Int}(c)) \neq 0 \), define

\[
\text{Arf}(c) := \text{Arf}_1(\phi \mid \text{Int}(c)) + 1 \pmod{2}.
\]

In the special case where \( \text{Arf}_1(\phi \mid \text{Int}(c)) = 0 \), we declare \( \text{Arf}(c) \) to be the symbol \( 1^+ \) (for the purposes of arithmetic, we treat this as \( 1 \in \mathbb{Z}/2\mathbb{Z} \)). Altogether, we define the type of a separating curve \( c \) to be the pair \((g(c), \text{Arf}(c))\).

**Lemma 4.2.** Let \( \phi \) be a framing of a surface \( \Sigma_{g,1} \). Let \( c \) be a separating curve of type \((g, \varepsilon)\). For the pairs of \((g, \varepsilon)\) listed below, the separating twist \( T_c \) is contained in \( T_\phi \).

\[
\begin{align*}
(1) & \quad (1 + 4k, 1) \text{ for } k \geq 1, \\
(2) & \quad (2 + 4k, 1) \text{ for } k \geq 0, \\
(3) & \quad (3 + 4k, 0) \text{ for } k \geq 0, \\
(4) & \quad (4k, 0) \text{ for } k \geq 0, \\
(5) & \quad (1, 1^+)\text{,} \\
(6) & \quad (3, 1). 
\end{align*}
\]

**Proof.** In cases (1)–(5), the Arf invariant of the surface agrees with the Arf invariant of a surface of the same genus which supports a maximal chain of admissible curves. By the change–of–coordinates principle for framed surfaces (c.f. Proposition 2.12), it follows that every such surface supports a maximal chain of admissible curves. Applying the chain relation (c.f. [FMI1 Proposition 4.12]) to this maximal chain shows that the separating twist about the boundary component is an element of \( T_\phi \).

Consider now case (6), where the subsurface \( S_c \) determined by \( c \) has genus 3 and Arf invariant 1. In this case, the framed change–of–coordinates principle implies that \( S_c \) supports a configuration \( a_0, \ldots, a_5 \) of admissible curves with intersection pattern given by the \( E_6 \) Dynkin diagram. By the “\( E_6 \) relation” (c.f. [Mat00] Theorem 1.4]), \( T_c \) can be expressed as a product of the admissible twists \( T_{a_0}, \ldots, T_{a_5} \).

\[ \square \]
In the proof of Proposition 4.1, it will be important to understand the additivity properties of the Arf invariant when a surface is decomposed into subsurfaces.

Lemma 4.3 (c.f. Lemma 2.11 of [RW13]). Suppose that $(\Sigma, \phi)$ is a framed surface and $c = \bigcup c_i$ is a separating multicurve. Let $(S_1, \psi_1)$ and $(S_2, \psi_2)$ denote the two components of $\Sigma \setminus c$, equipped with their induced framings. Then

$$\text{Arf}(\phi) = \text{Arf}(\psi_1) + \text{Arf}(\psi_2) + \sum_{c_i \in c} (\phi(c_i) + 1) \pmod{2}.$$ 

In particular, if each $c_i$ is separating and bounds a closed subsurface on one side, then each $\phi(c_i)$ is odd and hence the Arf invariant is additive.

The proof of Proposition 4.1 is built around the well-known lantern relation.

![Figure 7. The lantern relation. The arcs labeled $x, y, z$ determine curves by taking a regular neighborhood of the arc and the incident boundary components.](image)

Lemma 4.4 (Lantern relation). For the curves $a, b, c, d, x, y, z$ of Figure 7 there is a relation

$$T_aT_bT_cT_d = T_xT_yT_z.$$ 

We will use the lantern relation to "manufacture" new separating twists using an initially–limited set of twists.

Proof of Proposition 4.1. According to [Joh79, Theorem 1], it suffices to show that $T_c \in T_\phi$ for $c$ a separating curve of genus 1 or 2. If $c$ has type (2, 1), then $T_c \in T_\phi$ by Lemma 4.2. For the remaining types (1, 1), (1, 0), (2, 0), we will appeal to a sequence of lantern relations (Configurations (A), (B), (C)) as shown in Figure 8. Each of the configurations below occupy a surface of genus $\leq 4$, and by hypothesis, $g \geq 5$. Thus, in each configuration, the specified winding numbers do not constrain the Arf invariant. Therefore by the framed change–of–coordinates principle (Proposition 2.14), there is no obstruction to constructing such configurations. We also remark that we will use the additivity of the Arf invariant (Lemma 4.3) without comment throughout.

We say that separating curves $x, y \subset \Sigma_{g,1}$ are nested if there is a containment $\text{Int}(x) \subset \text{Int}(y)$ or $\text{Int}(y) \subset \text{Int}(x)$. Configuration (A) shows that $T_xT_y^{-1} \in T_\phi$ for any $x$ of type (1, 0) and $y$ of type (2, 0) such that $x$ and $y$ are not nested. Turning to Configuration (B), we apply the lantern relation to the curves on the subsurface bounded by $a, b, z, w$ to find that $T_aT_bT_c^{-1} \in T_\phi$; here $c$ is any curve of type (2, 0) and $a, b$ have type (1, 0) and are nested inside $c$. Applying Configuration (A) with $x = a, y = c$ and then with $x = b, y = c$ shows that $T_c^{-1}T_d^2 \in T_\phi$ for an arbitrary pair $c, d$ of curves of type (2, 0).
Consider now Configuration (C). The associated lantern relation shows that $T_c T_d^{-1} \in T_\phi$ for $c, d$ again both of type $(2, 0)$. As also $T_c^{-1} T_d^2 \in T_\phi$ by the above paragraph, it follows that $T_c \in T_\phi$ for $c$ an arbitrary curve of type $(2, 0)$.

Returning to Configuration (A), it now follows that $T_x \in T_\phi$ for $x$ an arbitrary curve of type $(1, 0)$. It remains only to show $T_a \in T_\phi$ for $a$ a curve of type $(1, 1)$. To obtain this, we return to Configuration (B), but replace the curve of type $(1, 1^+)\text{ with a general curve of type (1, 1)}$. The remaining twists in the lantern relation are now all known to be elements of $T_\phi$, and hence curves of type $(1, 1)$ are elements of $T_\phi$ as well. □

4.2. The minimal stratum. We are now prepared to prove the main result of the section.

Proposition 4.5. Let $g \geq 5$ be given, and let $\phi$ be a framing of $\Sigma_{g,1}$. Then there is an equality

$$T_\phi = \text{Mod}_{g,1}[\phi].$$

As discussed above, this will be proved by relating the framing $\phi$ on $\Sigma_{g,1}$ to a $(2g-2)$–spin structure on $\Sigma_g$ by way of a version of the Birman exact sequence. In the standard Birman exact sequence for the capping map $p : \Sigma_{g,1} \to \Sigma_g$, the kernel is given by the subgroup $\pi_1(UT \Sigma_g)$. In Lemma 4.6, the subgroup $H_g \leq \pi_1(U \Sigma_g)$ is defined to be the preimage in $\pi_1(U \Sigma_g)$ of $[\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \leq \pi_1(\Sigma_g)$.

Lemma 4.6. Let $\phi$ be a framing of $\Sigma_{g,1}$. Then there is $(2g-2)$–spin structure $\hat{\phi}$ on $\Sigma_g$ such that the boundary-capping map $p : \text{Mod}_{g,1} \to \text{Mod}_g$ induces the following exact sequence:

$$1 \to H_g \to \text{Mod}_{g,1}[\phi] \to \text{Mod}_g[\hat{\phi}].$$

Proof. The framing $\phi$ determines a nonvanishing vector field on $\Sigma_{g,1}$. Capping the boundary component, this can be extended to vector field on $\Sigma_g$ with a single zero. This vector field gives rise to the $(2g-2)$–spin structure $\hat{\phi}$ (c.f. Section 2.1), and if $f \in \text{Mod}_{g,1}$ preserves $\phi$, then necessarily $p(f)$ preserves $\hat{\phi}$.\vspace{1cm}

Figure 8. The lantern relations used in the proof of Proposition 4.1. Curves and arcs colored red correspond to twists known to be in $T_\phi$ from Lemma 4.2, while those in blue correspond to twists not yet known to be in $T_\phi$. Numbers inside subsurfaces indicate Arf invariants.
It remains to show that \( \ker(p) \cap \text{Mod}_{g,1}[\phi] = H_g \). Since \( H_g \leq K_{g,1} \leq \text{Mod}_{g,1}[\phi] \), one containment is clear. For the converse, we first consider the action of a simple based loop \( \gamma \in \pi_1(UT\Sigma_g) \) on the winding number of an arbitrary simple closed curve \( a \). Let \( \gamma_L \) (resp. \( \gamma_R \)) denote the curves on \( \Sigma_{g,1} \) lying to the left (resp. right) of \( \gamma \). Then \( \gamma \) acts via the mapping class \( \mathcal{P}(\gamma) = T_{\gamma_L} T_{\gamma_R}^{-1} \). The twist-linearity formula (Lemma 2.4.1) as applied to \( \mathcal{P}(\gamma) \) shows that

\[
\phi(\mathcal{P}(\gamma)(a)) = \phi(a) + \langle a, \gamma \rangle \phi(\gamma_L) - \langle a, \gamma_R \rangle \phi(\gamma_R)
\]

\[
= \phi(a) + \langle a, \gamma \rangle (\phi(\gamma_L) - \phi(\gamma_R))
\]

\[
= \phi(a) + \langle a, \gamma \rangle (2 - 2g).
\]

(5)

Here, the second equality holds since \( \gamma_L, \gamma_R, \) and \( \gamma \) all determine the same homology class, and the third equality holds by homological coherence (Lemma 2.4.2), since \( \gamma_L \cup \gamma_R \cup \partial \Sigma_{g,1} \) cobound a pair of pants and necessarily \( \phi(\Delta_1) = 1 - 2g \).

This formula will show that \( \mathcal{P}(\gamma) \in \text{Mod}_{g,1}[\phi] \) if and only if \( \gamma \in H_g \). To see this, let \( \gamma \) be an arbitrary curve, not necessarily simple, and factor \( \gamma = \gamma_1 \ldots \gamma_n \) with each \( \gamma_i \) simple. Since \( \mathcal{P}(\gamma_i) \) acts trivially on homology, there is an equality \( \{\mathcal{P}(\gamma_1 \ldots \gamma_i)(a)\} = \{a\} \) of elements of \( H_1(\Sigma_{g,1}; \mathbb{Z}) \) for each \( i = 1, \ldots, n \), and hence

\[
\langle \mathcal{P}(\gamma_1 \ldots \gamma_i)(a), \gamma_{i+1} \rangle = \langle a, \gamma_{i+1} \rangle.
\]

Thus applying (5) successively with \( \mathcal{P}(\gamma_i) \) acting on \( \mathcal{P}(\gamma_1 \ldots \gamma_{i-1})(a) \) for \( i = 1, \ldots, n \) shows that

\[
\phi(\mathcal{P}(\gamma)(a)) = \phi(a) + \langle a, \gamma \rangle (2 - 2g).
\]

(6)

If \( \gamma \) is not contained in \( H_g \), then there exists some simple curve \( a \) such that \( \langle a, \gamma \rangle \neq 0 \). Therefore, equation (6) shows that \( \phi(\mathcal{P}(\gamma)(a)) \neq \phi(a) \) and hence \( \mathcal{P}(\gamma) \notin \text{Mod}_{g,1}[\phi] \).

\( \square \)

**Proof of Proposition 4.5.** Consider the forgetful map \( p : \text{Mod}_{g,1} \to \text{Mod}_g, \) with kernel \( \ker(p) = \pi_1(UT\Sigma_g) \). By Lemma 4.6, it suffices to see that (a)

\[
p(T_\phi) = \text{Mod}_g[\phi],
\]

and (b)

\[
T_\phi \cap \pi_1(UT\Sigma_g) = H_g.
\]

To see that \( p(T_\phi) = \text{Mod}_g[\phi] \), we appeal to [CS19, Theorem B]. By the framed change-of-coordinates principle (Proposition 2.1.2), there exists a configuration of admissible curves on \( \Sigma_{g,1} \), either of type 1 or type 2 (in the sense of Theorem B). In type 1 (resp. 2), this configuration satisfies the hypotheses of [CS19, Theorem B.1] (resp. [CS19, Theorem B.2]), showing that \( p(T_\phi) = \text{Mod}_g[\phi] \).

The containment \( H_g \leq T_\phi \) will follow from the work of Section 4.1. By Proposition 4.1, there is a containment \( K_{g,1} \leq T_\phi \). According to [Put18, Theorem 4.1],

\[
K_{g,1} \cap \pi_1(UT\Sigma_g) = H_g,
\]

showing the claim.

\( \square \)

We observe that this proof also shows that (6) can be upgraded into the following:

\[
1 \to H_g \to \text{Mod}_{g,1}[\phi] \to \text{Mod}_g[\phi] \to 1.
\]
5. THE RESTRICTED ARC GRAPH AND GENERAL FRAMINGS OF HOLOMORPHIC TYPE

The goal of this section is to prove that for \( g \geq 5 \), given an arbitrary surface \( \Sigma_{g,n} \) equipped with a relative framing \( \phi \) of holomorphic type, there is an equality \( \text{Mod}_{g,n}[\phi] = T_\phi \). We will argue by induction on \( n \). The base case \( n = 1 \) was established in Proposition 4.5 above. To induct, we study the action of \( \text{Mod}_{g,n}[\phi] \) on a certain subgraph of the arc graph, and identify the stabilizer of a vertex with a certain \( \text{Mod}_{g,n-1}[\phi'] \). In Section 5.1, we introduce the \( s \)-restricted arc graph \( \mathcal{A}^s(\phi; p, q) \) and show that \( \text{Mod}_{g,n}[\phi] \) acts transitively on vertices and edges. In Section 5.3, we prove that \( \mathcal{A}^s(\phi; p, q) \) is connected, modulo a surgery argument. In Section 5.4, we prove this “admissible surgery lemma.” Finally in Section 5.5, we use these results to prove that \( \text{Mod}_{g,n}[\phi] = T_\phi \) for \( g \geq 5 \) and \( n \geq 0 \) arbitrary. We remark that it is only in this final step where the assumption that \( \phi \) be of holomorphic type is relevant.

5.1. The restricted arc complex. We must first clarify some conventions and terminology about configurations of arcs. If \( a \subset \Sigma_{g,n} \) is an arc and \( c \subset \Sigma_{g,n} \) is a curve, then the geometric intersection number \( i(a, c) \) is defined as for pairs of curves: \( i(a, c) \) denotes the minimum number of intersections of transverse representatives of the isotopy classes of \( a \) and \( c \). If \( a, b \) are both arcs (possibly based at one or more common point), we define \( i(a, b) \) to be the minimum number of intersections of transverse representatives of the isotopy classes of \( a, b \) on the interiors of \( a, b \). In other words, intersections at common endpoints are not counted. We say that arcs \( a, b \) are disjoint if \( i(a, b) = 0 \), so that “disjoint” properly means “disjoint except at common endpoints.” As usual, we say that an arc \( a \) is nonseparating if the complement \( \Sigma_{g,n} \setminus a \) is connected, and we say that a pair of arcs \( a, b \) is mutually nonseparating if the complement \( \Sigma_{g,n} \setminus \{a, b\} \) is connected (possibly after passing to well–chosen representatives of the isotopy classes in order to eliminate inessential components).

Our objective is to identify a suitable subgraph of the arc graph on which \( \text{Mod}_{g,n}[\phi] \) acts transitively. Before presenting the full definition (see Definition 5.2 below), we first provide a motivating discussion. By definition, an element of \( \text{Mod}_{g,n}[\phi] \) must preserve the winding number of every arc, and so to ensure transitivity we must restrict the vertices of this subcomplex to be arcs of a fixed winding number \( s \in \mathbb{Z} + \frac{1}{2} \). However, this alone is insufficient, as Lemma 5.1 below makes precise.

If \( \alpha \) and \( \beta \) are two disjoint (legal) arcs which connect the same points \( p \) and \( q \) on boundary components \( \Delta_p \) and \( \Delta_q \), then the action of \( \text{Mod}_{g,n}[\phi] \) must preserve the winding number of each boundary component of a neighborhood of \( \Delta_p \cup \alpha \cup \beta \cup \Delta_q \). The winding numbers of these curves depends on the \( \phi \) values of \( \alpha, \beta, \Delta_p, \) and \( \Delta_q \), but also on the configuration of \( \alpha \) and \( \beta \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sidedness.png}
\caption{Sidedness. Left: a one–sided pair. Right: a two–sided pair.}
\end{figure}
To that end, we say that a pair of arcs \( \{ \alpha, \beta \} \) as above is one–sided (respectively, two–sided) if \( \alpha \) leaves \( p \) and enters \( q \) on the same side (respectively, opposite sides) of \( \beta \) for every disjoint realization of \( \alpha \) and \( \beta \) on \( \Sigma \). See Figure 9.

A quick computation yields an equivalent formulation in terms of winding numbers, which for clarity of exposition we state only in the case when \( \phi(\alpha) = \phi(\beta) \). The proof follows by inspection of Figure 9.

**Lemma 5.1.** Let \( \{ \alpha, \beta \} \) be a pair of arcs as above with \( \phi(\alpha) = \phi(\beta) \). Let \( c^\pm \) denote the two curves forming the boundary of a neighborhood of \( \Delta_p \cup \alpha \cup \beta \cup \Delta_q \), oriented so that the subsurface containing \( \Delta_p \) and \( \Delta_q \) lies on their right. Then \( \{ \alpha, \beta \} \) is

- one–sided if and only if \( \{ \phi(c^+), \phi(c^-) \} = \{ 1, \phi(\Delta_p) + \phi(\Delta_q) + 1 \} \).
- two–sided if and only if \( \{ \phi(c^+), \phi(c^-) \} = \{ \phi(\Delta_p) + 1, \phi(\Delta_q) + 1 \} \).

In particular, if \( c \) is an admissible curve with \( i(\alpha, c) = 1 \), then \( \{ \alpha, T_c(\alpha) \} \) is two–sided.

Having identified sidedness as a further obstruction to transitivity, we come to the definition of the complex under discussion. For any \( s \in \mathbb{Z} + \frac{1}{2} \), we say that an arc \( \alpha \) is an \( s \)-arc if \( \phi(\alpha) = s \).

**Definition 5.2.** Let \( (\Sigma_{g,n}, \phi) \) be a framed surface with \( n \geq 2 \). Suppose that \( p \) and \( q \) are legal basepoints on distinct boundary components \( \Delta_p \) and \( \Delta_q \) and fix some \( s \in \mathbb{Z} + \frac{1}{2} \). Then the restricted \( s \)-arc graph \( A^s_\pm(\phi; p, q) \) is defined as follows:

- A vertex of \( A^s_\pm(\phi; p, q) \) is an isotopy class of \( s \)-arcs connecting \( p \) and \( q \).
- Two vertices \( \alpha \) and \( \beta \) are connected by an edge if they are disjoint and mutually nonseparating.

The two–sided restricted \( s \)-arc graph \( A^s(\phi; p, q) \) is the subgraph of \( A^s_\pm(\phi; p, q) \) such that:

- The vertex set of \( A^s(\phi; p, q) \) is the same as that of \( A^s_\pm(\phi; p, q) \).
- Two arcs \( \alpha \) and \( \beta \) are connected in \( A^s(\phi; p, q) \) if and only if they are connected in \( A^s_\pm(\phi; p, q) \) and the pair \( \{ \alpha, \beta \} \) is two–sided.

### 5.2. Transitivity

In this subsection we prove that the action of \( \text{Mod}_{g,n}[\phi] \) on \( A^s(\phi; p, q) \) is indeed transitive on both edges and vertices. The definition of \( A^s(\phi; p, q) \) above was rigged so that the proof of Lemma 5.3 follows as an extended consequence of the framed change–of–coordinates principle (Proposition 2.14). The length of the proof is thus a consequence more of careful bookkeeping than genuine depth.

**Lemma 5.3.** The action of \( \text{Mod}_{g,n}[\phi] \) on \( A^s(\phi; p, q) \) is transitive on vertices and on edges.

We caution the reader that the action of \( \text{Mod}_{g,n}[\phi] \) is not transitive on oriented edges of \( A^s(\phi; p, q) \).

**Proof.** We begin by showing that edge transitivity implies vertex transitivity. Suppose that \( \alpha \) and \( \beta \) are two vertices of \( A^s(\phi; p, q) \); we will exhibit an element of \( \text{Mod}_{g,n}[\phi] \) taking \( \alpha \) to \( \beta \).

By the framed change–of–coordinates principle (Proposition 2.14), there is some admissible curve \( c \) which meets \( \alpha \) exactly once, and so by Lemma 5.1 the arc \( T_c(\alpha) \) is adjacent to \( \alpha \) in \( A^s(\phi; p, q) \). Now choose some \( \gamma \in A^s(\phi; p, q) \) adjacent to \( \beta \). By edge transitivity, there exists a \( g \in \text{Mod}_{g,n}[\phi] \) which takes the \( \{ \alpha, T_c(\alpha) \} \) edge to the \( \{ \beta, \gamma \} \) edge. If \( g(\alpha) = \beta \), then we are done. Otherwise, \( g(T_c(\alpha)) = \beta \), and since \( c \) is admissible, \( gT_c \in \text{Mod}_{g,n}[\phi] \).
It remains to establish edge transitivity. Up to relabeling, we may assume that $p \in \Delta_1$ and $q \in \Delta_2$. Suppose that $\alpha = \{ \alpha_1, \alpha_2 \}$ and $\beta = \{ \beta_1, \beta_2 \}$ are two edges of $A^s(\phi; p, q)$. We also assume that $\alpha_1$ leaves $p$ from the right–hand side of $\alpha_2$ and enters $q$ to the left, and the same for $\beta_1$ and $\beta_2$.

For each $\bullet \in \{ \alpha, \beta \}$, let $c^\pm_\bullet$ denote the two boundary components of a neighborhood of $\bullet \cup \Delta_1 \cup \Delta_2$. Let $X_\bullet$ (respectively $Y_\bullet$) denote the component of $\Sigma \setminus c^\pm_\bullet$ containing (respectively, not containing) $\bullet$, equipped with the induced framing $\xi_\bullet$ (respectively, $\eta_\bullet$). Finally, orient each curve of $c^\pm_\bullet$ so that $Y_\bullet$ lies on its left–hand side. See Figure 10.

![Figure 10. The curves and subsurfaces determined by a two–sided pair of disjoint $s$–arcs.](image)

Since both $\alpha$ and $\beta$ are two–sided, Lemma 5.1 implies that

$$\{ \phi(c^+_\bullet), \phi(c^-_\bullet) \} = \{ \phi(\Delta_1) + 1, \phi(\Delta_2) + 1 \}$$

for $\bullet \in \{ \alpha, \beta \}$ and we fix the convention that $\phi(c^+_\bullet) = \phi(\Delta_1) + 1$.

The proof now follows by building homeomorphisms $X_\alpha \to X_\beta$ and $Y_\alpha \to Y_\beta$ and gluing them together. To that end, we must first describe these subsurfaces in more detail.

**Distinguished arcs in $X_\bullet$.** Recall that $c^\pm_\bullet$ are defined as the boundary components of a neighborhood of $\bullet \cup \Delta_1 \cup \Delta_2$. If this neighborhood is taken to be very small (with respect to some auxiliary metric on $\Sigma$), then away from $p$ and $q$ the framing restricted to $c^\pm_\bullet$ looks like the framing on segments of $\bullet \cup \Delta_1 \cup \Delta_2$. In particular, for each point $p' \neq p$ of $\Delta_1$ with an orthogonally inward– or outward–pointing framing vector there is a corresponding point of $c^+_\bullet$ with an orthogonally inward– or outward–pointing framing vector. The analogous statement of course also holds for $q' \neq q \in \Delta_2$ and $c^-_\bullet$.

Pick points $p'$ and $p'' \neq p$ on $\Delta_1$ such that the framing vector at $p'$ points orthogonally outwards and the framing vector at $p''$ points orthogonally inwards. For the sake of concreteness, we take $p'$ (respectively $p''$) so that the arc $\gamma'$ (respectively $\gamma''$) of $\Delta_1$ which runs clockwise connecting $p$ to $p'$ ($p''$) has winding number equal to $-1/2$ (respectively $-1$).

Now let $x^+_\bullet$ and $y^+_\bullet$ denote the corresponding points of $c^+_\bullet$; by construction, the framing vectors at these points point orthogonally into $X_\bullet$ and $Y_\bullet$, respectively. Using $q'$ and $q'' \neq q$ on $\Delta_2$, one may similarly construct $x^-_\bullet$ and $y^-_\bullet$. See Figure 11.

By our choice of $p'$ and $p''$, one may observe that there exist arcs $r^+_\bullet$ from $p$ to $x^+_\bullet$ with

$$\phi(r^+_\bullet) = -\frac{1}{2} \text{ and } \phi(r^-_\bullet) = s.$$
Similarly, there exist $s^\pm_\ast$ from $p$ to $y^\pm_\ast$ with
\[ \phi(s^+\ast) = -1 \text{ and } \phi(s^-\ast) = s - \frac{1}{2}. \]
Now $\{\ast_1, r^+\ast, r^-\ast\}$ forms a distinguished geometric basis for $X_\ast$, and hence one can compute that
\[ \operatorname{Arf}(\xi_\ast) = (s + \frac{1}{2})(\phi(\Delta_2) + 1) + (0)(\phi(\Delta_1) + 2) + (s + \frac{1}{2})(\phi(\Delta_2) + 2) \tag{8} \]
which in particular does not depend on $\ast \in \{\alpha, \beta\}$.

**Building homeomorphisms on subsurfaces.** By construction both $Y_\alpha$ and $Y_\beta$ are homeomorphic to $\Sigma_{g-1,n}$. By (7) their boundary signatures agree:
\[ \operatorname{sig}(\eta_\alpha) = (\phi(\Delta_1) + 1, \phi(\Delta_2) + 1, \phi(\Delta_3), \ldots, \phi(\Delta_n)) = \operatorname{sig}(\eta_\beta). \]
Moreover, by the additivity of the Arf invariant (Lemma 4.3) together with (7) and (8), we have that
\[ \operatorname{Arf}(\eta_\alpha) = \operatorname{Arf}(\phi) + \operatorname{Arf}(\xi_\alpha) + p + q = \operatorname{Arf}(\phi) + \operatorname{Arf}(\xi_\beta) + p + q = \operatorname{Arf}(\eta_\beta) \]
and so by the classification of Mod$_{g,n}$ orbits of framed surfaces (Proposition 2.10) there is a homeomorphism $f_Y : Y_\alpha \to Y_\beta$ such that $f_Y(\eta_\beta) = \eta_\alpha$. Moreover, $f_Y(c^\pm_\alpha) = c^\pm_\beta$ and in fact $f_Y(y^\pm_\alpha) = y^\pm_\beta$.

Now in order to extend $f_Y$ to a self–homeomorphism of $\Sigma$ which takes $\alpha$ to $\beta$, we need only specify a homeomorphism $f_X$ of $X_\alpha$ with $X_\beta$. This can be done easily by observing that $\alpha \cup r^\pm_\alpha$ cuts $X_\alpha$ into disks with the same combinatorics as $\beta \cup r^\pm_\beta$ cuts $X_\beta$, and hence there is a unique homeomorphism $f_X : X_\alpha \to X_\beta$ which takes $\alpha$ to $\beta$ and $r^\pm_\alpha$ to $r^\pm_\beta$.

Pasting $f_X$ and $f_Y$ together without twisting around $c^\pm_\alpha$, we therefore get a homeomorphism $\tilde{f} : \Sigma \to \Sigma$ which takes $\alpha$ to $\beta$.

**Preserving the framing.** It remains to show that $\tilde{f}$ preserves the framing $\phi$. Choose a distinguished geometric basis
\[ \mathcal{B}_\beta = \{x_1, y_1, \ldots, x_{g-1}, y_{g-1}\} \cup \{a_2, \ldots, a_n\} \]
for $Y_\beta$ such that all the arcs $a_i$ of $B_\beta$ emanate from $y_\beta^+ \in c_\beta^+$. By convention, suppose that $a_2$ runs from $y_\beta^+$ to $y_\beta^-$, and by twisting around $c_\beta^+$ if necessary, suppose that $a_2$ emerges to the left of all other $a_i$. Then $B_\beta$ extends to a distinguished geometric basis of $\Sigma$ in the following way:

$$\tilde{B}_\beta = \{x_1, y_1, \ldots, x_{g-1}, y_{g-1}, c_\beta^+, a_2 \cdot (s^-_\beta) \cdot s^+ \} \cup \{\beta_2, s^+ \cdot a_3, \ldots, s^+ \cdot a_n\}$$

where $a \cdot b$ represents the concatenation of the arcs $a$ and $b$ and $\pi$ represents the arc $a$ traveled backwards. See Figure 12. By concatenating with $s^\pm_\alpha$ arcs, the basis $f^{-1}_Y(B)$ on $Y_\alpha$ also extends to a basis of $\Sigma$ in a similar fashion:

$$\tilde{B}_\alpha = \{f^{-1}_Y(x_1), f^{-1}_Y(y_1), \ldots, f^{-1}_Y(x_{g-1}), f^{-1}_Y(y_{g-1}), c^-_\alpha, f^{-1}_Y(a_2) \cdot (s^-_\alpha) \cdot s^+_\alpha\}$$

$$\cup \{\alpha_2, s^+_\alpha \cdot f^{-1}_Y(a_3), \ldots, s^+_\alpha \cdot f^{-1}_Y(a_n)\}$$

(9)

![Figure 12. Extending a distinguished geometric basis from $Y_\beta$ to $\Sigma$ by using the arcs $s^\pm_\beta$ of $X_\beta$.](image)

Now by construction we have that $f_X(s^\pm_\alpha) = s^\pm_\beta$ and $\phi(s^\pm_\alpha) = s - 1/2 = \phi(s^\pm_\beta)$, so we have that for each element $x \in \tilde{B}_\alpha$,

$$\tilde{f}(x) \in \tilde{B}_\beta \text{ and } \phi(\tilde{f}(x)) = \phi(x).$$

Therefore $\tilde{f}$ preserves the winding numbers of a distinguished geometric basis, and so by Remark 2.7, $\tilde{f} \in \text{Mod}_{g,n}[\phi]$. □
5.3. Connectedness.

Lemma 5.4. Let \((\Sigma_{g,n},\phi)\) be a framed surface with \(g \geq 5\) and \(n \geq 2\). Let \(p,q\) be distinct boundary components, and let \(s \in \mathbb{Z}\) be arbitrary. Then \(\mathcal{A}^s(\phi;p,q)\) is connected.

This will require the preliminary Lemmas 5.5, 5.6, and 5.7. The first of these was proved in \[\text{Sal19}\]. There it was formulated only for closed surfaces, but the same proof applies for surfaces with an arbitrary number of punctures and boundary components.

Lemma 5.5 (c.f. Lemma 7.3 of \[\text{Sal19}\]). Let \(g \geq 5\) and \(n \geq 0\) be given. Let \(S\) and \(S'\) be subsurfaces of \(\Sigma_{g,n}\), each homeomorphic to \(\Sigma_{2,1}\). Then there is a sequence \(S = S_0, \ldots, S_n = S'\) of subsurfaces of \(\Sigma_{g,n}\) such that \(S_i \sim S_{i-1}\) and \(S_i \sim \Sigma_{2,1}\) for all \(i = 1, \ldots, n\).

Lemma 5.6 (Admissible surgery). Fix \(g \geq 3, n \geq 2\), and let \((\Sigma_{g,n},\phi)\) be a framed surface with distinguished legal basepoints \(p\) and \(q\) on boundary components \(\Delta_p\) and \(\Delta_q\). Let \(S \subset \Sigma_{g,n}\) be a subsurface homeomorphic to \(\Sigma_{2,1}\) (necessarily not containing \(p\) or \(q\)). Let \(\eta\) be an \(s\)-arc connecting \(p, q\) that is disjoint from \(S\). Let \(x \subset \Sigma_{g,n}\) be either a separating curve or an arc connecting \(p, q\), in either case disjoint from \(S\). Then there is a path \(\eta = \eta_0, \ldots, \eta_k\) in \(\mathcal{A}^s_\pm(\phi;p,q)\) such that \(i(\eta_k,x) = 0\).

Lemma 5.7. With hypotheses as above, if \(\mathcal{A}^s_\pm(\phi;p,q)\) is connected, then also \(\mathcal{A}^s(\phi;p,q)\) is connected.

The proofs of Lemma 5.6 and Lemma 5.7 are deferred to follow the proof of Lemma 5.4. To prove Lemma 5.4, we first introduce the notion of a “curve-arc sum”.

Curve–arc sums. We recall the notion of a “curve–arc sum” as discussed in \[\text{Sal19, Section 3.2}\] and \[\text{CS19, Definition 6.18}\]. Let \(a\) be an oriented curve or arc, \(b\) be an oriented curve, and \(\varepsilon\) be an embedded arc connecting the left sides of \(a\) and \(b\) and otherwise disjoint from \(a \cup b\) (if \(a\) is an arc we require \(\varepsilon \cap a\) to be a point on the interior of \(a\)). If \(a\) is a curve (resp. arc), the curve-arc sum \(a +_\varepsilon b\) is the curve (resp. arc) obtained by dragging \(a\) across \(b\) along the path \(\varepsilon\); see Figure 13.

![Figure 13. The curve-arc sum operation.](image)

Lemma 5.8 (c.f. Lemma 3.13 of \[\text{Sal19}\]). Let \(a, b, \varepsilon\) be as above and let \(\phi\) be a relative winding number function. Then

\[
\phi(a +_\varepsilon b) = \phi(a) + \phi(b) + 1.
\]

Proof of Lemma 5.4. Following Lemma 5.7, it suffices to show that \(\mathcal{A}^s_\pm(\phi;p,q)\) is connected. Let \(\alpha\) and \(\omega\) be \(s\)-arcs. Let \(S_\alpha \cong \Sigma_{2,1}\) be disjoint from \(\alpha\), and likewise choose \(S_\omega \cong \Sigma_{2,1}\) disjoint from \(\omega\). By
Lemma 5.5 there is a sequence $S_0 = S_0, \ldots, S_n = S_\omega$ of subsurfaces such that $S_i \cong \Sigma_{2,1}$ and such that $S_{i-1}$ and $S_i$ are disjoint for all $i = 1, \ldots, n$. We apply the Admissible Surgery Lemma (Lemma 5.6) taking $(S, \eta, x) = (S_0, \alpha, \partial S_1)$. This gives a path $\alpha = \alpha_0, \ldots, \alpha_m$ in $A^*_1(\phi; p, q)$ such that $\alpha_m$ is disjoint from $S_1$. We now repeat this process for each $S_i$ ($i \geq 1$), finding intermediate paths of $s$-arcs, beginning with one disjoint from $S_i$ and ending with one disjoint from $S_{i+1}$.

At the end of this process we have produced a path of $s$-arcs $\alpha, \ldots, \psi$ with the final arc $\psi$ disjoint from $S_\omega$. To complete the argument we apply the Admissible Surgery Lemma one final time with $(S, \eta, x) = (S_\omega, \psi, \omega)$. This produces a path $\psi = \psi_0, \ldots, \psi_k$ in $A^*_1(\phi; p, q)$ with $i(\psi_k, \omega) = 0$. If $\psi_k \cup \omega$ is nonseparating, then $\psi_k$ and $\omega$ are adjacent in $A^*_1(\phi; p, q)$, completing the path from $\alpha$ to $\omega$. If $\psi_k \cup \omega$ is separating, then at least one side of the complement has genus $h \geq 2$, and thus there exists a nonseparating oriented curve $d$ disjoint from $\psi_k \cup \omega$ that satisfies $\phi(d) = -1$. Define $\psi_{k+1} = \psi_k + \varepsilon d$ for a suitable arc $\varepsilon$. Then $\psi_k, \psi_{k+1}, \omega$ is a path in $A^*_1(\phi; p, q)$, completing the argument in this case. \[ \square \]

5.4. Proof of the Admissible Surgery Lemma. The proof will require the preliminary result of Lemma 5.9 below.

A change-of-coordinates lemma. We study the existence of suitable curves on genus 2 subsurfaces. The proof of Lemma 5.9 is a standard appeal to the framed change–of–coordinates principle (Proposition 2.14).

Lemma 5.9. Let $(\Sigma_{2,1}, \phi)$ be a framed surface, and let $\alpha$ be a nonseparating properly-embedded arc on $\Sigma_{2,1}$. For $t \in \mathbb{Z}$ arbitrary, there is an oriented nonseparating curve $c_t \subset \Sigma_{2,1}$ such that $\phi(c_t) = t$ and such that $\langle \alpha, c_t \rangle = 1$.

Proof of Lemma 5.9. The idea is to perform a sequence of surgeries on $\eta$ in order to successively reduce $i(\eta, x)$. Such surgeries will alter the winding number, but this will be repaired by using the “unoccupied” subsurface $S$ to fix the winding number while preserving the intersection pattern with $x$. The care we take below in selecting a suitable location for surgery ensures that the intersection pattern with $S$ remains unaltered. Throughout the proof we will refer to Figure 14.

Low intersection number. If $i(\eta, x) = 0$ there is nothing more to be done. If $i(\eta, x) = 1$, then there exists an arc $\eta'$ connecting $p$ to $q$ that is disjoint from $\eta \cup x \cup S$, and such that $\Sigma_{2,1} \setminus \{\eta \cup x \cup \eta'\}$ is connected. See Figure 14(A). Let $d \subset S$ be an oriented nonseparating curve satisfying $\phi(d) = s - \phi(\eta') - 1$. Let $\varepsilon$ be an arc disjoint from $\eta \cup x$ and such that $i(\varepsilon, \partial S) = 1$ that connects $\eta'$ to the left side of $d$, and let $\eta_1 = \eta' + \varepsilon d$. By Lemma 5.8 $\phi(\eta_1) = \phi(\eta') + \phi(d) + 1 = s$. Since there exists a curve $d' \subset S$ such that $i(d', \eta_1) = 1$ and $i(d', \eta) = 0$, it follows that $\eta \cup \eta_1$ is nonseparating, completing the argument in the case $i(\eta, x) = 1$.

The general case: outline. We now consider the case $i(\eta, x) = N \geq 2$. We will first pass to an adjacent $s$-arc $\eta_1$ that enters and exits $S$ exactly once. We will use this in combination with a surgery argument to produce an $s$-arc $\eta_2$ that is adjacent to $\eta_1$, satisfies $i(\eta_2, x) < i(\eta, x)$, and also enters and exits $S$ once. As the above arguments (treating the cases $N \leq 1$) can easily be adapted to the situation where $\eta$ passes once through $S$, this will complete the proof.
First steps; initial and terminal points. Let $c \subset S$ be an oriented nonseparating curve satisfying $\phi(c) = -1$. Let $\varepsilon$ be an arc disjoint from $x$ and such that $i(\varepsilon, \partial S) = 1$ connecting $\eta$ to the left side of $c$, and let $\eta_1 = \eta + \varepsilon$. See Figure 14(B). By Lemma 5.8, $\eta_1$ is an $s$–arc, and by construction, $\eta \cup \eta_1$ is isotopic to $c$ and is therefore nonseparating.

Enumerate the intersection points of $\eta_1 \cap x$ as $y_1, \ldots, y_N$, numbered consecutively as $\eta_1$ runs from $p$ to $q$; further set $y_0 = p$ and $y_{N+1} = q$. For some $0 \leq k \leq N$, the arc $\eta_1$ leaves $y_k$, enters $S$, and crosses back through $y_{k+1}$. The points $y_0, \ldots, y_k$ are called initial, and the points $y_{k+1}, \ldots, y_{N+1}$ are called terminal. We say that points $y_i$ and $y_j$ are $x$–adjacent if $y_i$ and $y_j$ appear consecutively when running along $x$ (in either direction).

Case 1: adjacent initial/terminal points. Suppose first that there is a pair of $x$–adjacent points $y_i, y_j$ that are either both initial or both terminal (if $x$ is a curve we consider $1 \leq i < j \leq N$, but if $x$ is an arc, the surgeries we describe below will work for all $0 \leq i < j \leq N + 1$). In this case, let $\eta'_2$ be obtained from $\eta_1$ by following $\eta_1$ from $p$ to $y_i$, then along $x$ to $y_j$, then finally along $\eta_1$ from $y_j$ to $q$. See Figure 14(C). Note first that $i(\eta_1, \eta'_2) = 0$ and that $i(\eta'_2, x) < i(\eta_1, x)$. It remains to alter $\eta'_2$ to an
arc $\eta_2$ that is also disjoint from $\eta_1$ but which has $\phi(\eta_2) = s$ and such that $\eta_1 \cup \eta_2$ is nonseparating, i.e., such that $\eta_1, \eta_2$ is an edge in $\mathcal{A}_\pm^t(\phi; p, q)$.

The method will be to find a curve on $S$ to twist along to correct the winding number of $\eta'_2$, but care must be taken to ensure that the twisted arc remains disjoint from $\eta_1$. Push $\eta'_2$ off of $\eta_1$ so that it runs parallel to $\eta_1$ except at the location of the surgery. As $\eta'_2$ and $\eta_1$ run along the segment between $y_k$ and $y_{k+1}$ through $S$, the pushoff of $\eta'_2$ lies to the left or to the right of $\eta_1$ in the direction of travel. We call the former case positive position and the latter negative position. If $c$ is a curve with $i(c, \eta'_2) = 1$, observe that $T_c^\pm(\eta'_2)$ is disjoint from $\eta_1$ so long as the sign of the twist coincides with the sign of the position.

Define $t := \phi(\eta'_2)$. By Lemma 5.9, there are nonseparating curves $d_\pm \subset S$ such that $\phi(d_\pm) = \pm(s - t)$ and such that $(\eta'_2, d_\pm) = 1$. Set $\eta_2 = T_{d_\pm}^\pm(\eta'_2)$, where the sign depends on the sign of the position $\eta'_2$. Then $\eta_2$ is an $s$-curve adjacent to $\eta_1$ in $\mathcal{A}^t(\phi; p, q)$ and $i(\eta_2, x) < i(\eta_1, x)$.

**Case 2: alternating initial/terminal points.** It remains to consider the case where every pair of $x$–adjacent points $y_i, y_j$ has one initial and one terminal element. Here there are two possibilities to consider: either there is exactly one terminal point (and hence $N = 2$), or else at least two. If there is exactly one terminal point and $x$ is an arc, then necessarily this terminal point is $q$. Then the unique initial point is $p$, and hence $\eta_1$ is disjoint from $x$ except at endpoints and there is nothing left to be done. If $x$ is a separating curve, then necessarily there are at least two terminal points, since if $\eta_1$ crosses into the subsurface bounded by $x$ at a terminal point, it must necessarily exit through another terminal point.

We therefore assume that every pair of $x$–adjacent points $y_i, y_j$ have one initial and one terminal element and that there are at least two terminal points. The first terminal point $y_{k+1}$ is $x$–adjacent to two distinct initial points $y_i, y_j$ ($i < j \leq k$), and likewise the last initial point $y_k$ is adjacent to two distinct terminal points $y_l, y_m$ ($k + 1 \leq l < m$).

A suitable surgery is illustrated in Figure 14(D). The surgered arc $\eta'_2$ begins by following $\eta_1$ forwards from $p$ to $y_1$, then along $x$ from $y_1$ to $y_{k+1}$, continuing backwards along $\eta_1$ from $y_{k+1}$ to $y_k$. At this point there is a choice: do we follow $x$ to $y_l$ or $y_m$ (in both cases continuing from here forwards along $\eta_1$ to $q$)? The orientation of $\eta_1$ endows each $y_l$ with a left and right side. If $y_l$ is adjacent to the left side of $y_{k+1}$, we continue $\eta'_2$ to whichever of $y_l, y_m$ lies to the right of $y_k$, and if $y_l$ lies to the right of $y_{k+1}$, we continue $\eta'_2$ along the point $y_l, y_m$ to the left. Observe that $i(\eta'_2, x) < i(\eta_1, x)$, even in the exceptional case where the chosen terminal point $y_l$ happens to be $y_{k+1}$.

The construction of $\eta'_2$ above facilitates the next step of the argument, which is to adjust $\eta'_2$ to an $s$–arc $\eta_2$ adjacent to $\eta_1$ in $\mathcal{A}^t_\pm(\phi; p, q)$. As in the prior case, let $\phi(\eta'_2) = t$, and select (by Lemma 5.9) nonseparating curves $d_\pm \subset S$ such that $\phi(d_\pm) = \pm(s - t)$ and such that $(\eta, d_\pm) = 1$. If $y_l$ is adjacent to the left side of $y_{k+1}$, define

$$\eta_2 = T_{d_-}^{-1}(\eta'_2),$$

and otherwise define

$$\eta_2 = T_{d_+}(\eta'_2).$$

In both cases, $\eta_2$ is an $s$–arc adjacent to $\eta_1$ in $\mathcal{A}^t_\pm(\phi; p, q)$ and $i(\eta_2, x) < i(\eta_1, x)$. 

Having established the admissible surgery lemma (Lemma 5.6), it remains only to give the proof of Lemma 5.7, showing that connectivity of the restricted s–arc graph $A_{s}(\phi; p, q)$ implies the connectivity of the two-sided restricted s–arc graph $A^{s}(\phi; p, q)$.

Figure 15. Connecting a one–sided pair $\{\alpha, \beta\}$ with two two–sided pairs $\{\alpha, \gamma\}$ and $\{\gamma, \beta\}$.

Proof of Lemma 5.7. We will refer to Figure 15 throughout. It suffices to show that if $\{\alpha, \beta\}$ is a one-sided edge in $A_{s}(\phi; p, q)$, there is a path in $A^{s}(\phi; p, q)$ connecting $\alpha$ to $\beta$. Without loss of generality, suppose that $\alpha$ exits $p$ and enters $q$ on the left–hand side of $\beta$.

As $\Sigma \setminus (\alpha \cup \beta)$ is a framed surface with genus $g - 1 \geq 2$ and $n$ boundary components, we can apply the framed change–of–coordinates principle (Proposition 2.14) to deduce that there exists a nonseparating curve $c$ on $\Sigma$, disjoint from $\alpha \cup \beta$, such that

$$\phi(c) = s - \phi(\alpha) - \phi(\Delta_p) - 1.$$ 

As $\alpha \cup \beta$ and $c$ are both nonseparating, there exists an arc $\varepsilon$ from the left–hand side of $\alpha$ to the left–hand side of $c$. Therefore, by Lemmas 5.8 and 2.4.1, we see that for $\gamma := T_{\Delta_p}^{-1}(\alpha + \varepsilon c)$,

$$\phi(\gamma) = \phi(\alpha + \varepsilon c) + \phi(\Delta_p) = \phi(c) + \phi(\alpha) + 1 + \phi(\Delta_p) = s.$$ 

Now since $\alpha + \varepsilon c$ leaves $p$ and enters $q$ on the left–hand side of $\alpha$ (by construction), we see that $\gamma$ leaves $p$ to the right of both $\alpha$ and $\beta$, but enters $q$ on the left of both $\alpha$ and $\beta$.

Therefore $\{\alpha, \gamma\}$ and $\{\gamma, \beta\}$ are edges in $A^{s}(\phi; p, q)$. □

5.5. Completing Theorem B. Having completed the proof of the Admissible Surgery Lemma, we can proceed with the proof of Theorem B. The remainder of the proof is built around the following standard technique of geometric group theory.

Lemma 5.10. Let $G$ be a group acting on a connected graph $X$. Suppose that $G$ acts transitively on vertices and edges of $X$. For a vertex $v$, let $G_v$ denote the stabilizer of $v$. Let $e$ be an edge connecting vertices $v, w$, and let $h \in G$ satisfy $h(w) = v$. Then $G = \langle G_v, h \rangle$.

Proof. This is very similar to [FM11] Lemma 4.10|}. The argument there can easily be adapted to prove this slightly stronger statement. □

Proof (of Theorem B). Proposition 3.1 shows that the finite collections of twists shown in Figure B generate the admissible subgroup $T_{\phi}$. Our objective is now to show that for $g \geq 5$ and $\phi$ an arbitrary framing of $\Sigma_{g,n}$, there is an equality $T_{\phi} = \text{Mod}_{g,n}[\phi]$. We argue by induction on the number $n$ of
boundary components. The base case $n = 1$ was established above as Proposition 4.5. To proceed, we appeal to Lemma 5.10, taking $G = \text{Mod}_{g,n}[\phi]$ and $X = \mathcal{A}^s(\phi; p, q)$ for $s \in \mathbb{Z} + \frac{1}{2}$ arbitrary.

Lemmas 5.3 and 5.4 combine to show that the hypotheses of Lemma 5.10 are satisfied for $G = \text{Mod}_{g,n}[\phi]$ and $X = \mathcal{A}^s(\phi; p, q)$. Let $\alpha$ be any $s$-arc connecting $p$ to $q$. By the framed change-of-coordinates principle (Proposition 2.14), there exists an admissible curve $a$ such that $i(\alpha, a) = 1$. The arc $T_a(\alpha)$ is disjoint from $\alpha$, the adjacency is two-sided, and the union $T_a(\alpha) \cup \alpha$ is nonseparating. As $\alpha$ is admissible, it follows that $\phi(T_a(\alpha)) = \phi(\alpha) = s$, so that $T_a(\alpha)$ is adjacent to $\alpha$ in $\mathcal{A}^s(\phi; p, q)$.

By Lemma 5.10 it now follows that $\text{Mod}_{g,n}[\phi]$ is generated by $T_a$ and the stabilizer $(\text{Mod}_{g,n}[\phi])_a$. By hypothesis, $T_a \in \mathcal{T}_g$. To complete the inductive step, it remains to see that $(\text{Mod}_{g,n}[\phi])_a \leq \mathcal{T}_g$. Cutting $\Sigma_{g,n}$ open along a neighborhood $\Delta$ of $\alpha \cup \Delta_p \cup \Delta_q$ yields a framed surface $(\Sigma_{g,n-1}, \phi')$ with a new boundary component, which by abuse of notation we also call $\Delta$. Homological coherence (Lemma 2.4.2) implies that $\phi(\Delta) = \phi(\Delta_p) + \phi(\Delta_q) + 1 \leq -1$ so that $\phi'$ is of holomorphic type. Consider the short exact sequence

$$1 \to \langle T_\Delta \rangle \to (\text{Mod}_{g,n})_a \to \text{Mod}_{g,n-1} \to 1.$$ 

Restricting to the stabilizers of $\phi$ and $\phi'$ yields the exact sequence

$$1 \to \langle T_\Delta \rangle \cap (\text{Mod}_{g,n}[\phi])_a \to (\text{Mod}_{g,n}[\phi])_a \to \text{Mod}_{g,n-1}[\phi'].$$

Observe that cutting open along $\Delta$ induces a bijection between $\phi'$-admissible curves on $\Sigma_{g,n-1}$ and $\phi$-admissible curves on $\Sigma_{g,n}$ that are disjoint from $\alpha$. Now by the inductive hypothesis, $T_{\phi'} = \text{Mod}_{g,n-1}[\phi']$ and so it follows that the cutting map induces a surjection

$$\mathcal{T}_{\phi} \cap (\text{Mod}_{g,n}[\phi])_a \to \text{Mod}_{g,n-1}[\phi'].$$

From here, we treat the cases $n = 2$ and $n \geq 3$ separately. For $n = 2$, we claim that $T_\Delta \in \mathcal{T}_{\phi'}$. Indeed, $\Delta$ can be identified with the boundary component of the cut-open surface $\Sigma_{g,2} \setminus \alpha \cong \Sigma_{g,1}$, and hence $T_\Delta \in \mathcal{T}_{\phi'}$ by Proposition 4.1. For $n \geq 3$ we claim that there is an isomorphism $(\text{Mod}_{g,n}[\phi])_a \cong \text{Mod}_{g,n-1}[\phi']$, or equivalently that $(\text{Mod}_{g,n}[\phi])_a \cap \langle T_\Delta \rangle = \{1\}$. Let $\beta$ be an arc connecting $p$ to a third boundary component $p' \neq q$. As observed above, $\phi(\Delta) \leq -3 < 0$. Thus for $k \neq 0$ arbitrary, $\phi(T_\Delta(\beta)) \neq \phi(\beta)$ and the claim follows.

6. Other framed mapping class groups

In this section we leverage our work on the framed mapping class group $\text{Mod}_{g,n}[\phi]$ in order to study two variants we will encounter in our investigation of the monodromy groups of strata of abelian differentials. The most straightforward variant we consider is the stabilizer of the framing up to absolute isotopy, i.e., where the isotopy is not necessarily trivial on the boundary. In this case, we will see that there is a sensible theory even when boundary components are replaced by marked points. We carry this out in Section 6.2 culminating in Proposition 6.8.

Our analysis of this case is built on a study of an intermediate refinement we call the “pronged mapping class group.” This group was introduced in a slightly different form in [BSW16], where it was called the “mapping class group rel boundary” of the surface. In Section 6.1, we lay out the basic
theory of prong structures, pronged mapping class groups, and framings on pronged surfaces, leading
to the structural result Proposition 6.6. The material in Section 6.2 then follows as an easy corollary.

Section 6.3 contains an analysis of the relationship between the (relative) framed mapping class
group $\text{Mod}_{g,n}^\phi$ and its absolute counterpart $\text{PMod}_{g}^n[\bar{\phi}]$. The main result here is Proposition 6.11,
which identifies an obstruction for $\text{Mod}_{g,n}^\phi$ to surject onto $\text{PMod}_{g}^n[\bar{\phi}]$.

## 6.1. Pronged surfaces and pronged mapping class groups

In our study of the monodromy of strata of abelian differentials, we will encounter a variant of a puncture/boundary component known as a **prong structure**. Here we outline the basic theory of surfaces with prong structure and their mapping class groups.

**Definition 6.1** (Prong structure, pronged mapping class group). Let $\Sigma$ be a surface of genus $g$ equipped with a Riemannian metric and $p_i \in \Sigma$ a marked point. A prong point of order $k_i$ at $p_i$ is a choice of $k_i$ distinct unit vectors (prongs) $v_1, \ldots, v_{k_i} \in T_{p_i}\Sigma$ spaced at equal angles. With this data specified, we will write $\vec{p}_i$ to refer to the set of prongs based at $p_i$, and $\vec{P}$ to indicate a set of prong points $\{\vec{p}_1, \ldots, \vec{p}_n\}$ with underlying points $P = \{p_1, \ldots, p_n\}$.

Let $\vec{P}$ be a collection of prong points. Define $\text{Diff}^+(\Sigma; \vec{P})$ to be the group of orientation-preserving diffeomorphisms of $\Sigma$ that preserve each prong point (elements must fix the points $p_1, \ldots, p_n$ pointwise, but can induce a (necessarily cyclic) permutation of the overlying tangent vectors). The **pronged mapping class group** is then defined as

$$\text{Mod}(\Sigma, \vec{P}) := \pi_0(\text{Diff}^+(\Sigma; \vec{P})).$$

In the interest of compressing notation, this will often be written simply $\text{Mod}^\ast_{g,n}$, with the underlying prong structure understood from context.

**Prongs vs. boundary components.** Here we outline the relationship between prongs and boundary components. First note that a prong point of order 1 is simply a choice of fixed tangent vector. Also recall that in the case where all prong points have order 1, there is a natural isomorphism

$$\text{Mod}(\Sigma, \vec{P}) \cong \pi_0(\text{Diff}^+(\Sigma; \vec{P})).$$

where $\Sigma^* \cong \Sigma_{g,n}$ is the surface with $n$ boundary components obtained by performing a real oriented blowup at each $p_i$ (see Construction 7.4 below). More generally, let $(\Sigma, \vec{P})$ be an arbitrary pronged surface with $\vec{p}_i$ a prong point of order $k_i$. Let $\mu_k \subseteq \mathbb{C}^\times$ denote the group of $k^{th}$ roots of unity, and define the “prong rotation group”

$$PR := \prod_{i=1}^{n} \mu_{k_i}.$$

For each $1 \leq i \leq n$, there is a map $D_i : \text{Mod}^\ast_{g,n} \to \mu_{k_i}$ given by taking the rotational part of the derivative at $T_{p_i}\Sigma \cong \mathbb{C}$. We define

$$D := \prod_{i=1}^{n} D_i : \text{Mod}^\ast_{g,n} \to PR$$
to be the product. Then $D$ induces the following short exact sequence (compare [BSW16 (6)]):

$$1 \to \text{Mod}_{g,n} \to \text{Mod}^*_g \to PR \to 1.$$  

(10)

**Fractional twists.** There is an explicit set-theoretic splitting of (10). We define a *fractional twist* $T_{\vec{p}_i}$ at the prong point $\vec{p}_i$ of order $k_i$ to be the mapping class specified in a local complex coordinate $z \leq 1$ near $p_i$ by

$$T_{\vec{p}_i}(z) = z e^{(2\pi i (1-|z|))/k_i}.$$ 

Intuitively, $T_{\vec{p}_i}$ acts by applying a “screwing motion” of angle $2\pi/k_i$ at $p_i$, viewing a small neighborhood of $p_i$ as being constructed from an elastic material connected to a rigid immobile boundary component. It is then clear that $PR$ embeds into $\text{Mod}^*_g$ as the set of fractional twists $\{T_{\vec{p}_i}^j \mid 0 \leq j < k_i, 1 \leq i \leq n \}$. Define

$$FT = \langle T_{\vec{p}_i} \mid 1 \leq i \leq n \rangle \subseteq \text{Mod}^*_g$$

to be the group generated by the fractional twists.

On the blowup $\Sigma^*$, the fractional twist $T_{\vec{p}_i}$ acts as a fractional rotation of the corresponding boundary component. It will be convenient to introduce the following notation: if $\Delta_i$ is the boundary component corresponding to $\vec{p}_i$, write $T^{1/k_i}_{\Delta_i}$ in place of $T_{\vec{p}_i}$. Note that $T^{k_i}_{\vec{p}_i}$ can be identified with the full Dehn twist about $\Delta_i$, so that the equation

$$(T^{1/k_i}_{\Delta_i})^{k_i} = T_{\Delta_i}$$

holds as it should. We will therefore allow $T_{\Delta_i}$ to assume fractional exponents with denominator $k_i$.

**Relative framings of pronged surfaces.** Let $(\Sigma, \vec{P})$ be a pronged surface. For simplicity’s sake, we will formulate the discussion in this paragraph in terms of the blow-up $\Sigma^*$. Consider now a nonvanishing vector field $\xi$ on $\Sigma^*$. As in Section 2 when a Riemannian metric is fixed, $\xi$ gives rise to a framing $\phi$ of $\Sigma^*$. In the presence of a prong structure, we will impose a further “compatibility” requirement on $\phi$ at the boundary.

**Definition 6.2** (Compatible framing). Let $\vec{p}_i$ be a prong point of order $k_i$ on $\Sigma$ and let $\Delta_i$ be the associated boundary component of $\Sigma^*$. There is a canonical identification $\Delta_i \cong U\text{t}_{\vec{p}_i} \Sigma$ between $\Delta_i$ and the space of unit tangent directions at $p_i$. We say that a framing $\phi$ is compatible with $\vec{p}_i$ if the following conditions hold:

1. For $v \in U\text{t}_{p_i} \Sigma$, the framing vector $\phi(v)$ is orthogonally inward-pointing on $\Sigma^*$ if and only if $v$ is a prong.

2. The restriction of $\phi$ to $U\text{t}_{p_i} \Sigma$ is invariant under the action of $\mu_{k_i}$ on $U\text{t}_{p_i} \Sigma$.

If $\vec{P}$ is a prong structure, we say that $\phi$ is compatible with $\vec{P}$ if $\phi$ is compatible with each $\vec{p} \in \vec{P}$ in the above sense.

**Remark 6.3.** Observe that if $\phi$ is compatible with a prong point $\vec{p}_i$ of order $k_i$, then the winding number of the associated boundary component $\Delta_i$ of $\Sigma^*$ is determined: $\phi(\Delta_i) = -k_i$ (recalling the standing convention that boundary components are oriented with the interior of the surface lying to the left). In particular, $\phi$ is necessarily of holomorphic type.
Observe that the notion of relative isotopy of framings still makes sense on a pronged surface. If φ is compatible with \( \vec{P} \), then there is a well-defined action of \( \text{Mod}^*_{g,n} \) on the set of relative isotopy classes of framings. Exactly as in Section 2, we define the framed mapping class group of the framed pronged surface \( (\Sigma^*, \phi) \) to be the stabilizer of φ:

\[
\text{Mod}^*_{g,n}[\phi] = \{ f \in \text{Mod}^*_{g,n} \mid f \cdot \phi = \phi \}.
\]

**Winding number functions.** Exactly as in the setting of Section 2, relative isotopy classes of framings on pronged surfaces are in bijection with suitably-defined winding number functions. The definition of winding number of a closed curve needs no modification. To set up a theory of winding numbers for arcs on pronged surfaces, we adopt the natural counterparts of the definitions of legal basepoint and legal arc from Section 2.

Suppose \( (\Sigma, \vec{P}) \) is a pronged surface equipped with a compatible framing \( \phi \). Let \( \vec{p}_i \) be a prong point of order \( k_i \). The prongs \( v_1, \ldots, v_{k_i} \in T_{p_i} \Sigma \) correspond to \( k_i \) distinct points on the corresponding boundary component \( \Delta_i \) of \( \Sigma^* \), and by the compatibility assumption, each \( v_i \) is a legal basepoint in the sense of Section 2. We say that an arc \( \alpha \) on the pronged surface \( (\Sigma, \vec{P}) \) (or equivalently on the blowup \( \Sigma^* \)) is legal if \( \alpha \) is properly embedded, each endpoint is some legal basepoint on \( \Sigma^* \), and if \( \alpha'(0) \) is orthogonally inward-pointing and \( \alpha'(1) \) is orthogonally outward-pointing. In short, the theory of legal arcs on pronged surfaces differs from the theory on surfaces with boundary only in that we allow arcs to be based at any legal basepoint, not at one fixed point per boundary component. Fractional twists about the boundary may change the basepoint, and so we must consider all legal basepoints at once, instead of a single one.

Under this definition, legal arcs have half-integral winding number as before, and moreover \( \text{Mod}^*_{g,n} \) acts on the set of isotopy classes of relative arcs. The twist–linearity formula (Lemma 2.4.1) generalizes to fractional twists as follows (the proof is straightforward and is omitted).

**Lemma 6.4.** Let \( a \) be a legal arc on a pronged surface \( (\Sigma, \vec{P}) \) equipped with a compatible framing \( \phi \). If \( a \) has an endpoint at a legal basepoint on \( \Delta_i \), then

\[
\phi(T_{\Delta_i}^{1/k_i}(a)) = \phi(a) \pm 1,
\]

with the sign positive if and only if \( a \) is oriented as to be incoming at \( \Delta_i \).

We also have the following straightforward extension of Lemma 2.2.

**Lemma 6.5.** Let \( \phi \) and \( \psi \) be two framings of the pronged surface \( (\Sigma, \vec{P}) \), both compatible with the prong structure. Then \( \phi \) and \( \psi \) are relatively isotopic if and only if the associated relative winding number functions are equal. Moreover, \( \phi = \psi \) as relative winding number functions if and only if \( \phi(b) = \psi(b) \) for all elements \( b \) of a distinguished geometric basis \( \mathcal{B} \).

We come now to the main result of the section. This describes the relationship between the stabilizer \( \text{Mod}^*_{g,n}[\phi] \) in the pronged mapping class group, and its subgroup \( \text{Mod}^*_{g,n}[\phi] \) where each prong is required to be individually fixed. In order to do so, we define a certain subgroup of \( PR \). For convenience, we will switch to additive notation and identify \( \mu_k \cong \mathbb{Z}/k \mathbb{Z} \), writing

\[
PR = \left\{ \sum c_i e_i \mid c_i \in \mathbb{Z}/k_i \mathbb{Z} \right\}.
\]
We furthermore write \( \sum' \) to indicate a sum over all indices \( i \) such that \( k_i \) is even. Then define

\[
PR' := \left\{ \sum c_i e_i \in PR \mid \sum' c_i \equiv 0 \pmod{2} \right\}.
\]  

(11)

Observe that if all \( k_i \) are odd then \( PR' = PR \).

**Proposition 6.6.** Let \( (\Sigma, \bar{P}) \) be a pronged surface and let \( \phi \) be a compatible framing. Then the map \( D : \text{Mod}^*_{g,n} \to PR \) induces the short exact sequence

\[
1 \to \text{Mod}_{g,n}[\phi] \to \text{Mod}^*_{g,n}[\phi] \to PR' \to 1.
\]

(12)

Before we begin the proof, we introduce the notion of an **auxiliary curve**.

**Definition 6.7 (Auxiliary curves).** Let \( (\Sigma_{g,n}, \phi) \) be a framed surface with boundary components \( \Delta_1, \ldots, \Delta_n \). An **auxiliary curve** for \( \Delta_k \) is a separating curve \( d_k \) such that \( d_k \) separates \( \Delta_k \) from the remaining boundary components, and so that \( \phi(d_k) = \pm 1 \) or \( \pm 2 \) according to whether \( \phi(\Delta_k) \) is odd or even. An **auxiliary curve** for \( \Delta_i \) and \( \Delta_j \) is any separating curve \( c_{i,j} \) that separates boundary components \( \Delta_i, \Delta_j \) from the remaining components.

**Proof of Proposition 6.6.** By definition,

\[
\text{Mod}_{g,n}[\phi] := \text{Mod}_{g,n} \cap \text{Mod}^*_{g,n}[\phi],
\]

and \( \text{Mod}_{g,n} \) is the kernel of \( D \). This establishes exactness at \( \text{Mod}_{g,n}[\phi] \) and at \( \text{Mod}^*_{g,n}[\phi] \).

It remains to be seen that \( D(\text{Mod}^*_{g,n}[\phi]) = PR' \). We first show that \( PR' \leq D(\text{Mod}^*_{g,n}[\phi]) \) by explicit construction. Observe that \( PR' \) is generated by elements of three kinds:

(G1) \( e_i \) for \( i \) such that \( k_i \) is odd,

(G2) \( 2e_i \) for \( i \) such that \( k_i \) is even,

(G3) \( m(e_i + e_j) \) for \( m \) odd and \( i, j \) such that \( k_i \) and \( k_j \) are even.

Let \( \Delta_i \in \partial \Sigma^* \) be given. Choose a curve \( d_i \) in the following way: if \( k_i \) is odd, pick \( d_i \subset \Sigma^* \) such that \( d_i \) separates \( \Delta_i \) from all remaining boundary components and the subsurface bounded by \( \Delta_i \) and \( d_i \) has genus \((k_i + 1)/2\). If \( k_i \) is even, \( d_i \) may be defined identically except that \( \Delta_i \cup d_i \) must cobound a surface of genus \((k_i + 2)/2\). By Remark 6.3 if \( \phi \) is compatible with \( \bar{P} \), then \( \phi \) is of holomorphic type, and hence \( k_i \leq 2g - 1 \) for all \( i \). Thus the genus of the surface cobounded by \( \Delta_i \cup d_i \) is at most \( g \), and hence \( d_i \) exists for all boundary components \( \Delta_i \). In both cases, orient \( d_i \) so that \( \Delta_i \) is on its left.

By Remark 6.3 and the homological coherence property (Lemma 2.4), if \( k_i \) is odd, then \( \phi(d_i) = -1 \) and if \( k_i \) is even, then \( \phi(d_i) = -2 \). Therefore, \( d_i \) is an auxiliary curve for \( \Delta_i \).

If \( k_i \) is odd, we define an **auxiliary twist of type 1** to be the mapping class

\[
A_i := T_{\Delta_i}^{1/k_i} T_{d_i}^{-1},
\]

(where \( d_i \) is as above), and if \( k_i \) is even, we define it to be

\[
A_i := T_{\Delta_i}^{2/k_i} T_{d_i}^{-1}.
\]

Observe that \( D(A_i) = e_i \) if \( k_i \) is odd, and \( D(A_i) = 2e_i \) if \( k_i \) is even. By the twist-linearity formula (Lemma 2.4) and its extension to fractional twists Lemma 6.4, one verifies that \( A_i \in \text{Mod}^*_{g,n}[\phi] \).
Thus we have exhibited generators of the form \((G1), (G2)\) for \(PR'\). It remains to construct elements mapping to generators of type \((G3)\). Let \(k_i, k_j\) be even and choose an auxiliary curve \(c_{i,j}\) for \(\Delta_i\) and \(\Delta_j\). By homological coherence (Lemma 2.42), \(\phi(c_{i,j})\) is odd. Define the auxiliary twist of type 2 to be the mapping class
\[
B_{i,j} := T_{\Delta_i}^{\phi(c_{i,j})/k_i} T_{\Delta_j}^{\phi(c_{i,j})/k_j} T_{c_{i,j}}^{-1}.
\]
Then \(D(B_{i,j}) = \phi(c_{i,j})(e_i + e_j)\) represents a generator of type \((G3)\), and as before, the twist–linearity formula shows that \(B_{i,j} \in \text{Mod}^*_{g,n}[\phi]\).

We now establish the converse assertion \(D(\text{Mod}^*_{g,n}[\phi]) \leq PR'\). For this, we recall that there is a set–theoretic splitting \(s : PR \to \text{Mod}^*_{g,n}\) given by fractional twists, and define the set–theoretic retraction \(r : \text{Mod}^*_{g,n} \to \text{Mod}_{g,n}\) by \(r(f) = sD(f^{-1})f\).

Let \(f \in \text{Mod}^*_{g,n}[\phi]\) be given. Then \(r(f) \in \text{Mod}_{g,n}\) by construction. The Arf invariant classifies orbits of relative framings under the action of \(\text{Mod}_{g,n}\), and hence we must have
\[
\text{Arf}(r(f) \cdot \phi) = \text{Arf}(\phi).
\]
Let \(\mathcal{B} = \{x_1, \ldots, y_g\} \cup \{a_2, \ldots, a_n\}\) be a distinguished geometric basis. By hypothesis, \(\phi(f \cdot b) = \phi(b)\) for all \(b \in \mathcal{B}\). Also note that
\[
\phi(T_{\Delta_i}^{1/k_i}(a_i)) = \phi(a_i) + 1
\]
while fixing the \(\phi\) values of all other elements of \(\mathcal{B}\). Likewise,
\[
\phi(T_{\Delta_i}^{1/k_i}(a_i)) = \phi(a_i) - 1
\]
for \(i = 2, \ldots, n\) while \(T_{\Delta_i}^{1/k_i}\) fixes the \(\phi\) value of each of the curves \(x_1, \ldots, y_g\). Considering the Arf invariant formula [1], it follows that
\[
\text{Arf}(r(f) \cdot \phi) - \text{Arf}(\phi) = \sum D_i(f).
\]
Thus \(\text{Arf}(r(f) \cdot \phi) = \text{Arf}(\phi)\) exactly when \(\sum D_i(f) = 0\), i.e., when \(D(f) \in PR'\). 

6.2. Pointed surfaces and absolute framed mapping class groups. The second variant of a framed mapping class group we consider is the most coarse. As in the pronged setting, we consider a closed genus \(g\) surface \(\Sigma\) with a collection \(P = \{p_1, \ldots, p_n\}\) of marked points, but we do not equip each \(p_i\) with the structure of a prong point. Thus the mapping class group acting up to isotopy on \((\Sigma, P)\) is the familiar punctured mapping class group \(\text{Mod}^n_g\).

Forgetting the prong structure induces the following short exact sequence of mapping class groups (c.f. [BSW16], Lemma 2.4):
\[
1 \to FT \to \text{Mod}^*_{g,n} \to \text{PMOD}^n_g \to 1,
\]
where we recall that \(FT\) is the group generated by the fractional twists at each \(p_i\).

Suppose that \(\xi\) is a vector field on \(\Sigma\) vanishing only at \(P\). Then \(\xi\) determines a framing of the punctured surface \(\Sigma \setminus P\). Since we do not fix any boundary data, the notion of relative isotopy is ill–defined. To emphasize this, we use the term absolute in this setting, and so we speak of absolute isotopy classes of framings, absolute winding number functions (which measure winding numbers only of oriented simple closed curves, not arcs), and the absolute framed mapping class group. In this language, the pointed mapping class group \(\text{Mod}^n_g\) acts on the set of isotopy classes of absolute framings,
or equivalently on the set of absolute winding number functions. If \( \phi \) is an absolute framing/winding number function, we write \( \text{Mod}^n_g[\phi] \) to denote the stabilizer. The main result concerning \( \text{Mod}^n_g[\phi] \) that we will need for later use is the following.

**Theorem 6.8.** Let \((\Sigma, \vec{P})\) be a pronged surface equipped with a compatible framing \( \phi \). The forgetful map \( p : (\Sigma, \vec{P}) \to (\Sigma, P) \) induces a surjection

\[
p_* : \text{Mod}^*_{g,n}[\phi] \to \text{PMod}^n_g[\phi].
\]

**Proof.** Let \( \vec{f} \in \text{Mod}^n_g[\phi] \) be given, and choose a lift \( f \in \text{Mod}^*_{g,n} \). The set of lifts is a torsor on the kernel \( \text{FT} \) of the forgetful map \( p_* : \text{Mod}^*_{g,n} \to \text{Mod}^n_g \). The group of fractional twists \( \text{FT} \) preserves all absolute winding numbers. If \( B = \{x_1, \ldots, y_g\} \cup \{a_2, \ldots, a_n\} \) is a distinguished geometric basis, then by Lemma 6.4, \( \text{FT} \) acts transitively on the set of values \( (\phi(a_2), \ldots, \phi(a_n)) \). Thus there is \( g \in \text{FT} \) such that \( \phi(gf(b)) = \phi(b) \) for all \( b \in B \). By Lemma 6.5, such \( gf \) is an element of \( \text{Mod}^*_{g,n}[\phi] \), and by construction \( p_*(gf) = \vec{f} \). \( \square \)

Combining this result with Proposition 6.6 yields an explicit generating set for \( \text{PMod}^n_g[\bar{\phi}] \).

**Definition 6.9.** Let \( \Sigma_{g,n} \) have boundary components \( \Delta_1, \ldots, \Delta_n \). An **auxiliary curve system** is a collection of the following auxiliary curves:

- Auxiliary curves \( c_{i,j} \) for all pairs \( i, j \) such that both \( \phi(\Delta_i), \phi(\Delta_j) \) are even,
- Auxiliary curves \( d_k \) for all indices \( k \)

No requirements are imposed on the intersection pattern of curves in an auxiliary curve system.

**Corollary 6.10.** Let \( \kappa \) be a partition of \( 2g - 2 \). Let \( \phi \) be a relative framing with signature \( -1 - \kappa \) and let \( \bar{\phi} \) denote the absolute framing induced on \( \Sigma^n_g \) by capping off the boundary components of \( \Sigma_{g,n} \) with punctured disks. Then \( \text{PMod}^n_g[\bar{\phi}] \) is generated by \( p_*(\text{Mod}_{g,n}[\phi]) \) together with the twists about an auxiliary curve system \( A \).

In particular, when \( g \geq 5 \), then \( \text{PMod}^n_g[\bar{\phi}] \) is generated by the Dehn twists in the curves of \( C \cup A \), where \( C \) is as in Figure 7.

**Proof.** As in the proof of Proposition 6.6, the group of auxiliary twists \( \langle A_k, B_{i,j} \rangle \) surjects onto \( PR' \) and hence by Proposition 6.6

\[
\langle \text{Mod}_{g,n}[\phi], A_k, B_{i,j} \rangle = \text{Mod}^*_{g,n}[\phi].
\]

Now it remains to observe that by Theorem 6.8 this group surjects onto \( \text{PMod}^n_g[\bar{\phi}] \) and that by construction \( p_*(A_k) = T_{d_k}^{-1} \) and \( p_*(B_{i,j}) = c_{i,j}^{-1} \). \( \square \)

### 6.3. The image of the relatively framed mapping class group.

The final result we consider here determines the image of \( \text{Mod}_{g,n}[\phi] \) in \( \text{PMod}^n_g[\bar{\phi}] \) induced by the boundary-capping map \( \Sigma_{g,n} \to \Sigma^n_g \).

**Proposition 6.11.** The image of \( \text{Mod}_{g,n}[\phi] \) in \( \text{PMod}^n_g[\bar{\phi}] \) is a normal subgroup with quotient isomorphic to \( PR'/\langle(1, \ldots, 1)\rangle \).
Proof. By Proposition 6.6, \( \text{Mod}_{g,n}[\phi] \triangleleft \text{Mod}^*_{g,n}[\phi] \) is a normal subgroup with quotient \( PR' \) induced by the “prong rotation map”

\[
D : \text{Mod}^*_{g,n} \rightarrow PR.
\]

By Theorem 6.8, the boundary-capping map \( p_* : \text{Mod}^*_{g,n} \rightarrow \text{Mod}^n_{g} \) restricts to a surjection \( p_* : \text{Mod}^*_{g,n}[\phi] \rightarrow \text{Mod}^n_{g}[\bar{\phi}] \). Thus the image of \( \text{Mod}_{g,n}[\phi] \) is a normal subgroup of \( \text{Mod}^n_{g}[\bar{\phi}] \).

The quotient \( \text{Mod}^n_{g}[\bar{\phi}]/p_*(\text{Mod}_{g,n}[\phi]) \) can be identified by the Isomorphism Theorems. Suppose that \( G \) is a group and \( N_1, N_2 \) are normal subgroups. Then \( N_1 N_2 \) is normal in \( G \), and by the third isomorphism theorem,

\[
(G/N_1)/(N_1N_2/N_1) \cong G/(N_1N_2) \cong (G/N_2)/(N_1N_2/N_2).
\]

We apply this here with

\[
G = \text{Mod}^*_{g,n}[\phi], \quad N_1 = \ker p_*, \quad N_2 = \text{Mod}_{g,n}[\phi].
\]

Then

\[
G/N_1 \cong \text{Mod}^n_{g}[\bar{\phi}], \quad G/N_2 \cong PR',
\]

\[
N_1 N_2/N_1 \cong p_*(N_2) \cong p_*(\text{Mod}_{g,n}[\phi]), \quad N_1 N_2/N_2 \cong D(N_1) \cong D(\ker p_*).
\]

Altogether,

\[
\text{Mod}^n_{g}[\bar{\phi}]/p_*(\text{Mod}_{g,n}[\phi]) \cong PR'/D(\ker p_*).
\]

To complete the argument, it therefore suffices to show that \( D(\ker p_*) \cong \langle (1, \ldots, 1) \rangle \). According to (13), the kernel of \( p_* \) on \( \text{Mod}^*_{g,n} \) is the group \( FT \) of fractional twists. Thus we must identify \( FT \cap \text{Mod}^*_{g,n}[\phi] \). We claim that

\[
FT \cap \text{Mod}^*_{g,n}[\phi] \cong \mathbb{Z}
\]

generated by the fractional twist \( \prod_{i=1}^{n} T_{\Delta_i}^{1/k_i} \). Note that

\[
D(\prod_{i=1}^{n} T_{\Delta_i}^{1/k_i}) = (1, \ldots, 1),
\]

so that showing this isomorphism will complete the argument.

To show this claim, consider an arbitrary element

\[
f := \prod_{i=1}^{n} T_{\Delta_i}^{a_i/k_i}
\]

of \( FT \cap \text{Mod}^*_{g,n}[\phi] \). Let \( \alpha_{i,j} \) be a legal arc connecting \( \Delta_i \) to \( \Delta_j \). By Lemma 6.4,

\[
\phi(f(\alpha_{i,j})) - \phi(\alpha_{i,j}) = a_j - a_i,
\]

so that if \( f \in \text{Mod}^*_{g,n}[\phi] \), necessarily \( a_i = a_j \) for all pairs \( i, j \) as claimed. \( \square \)

We now unravel the condition that \( PR'/\langle (1, \ldots, 1) \rangle \) is trivial using some elementary group theory.
Corollary 6.12. Let $\kappa$ be a partition of $2g - 2$ and write $\kappa = (\eta_1, \ldots, \eta_p, v_1, \ldots, v_q)$ where $\eta_i$ are even, $v_j$ are odd, and $p + q = n$. Then $\Mod_{g,n}[\phi]$ surjects onto $\PMod_n^g[\bar{\phi}]$ if and only if $q \leq 2$ and
\[
\left\{ \eta_1 + 1, \ldots, \eta_p + 1, \frac{v_1 + 1}{2}, \ldots, \frac{v_q + 1}{2} \right\}
\]
are pairwise coprime.

Proof. By Proposition 6.11, it suffices to determine when $PR'$ is cyclic (with generator $(1, \ldots, 1)$).

Using the additive notation introduced above, write
\[
PR_e := \left\{ \sum_{j=1}^{q} c_j e_j \mid c_j \in \mathbb{Z}/(v_j + 1)\mathbb{Z} \right\} \quad \text{and} \quad PR_o := \left\{ \sum_{i=1}^{p} c_i e_i \mid c_i \in \mathbb{Z}/(\eta_i + 1)\mathbb{Z} \right\}.
\]

Even though $PR_e$ is a product over the odd $v_j$'s, our notation reflects the fact that each of its factors has even order.

Now by definition $PR = PR_e \times PR_o$, and likewise we can write $PR' = PR'_e \times PR_o$ where
\[
PR'_e := \left\{ \sum_{j} c_j e_j \in PR_e \mid \sum_{j} c_j \equiv 0 \pmod{2} \right\}
\]
as in (11). We observe that $PR'$ is cyclic if and only if $PR_o$ and $PR'_e$ are cyclic of coprime order, and that $PR_e$ is cyclic if and only if the set of $\eta_i + 1$ are all pairwise coprime.

Suppose that $PR'_e$ is cyclic. If $PR_e$ (and hence also $PR'_e$) is trivial, then the claim holds. Otherwise, there is a short exact sequence
\[
1 \to PR'_e \to PR_e \to \mathbb{Z}/2 \to 1.
\]
It follows that $PR_e$ is either cyclic of order $2|PR'_e|$ or else is isomorphic to $\mathbb{Z}/2 \times PR'_e$. Now as each factor of $PR_e$ is a cyclic group of even order, this implies that $PR_e$ has at most two factors, i.e., $q \leq 2$. Since $PR_e$ is a product over the odd $v_j$'s and we have that $\sum \eta_i + \sum v_j = 2g - 2$, this implies that $q$ is even and is therefore either 0 or 2. So if $PR_e$ is nontrivial it must be isomorphic to $(PR'_e) \times \mathbb{Z}/2$. Necessarily then $(v_1 + 1)/2$ and $(v_2 + 1)/2$ are coprime and
\[
PR'_e \cong \mathbb{Z}/\left(\frac{v_1 + 1}{2}\right)\mathbb{Z} \times \mathbb{Z}/\left(\frac{v_2 + 1}{2}\right)\mathbb{Z}.
\]

The remaining hypothesis that $PR_o$ and $PR'_e$ be cyclic of coprime order readily implies the claim that the elements of
\[
\left\{ \eta_1 + 1, \ldots, \eta_p + 1, \frac{v_1 + 1}{2}, \ldots, \frac{v_q + 1}{2} \right\}
\]
are pairwise coprime as required. □

7. Monodromy of strata

In this section, we fuse our discussion of framings and mapping class groups with the theory of abelian differentials to deduce Theorem [A] from Theorem [B].

We begin in Section [7.1] by collecting basic results about abelian differentials and their strata. We also record Kontsevich and Zorich’s seminal classification (Theorem [7.2]) of the connected components of strata, together with a slight variation in which one labels the zeros of the differential (Lemma [7.3]).
With these foundations laid, we proceed to discuss the relationship between abelian differentials and framings on punctured, bordered, and pronged surfaces in Section 7.2. This section also contains a detailed description of the real oriented blow-up of an abelian differential along its zero locus (Construction 7.4). The main result of this subsection is Boissy’s classification of the components of strata of prong–marked differentials (Theorem 7.5) and its implications for monodromy (Corollary 7.6).

Mapping class groups enter the picture in Section 7.3, in which we define a family of coverings of strata (first introduced in [BSW16]) whose deck groups are mapping class groups of punctured, bordered, and pronged surfaces (see Diagram 7.1). Using the relations between these spaces, we then prove that the monodromy of each covering must preserve the appropriate framing datum (Lemma 7.10 and Corollaries 7.11 and 7.12).

We establish the reverse inclusions (that the monodromy is the entire stabilizer of the framing) in Section 7.4 as Theorems 7.13, 7.14, and A.

### 7.1. Abelian differentials

An abelian differential $\omega$ is a holomorphic 1–form on a Riemann surface $X$. The collection of all abelian differentials forms a vector bundle (in the orbifold sense) $\Omega M_g$ over the moduli space of curves. The complement of the zero section of this bundle is naturally partitioned into disjoint subvarieties called strata which have a fixed number and order of zeros. For $\kappa = (\kappa_1, \ldots, \kappa_n)$, we will let $\Omega M_g(\kappa)$ denote the space of all pairs $(X, \omega)$ where $\omega$ is an abelian differential on $X$ which has zeros of order $\kappa_1, \ldots, \kappa_n$. Throughout this section, we will use $Z$ to denote the set of zeros of $\omega$.

Away from its zeros, an abelian differential has canonical local coordinates in which it can be written as $dz$; the transition maps between these coordinate charts are translations, so the data of an abelian differential $\omega$ on a Riemann surface $X$ is also sometimes called a translation structure. Pulling back the Euclidean metric of $\mathbb{C}$ along the canonical coordinates defines a flat metric on $X$ with a cone point of angle $2\pi(k + 1)$ at each zero of order $k$.

Every abelian differential $\omega$ also defines a horizontal vector field $H_\omega := 1/\omega$ on $X$ with singularities of order $-\kappa_1, \ldots, -\kappa_n$, and hence gives rise to a prong structure $\tilde{Z}$ with a prong point of order $\kappa_i + 1$ at the $i$th zero of $\omega$. Forgetting the prong structure and the marked points, the differential induces a $\gcd(\kappa)$–spin structure $\phi$ on $X$ (see Definition 2.6 and the discussion which follows it). In particular, if $\gcd(\kappa)$ is even, then there is a well-defined mod-2 reduction of the spin structure.

**Remark 7.1.** For a more thorough treatment of the relationship between (higher) spin structures and abelian differentials, the reader is directed to [Cal19] and [CS19].

While it does not make sense to compare spin structures on different (unmarked) surfaces, when $\gcd(\kappa)$ is even the Arf invariant of $\phi$ is well-defined even without choice of marking. Moreover, Arf($\phi$) is invariant under deformation and classifies the non-hyperelliptic components of $\Omega M_g(\kappa)$.

**Theorem 7.2** (Theorem 1 of [KZ03]). Let $g \geq 4$ and $\kappa = (\kappa_1, \ldots, \kappa_n)$ be a partition of $2g - 2$. Then $\Omega M_g(\kappa)$ has at most three components:
• If \( \kappa = (2g - 2) \) or \( (g - 1, g - 1) \) then there is a unique component of \( \Omega M_g(\kappa) \) that consists entirely of hyperelliptic differentials.

• If \( \gcd(\kappa) \) is even then there are two components containing non-hyperelliptic differentials, classified by the Arf invariant of their induced 2–spin structure.

• If \( \gcd(\kappa) \) is odd then there is a unique component containing non-hyperelliptic differentials.

We will focus our attention on the non-hyperelliptic components of \( \Omega M_g(\kappa) \); the hyperelliptic components are \( K(G, 1) \)'s for (finite extensions of) spherical braid groups \([LM14, \S 1.4]\) and therefore their monodromy can be understood entirely through Birman–Hilden theory. See \([Cal19, \S 2]\) for a more thorough discussion.

We will often find it convenient to label the zeros of an abelian differential so as to distinguish them. The corresponding stratum \( \Omega M_g^{\text{lab}}(\kappa) \) of abelian differentials with labeled singularities is clearly a finite cover of \( \Omega M_g(\kappa) \) with deck group

\[
\text{Sym}(\kappa) = \prod_{j=1}^{m} \text{Sym}(r_j)
\]

where \( \kappa = (k_1^{r_1}, \ldots, k_m^{r_m}) \) and \( \text{Sym}(n) \) is the symmetric group on \( n \) letters. Moreover, it is not hard to show that each preimage of a connected component of \( \Omega M_g(\kappa) \) is itself connected (see, e.g., \([Boi12, \text{Proposition 4.1}]\)), and hence the monodromy of this covering map is the entirety of the deck group.

**Lemma 7.3.** Let \( \mathcal{H} \) be a component of \( \Omega M(\kappa) \). Then the monodromy homomorphism

\[
\pi_1^{\text{orb}}(\mathcal{H}) \to \text{Sym}(\kappa)
\]

associated to the covering \( \Omega M_g^{\text{lab}}(\kappa) \to \Omega M_g(\kappa) \) is surjective.

7.2. From differentials to framings. To relate our results on framed mapping class groups to strata, we must first understand the different types of framings which an abelian differential induces.

As observed above, the horizontal vector field of an abelian differential \((X, \omega) \in \Omega M_g(\kappa)\) induces a \( \gcd(\kappa) \)-spin structure on the underlying closed surface \( X \). Moreover, since \( \omega \) has canonical coordinates (in which it looks like \( dz \)) away from its zeros, we see that \( \omega \) induces a trivialization of \( T(X \setminus Z) \) and hence an absolute framing of \( X \setminus Z \) (in the sense of \( \S 6.2 \)).

To obtain a relative framing from \( \omega \), we must first identify the surface \( X^* \) which is to be framed. Informally, this “real oriented blow-up” \( X^* \) is obtained by replacing each zero of \( \omega \) by the circle of directions at that point; the horizontal vector field then extends by continuity along rays to a vector field (and eventually, a framing) on \( X^* \) whose boundary data can be read off from the order of the singularities. For more on the real oriented blow-up construction in the context of translation surfaces, see \([BSW16, \S 2.5]\).

**Construction 7.4** (Real oriented blow-ups). We begin by first describing the real oriented blow-up of \( 0 \in \mathbb{C} \); this toy example will provide a local model for the blow-up of a translation surface.

\[\text{Recall that an abelian differential is hyperelliptic if arises as the global square root of a quadratic differential on \( \hat{\mathbb{C}} \) with at worst simple poles.}\]
Equipping \( \mathbb{C} \) with polar coordinates \( z = re^{i\theta} \) gives a parametrization of \( \mathbb{C} \setminus \{0\} \) by the infinite open half–cylinder \( \mathbb{R}_{>0} \times [0, 2\pi]/(0 = 2\pi) \). The real oriented blow-up of \( 0 \in \mathbb{C} \) is the closed half–cylinder \( \mathbb{R}_{\geq 0} \times [0, 2\pi]/(0 = 2\pi) \), which has a natural surjective map onto \( \mathbb{C} \) extending polar coordinates. The fiber of this map above 0 is therefore identified with the circle of directions at 0.

To blow up a cone point, let \( k \geq 1 \) and consider the branched cover of \( \mathbb{C} \) given by \( z \mapsto z^k \). The Euclidean metric of \( \mathbb{C} \) pulls back to a cone metric with cone angle \( 2k\pi \) at 0, and similarly the polar parametrization of \( \mathbb{C} \setminus \{0\} \) pulls back to a parametrization by \( \mathbb{R}_{>0} \times [0, 2k\pi]/(0 = 2k\pi) \). Therefore, the blow-up of a cone point of angle \( 2k\pi \) is the corresponding closed cylinder \( \mathbb{R}_{\geq 0} \times [0, 2k\pi]/(0 = 2k\pi) \), and the fiber above 0 corresponds to the \( 2k\pi \)'s worth of directions at 0.

Now suppose that \((X, \omega) \in \Omega M_{g,\text{lab}}^\text{pr} (\mathcal{K})\) with zeros at points \( p_1, \ldots, p_n \). The real oriented blow-up \( X^* \) of \( X \) is the space obtained after blowing up each cone point \( p_i \) via the above construction.

Observe that \( X^* \) is naturally a surface of the same genus as \( X \) with boundary components \( \Delta_1, \ldots, \Delta_n \), the \( i^{th} \) of which comes with an identification with the \((k_i + 1)\)-fold cover of the circle (which is of course just a circle itself, equipped with a cyclic symmetry of order \((k_i + 1)\)).

Moreover, the unit horizontal vector field \( H \) of \( \omega \) induces a framing \( \omega \) on \( X^* \) by extending \( H \) continuously along rays into each cone point. For each boundary component \( \Delta_i \), the vector field \( H^*|_{\Delta_i} \) is invariant under the cyclic symmetry described above, and its winding number is \(-1 - k_i\).

Hence \( H^* \) induces a framing \( \phi \) of \( X^* \) with boundary signature

\[
\text{sig}(\phi) = (-1 - k_1, \ldots, -1 - k_n)
\]

which is compatible (in the sense of Definition 6.2) with the prong structure \((X, \vec{Z})\) induced by \( \omega \).

**Prong markings.** While the blowup \( X^* \) of \((X, \omega)\) is topologically a surface with boundary, it is more accurate to view \( X^* \) as (the blow-up of) a surface with prong structure. In particular, there exist loops \( \gamma \) in \( \Omega M_{g,\text{lab}}^\text{pr} (\mathcal{K}) \) which rotate the prongs of \((X, \vec{Z})\) and hence act by fractional twists on \( \partial X^* \). This phenomenon can be interpreted as the monodromy of \( \Omega M_{g,\text{lab}}^\text{pr} (\mathcal{K}) \) taking values in \( \text{Mod}^*_{g,\text{pr}} \) rather than \( \text{Mod}^*_{g,\text{n}} \) (see Section 7.3 below).

By passing to a finite cover of \( \Omega M_{g,\text{lab}}^\text{pr} (\mathcal{K}) \) which remembers more information, we may therefore constrain the monodromy to lie in \( \text{Mod}^*_{g,\text{pr}} \). To that end, define a **prong marking** of an abelian differential \( \omega \) to be a labeling of its zeros together with a choice of positive horizontal separatrix at each zero (these are also sometimes called **framings of** \( \omega \), as in \cite{Boi15}). In terms of the prong structure \( \vec{Z} \) on \( X \) induced by \( \omega \), a prong marking chooses a prong at each zero.

The (components of the) space \( \Omega M_{g,\text{lab}}^\text{pr} (\mathcal{K}) \) of prong–marked abelian differentials are finite covers of (the components of) a stratum \( \Omega M_{g,\text{lab}}^\text{lab} (\mathcal{K}) \). Moreover, the deck transformations of this covering rotate the choice of specified prong at each zero and hence the deck group is exactly \( PR \).

Any loop in \( \Omega M_{g,\text{lab}}^\text{pr} (\mathcal{K}) \) preserves the prong marking and so acts as the identity on \( \partial X^* \) by the correspondence outlined in Section 6.3. Therefore, the real oriented blow-up of a prong–marked abelian differential...
differential can be consistently interpreted as a surface with boundary (and hence the monodromy of $\Omega M_g^{pr}(\kappa)$ is in $\text{Mod}_{g,n}$, as we will see in §7.3).

We now record Boissy's classification of the components of $\Omega M_g^{pr}(\kappa)$. Our statement of following theorem looks rather different than that which appears in [Boi15]; we reconcile these differences in Remark 7.7 at the end of this section.

**Theorem 7.5** (c.f. Theorem 1.3 of [Boi15]). Suppose that $\mathcal{H}$ is a non-hyperelliptic connected component of $\Omega M_g^{ab}(\kappa)$. Then the preimage of $\mathcal{H}$ in $\Omega M_g^{pr}(\kappa)$ has

- one connected component if $\gcd(\kappa)$ is even, and
- two connected components if $\gcd(\kappa)$ is odd, distinguished by the generalized Arf invariant $(\text{l})$ of the relative framing on the real oriented blow-up.

Combining this classification with Theorem 7.2 immediately implies that for $g \geq 4$ there are exactly two non-hyperelliptic components of $\Omega M_g^{pr}(\kappa)$, classified by generalized Arf invariant (compare with Proposition 2.10).

Translating Theorem 7.5 into the action of the deck group $PR$ therefore identifies the monodromy of the covering $\Omega M_g^{pr}(\kappa) \to \Omega M_g^{ab}(\kappa)$:

**Corollary 7.6.** Suppose that $g \geq 3$ and let $\mathcal{H}$ be a non-hyperelliptic connected component of $\Omega M_g^{ab}(\kappa)$. Then the image of the monodromy homomorphism $\pi_1^{\text{orb}}(\mathcal{H}) \to PR$ is exactly the subgroup $PR'$ of (11).

In particular, when $\gcd(\kappa)$ is even the monodromy is all of $PR$.

**Proof.** This is an easy consequence of Lemma 6.4 together with the formula for the generalized Arf invariant (1); simply observe that a prong rotation changes the winding number of every arc incident to the prong point by $\pm 1$.

More explicitly, pick $\omega \in \mathcal{H}$ and a preimage $\omega_1^{pr} \in \Omega M_g^{pr}(\kappa)$. Choose a distinguished geometric basis $B_1$ on $X^*$ with its basepoints specified by $\omega_1^{pr}$. Then given another preimage $\omega_2^{pr}$ of the same $\omega \in \mathcal{H}$ there is a corresponding basis $B_2$ which differs from $B_1$ by a fractional multitwist $\tau$ (this choice is not unique, but is determined up to full twists about the boundary of the blowup $X^*$ of $\omega$).

Computing the Arf invariants of $\omega_1^{pr}$ and $\omega_2^{pr}$ with respect to these bases, we see that the two Arf invariants agree (and hence $\omega_1^{pr}$ and $\omega_2^{pr}$ live in the same component of $\Omega M_g^{pr}(\kappa)$) if and only if

$$\sum_{i=2}^n \phi(a_i)\kappa_i = \sum \phi'(a_i) = \sum \phi(\tau(a_i)) = \sum_{i=2}^n \phi(\tau(a_i))\kappa_i \pmod{2}.$$  

In particular, this equality holds if and only if $D(\tau) \in PR'$.

We observe that the choice of $\tau$ does not matter as any two choices $\tau$ and $\tau'$ differ by full twists about the boundary of $X^*$: twists about boundary components with even winding number do not change the parity of $\phi(\tau(a_i))$ while twists about odd boundaries change parity of arcs which do not end up contributing to $\sum'$. \Box

**Remark 7.7.** In addition to the differences in terminology, the statement and proof of Theorem 7.5 in [Boi15] distinguishes the two non-hyperelliptic components of $\Omega M_g^{pr}(\kappa)$ by a different invariant. There, Boissy differentiates the two by first choosing a set of arcs on $\omega \in \Omega M_g^{pr}(\kappa)$ which pair up the
odd order zeros, are transverse to the horizontal foliation, and are tangent to the specified prongs. Applying the “parallelogram construction” of [EMZ03] to \( \omega \) along these arcs results in a new differential \( \omega' \) of higher genus with all even zeros; the Arf invariant of the 2–spin structure induced by \( \omega' \) then distinguishes the components of \( \Omega M_g^{pr}(\kappa) \).

The reader may verify that Boissy’s invariant coincides with the generalized Arf invariant by computing the contribution to Arf(\( \omega' \)) of each new handle and comparing it to the corresponding term in the expression for Arf(\( \phi \)) (where \( \phi \) is interpreted as a framing of \( X^* \)).

### 7.3. Markings and monodromy.

In order to compare the framings induced by differentials on different Riemann surfaces, we pull them back to a framings of a reference topological surface. To that end, we need to understand markings of \( X, X^* \), and \( (X, \bar{Z}) \), together with the corresponding spaces of marked differentials.

The coarsest type of marking data we consider in this section are homeomorphisms from a closed surface \( \Sigma_g \) to \( X \) which take a specified set of (labeled) marked points \( P \) to the (labeled) zeros \( Z \) of \( \omega \). With this data, we can define the corresponding space \( \Omega T^{lab}_g(\kappa) \) of marked differentials with marked points as the space of triples \((X, \omega, f)\), where \((X, \omega) \in \Omega M_g^{lab}(\kappa) \) and \( f : (\Sigma, P) \to (X, Z) \) is an isotopy class of homeomorphisms of pairs. The space of marked differentials with marked points is naturally a (disconnected, orbifold) covering space of \( \Omega M_g^{lab}(\kappa) \) whose deck transformations correspond to changing the marking, so its deck group is \( P\text{Mod}_g^{lab} \).

On the other end of the spectrum, we may also mark a differential \( \omega \in \Omega M_g^{lab}(\kappa) \) by a pronged surface. Fix a topological pronged surface \((\Sigma, \vec{P})\) with prong points of order \( \kappa_1 + 1, \ldots, \kappa_n + 1 \), and recall that that any differential \( \omega \) naturally equips its underlying surface \( X \) with a prong structure \( \vec{Z} \) of the same prong type. Then a marking of \((X, \omega)\) by \((\Sigma, \vec{P})\) is a diffeomorphism of pairs \( f \) from \((\Sigma, P)\) to \((X, Z)\) such that \( Df \) takes the prong structure of \( \vec{P} \) to that of \( \vec{Z} \). Equivalently, \( f \) is a diffeomorphism from the real oriented blow-up \( \Sigma^* \) of \( \Sigma \) to \( X^* \) which takes the distinguished points \( \partial \Sigma^* \) to those of \( \partial X^* \).

We may now record the definition of the corresponding space of marked differentials:

**Definition 7.8** (c.f. §2.9 of [BSW16]). \( \Omega T^{pr}_g(\kappa) \) is the space of marked differentials with a prong–marking, that is, the space of triples \((X, \omega, f)\) where \((X, \omega) \in \Omega M_g^{lab}(\kappa) \) and \( f : (\Sigma, \vec{P}) \to (X, \vec{Z}) \) is an isotopy class of marking of pronged surfaces.

Forgetting the the prong structure induces a covering map from \( \Omega T^{pr}_g(\kappa) \) to \( \Omega T^{lab}_g(\kappa) \), the deck group of which is exactly \( FT \). Similarly, forgetting the marking of the surface but remembering the marking of its boundary (i.e., remembering \( \partial f : \partial \Sigma_{g,n} \to \partial X^* \)) induces a map from \( \Omega T^{pr}_g(\kappa) \) to \( \Omega M_g^{pr}(\kappa) \). Since this map remembers the boundary marking, hence the specified prong, the deck group of this covering is thus a change–of–marking group which preserves the boundary pointwise.

**Lemma 7.9** (Corollary 2.7 of [BSW16]). The space \( \Omega T^{pr}_g(\kappa) \) is a (disconnected, orbifold) covering of both \( \Omega M_g^{pr}(\kappa) \) and \( \Omega T^{lab}_g(\kappa) \). Moreover, the deck group of the former covering is \( \text{Mod}_{g,n} \), and the latter \( FT \).

\(^7 \)Recall that \( \partial \Sigma^* \) can be identified with the circle of directions above each blown–up point, hence a prong point of order \( k \) corresponds to a boundary component with \( k \) distinguished points.
Putting this Lemma together with either (10) or (13), we see that the deck group of the covering \(\Omega T^\text{pr}_g(\kappa) \rightarrow \Omega M^\text{lab}_g(\kappa)\) is exactly the pronged mapping class group \(\text{Mod}^*_{g,n}\).

We summarize the relationship between all of these spaces (and their deck groups) in the following diagram, in which arrow labels correspond to the deck group of the covering:

\[
\begin{array}{cccc}
\Omega T^\text{pr}_g(\kappa) & \xrightarrow{FT} & \Omega T^\text{lab}_g(\kappa) \\
\downarrow \text{Mod}_{g,n} & \swarrow \text{Mod}^*_{g,n} & \searrow \text{PMod}^*_{g,n} \\
\Omega M^\text{pr}_g(\kappa) & & \Omega M^\text{lab}_g(\kappa) & \xrightarrow{\text{Sym}(\kappa)} \Omega M_g(\kappa)
\end{array}
\]

(14)

Observe that the furthest left triangle demonstrates the exact sequence (10), while the center triangle corresponds to (13).

**Constraining the monodromy.** While the deck groups of the coverings in (14) are easy to describe, their elements generally permute the components of the corresponding covers. We now shift our focus to the stabilizer of a component of one of these covers; for concreteness, throughout the rest of this section we will focus on the covering \(\Omega T^\text{pr}_g(\kappa) \rightarrow \Omega M^\text{lab}_g(\kappa)\) and use this discussion to deduce the corresponding results for each of the intermediate coverings.

We observe that by the path–lifting property, understanding the stabilizer of a component of \(\Omega T^\text{pr}_g(\kappa)\) is equivalent to understanding the monodromy of the component of \(\Omega M^\text{lab}_g(\kappa)\) which it covers (c.f. [CS19, Proposition 3.7]).

Monodromy groups are always only defined up to conjugacy, so we fix some reference marking \(f : (\Sigma, \vec{P}) \rightarrow (X, \vec{Z})\) (equivalently, a lift of a basepoint in \(\Omega M^\text{lab}_g(\kappa)\) to \(\Omega T^\text{pr}_g(\kappa)\)). Pulling back the framing of \((X, \vec{Z})\) induced by \(\omega\) along \(f\) induces a framing \(\phi\) of \(\Sigma\) compatible with \(\vec{P}\). Path–lifting then allows us to place the following constraint on the monodromy, which is just a version of [Cal19, Corollary 4.8] adapted to the setting of framings rather than \(r\)–spin structures.

**Lemma 7.10.** Let \(H\) be a component of \(\Omega M^\text{lab}_g(\kappa)\) and fix some basepoint \((X, \omega) \in H\). Then the monodromy of the covering of \(H\) by (a component of) \(\Omega T^\text{pr}_g(\kappa)\) preserves the induced framing \(\phi\) on the pronged surface \(X\). In other words, the image of

\[
\rho : \pi_1^\text{orb}(H, (X, \omega)) \rightarrow \text{Mod}^*_{g,n}
\]

is contained inside of \(\text{Mod}^*_{g,n}[\phi]\).

**Proof.** Let \(f\) be a marking of \((X, \vec{Z})\) by \((\Sigma, \vec{P})\) and let \(\gamma\) be a loop in \(H\) based at \((X, \omega)\).

By Lemma 7.9, the loop \(\gamma\) lifts to a path \(\tilde{\gamma}\) in \(\Omega T^\text{pr}_g(\kappa)\), and hence a path of marked prong–marked abelian differentials \((X_t, \omega_t, f_t)\) with horizontal vector fields \(H_t\). Pulling back \(H_t\) by \(f_t\) yields a continuous family of vector fields on \((\Sigma, \vec{P})\), all compatible with the prong structure. Let \(\phi_t\) denote the associated framing of \((\Sigma, \vec{P})\).
Then since the vector fields vary continuously (and do not vanish on \( \Sigma \setminus P \)), the winding number \( \phi_t(a) \) of every simple closed curve or legal arc \( a \) is continuous in \( t \). However, \( \phi_t(a) \) takes values only in \( \mathbb{Z} \) or \( \mathbb{Z} + (1/2) \) and must therefore be constant over the entire path \( \gamma \). Thus \( \rho(\gamma) \) preserves the winding number of every simple closed curve and legal arc, hence the entire framing.

In view of the sequences (12) and (13), together with (14), this implies the following two results, where \( \phi \) and \( \bar{\phi} \) denote the relative and absolute framings induced on \( X^* \) and \( X \setminus Z \), respectively.

**Corollary 7.11.** The monodromy of the covering \( \Omega T^\text{pr}_g(\kappa) \to \Omega M^\text{pr}_g(\kappa) \) lies inside of \( \text{Mod}_g,n[\phi] \).

**Corollary 7.12.** The monodromy of the covering \( \Omega T^\text{lab}_g(\kappa) \to \Omega M^\text{lab}_g(\kappa) \) lies inside of \( \text{PMod}_g^n[\bar{\phi}] \).

We note that these corollaries can also be proven directly using the same argument as Lemma 7.10. For brevity, in the sequel we use \( G^*, G^\text{pr}, \) and \( G^\text{lab} \) to denote the monodromy groups of the coverings appearing in Lemma 7.10 and Corollaries 7.11 and 7.12 respectively.

7.4. **Generating framed mapping class groups.** Now that we have shown that the monodromy of each of the coverings under consideration stabilizes the relevant framing, we can use Theorem B to show that the group is the entire stabilizer of the framing:

**Theorem 7.13.** Suppose that \( g \geq 5 \) and let \( \mathcal{H} \) be a non-hyperelliptic component of \( \Omega M^\text{pr}_g(\kappa) \). Then

\[
G^\text{pr}(\mathcal{H}) \cong \text{Mod}_{g,n}[\phi]
\]

where \( \phi \) is the relative framing of the real oriented blow-up \( X^* \) induced by the horizontal vector field on any \( (X, \omega) \in \Omega M^\text{pr}_g(\kappa) \).

Assuming this theorem, we can leverage our understanding of the relationship between framing stabilizers to characterize both \( G^* \) and \( G^\text{lab} \).

**Theorem 7.14.** Suppose that \( g \geq 5 \) and let \( \mathcal{H} \) be a non-hyperelliptic component of \( \Omega M^\text{lab}_g(\kappa) \). Then

\[
G^*(\mathcal{H}) \cong \text{Mod}_{g,n}^*[\phi]
\]

where \( \phi \) is the relative framing of a pronged surface induced by the horizontal vector field of any \( (X, \omega) \in \Omega M^\text{lab}_g(\kappa) \).

**Proof.** By Lemma 7.10 \( G^* \leq \text{Mod}_{g,n}^*[\phi] \). By Theorem 7.5 \( G^* \) surjects onto \( PR' \), and so by Theorems 7.13 and 6.6 it follows that \( \text{Mod}_{g,n}^*[\phi] \leq G^* \). \( \square \)

Forgetting the prongs, we can push this result down to the mapping class group with marked points to complete the proof of Theorem A.

**Proof of Theorem A.** Observe that \( G^\text{lab} \) is the image of \( G^* \) under the surjection of (13). Therefore, Theorems 7.14 and 6.8 together imply that

\[
G^\text{lab} = \text{PMod}_g^n[\bar{\phi}].
\]
Combining this result with Lemma 7.3 and the short exact sequence
\[ 1 \to \text{PMod}_{g}^{n}[\phi] \to \text{Mod}_{g}^{n}[\phi] \to \text{Sym}(\kappa) \to 1 \]
completes the proof of the theorem. □

**Cylinder shears and prototypes.** It therefore remains to show that \( \text{Mod}_{g,n}^{n}[\phi] \leq \mathcal{G}^{pr} \). In order to demonstrate this, we will need to build a collection of loops of abelian differentials with prescribed monodromy.

Recall that a cylinder on an abelian differential is an embedded Euclidean cylinder with no singularities in its interior. Shearing the cylinder while leaving the rest of the surface fixed results in a loop in a stratum whose monodromy is the Dehn twist of the core curve of the cylinder.

**Lemma 7.15** (c.f. Lemma 6.2 of [Cal19]). Let \( H \) be a component of \( \Omega M_{g}^{pr}(\kappa) \) and suppose that \( \omega \in H \) has a cylinder with core curve \( c \). Then \( T_{c} \in \mathcal{G}^{pr} \).

Now that we can use cylinder shears to exhibit Dehn twists, it remains to show that there exist differentials in a stratum with a configuration of cylinders to which we can apply Theorem B.

**Lemma 7.16** (Lemma 3.14 of [CS19]). Suppose that \( g \geq 5 \) and \( \kappa = (\kappa_{1}, \ldots, \kappa_{n}) \) is a partition of \( 2g - 2 \). Let \( H \) be a non-hyperelliptic component of \( \Omega M_{g}(\kappa) \). Set \( C \) to be the curve system specified in Theorem B of type 1 if \( \text{gcd}(\kappa_{1}, \ldots, \kappa_{n}) \) is even and
\[ \text{Arf}(H) = \begin{cases} 1 & g \equiv 0, 3 \pmod{4} \\ 0 & g \equiv 1, 2 \pmod{4}, \end{cases} \]
and of type 2 otherwise. Then there exists some \( \omega \in H \) whose horizontal and vertical cylinders are exactly the curves of \( C \).

By labeling a separatrix at each zero, these prototype surfaces also give rise to prong–marked abelian differentials with specified Arf invariant.

**Lemma 7.17.** Let \( g, \kappa, H, \) and \( C \) be as above. Let \( H^{pr} \) be a (non-hyperelliptic) component of the stratum of prong–marked abelian differentials which covers \( H \). Then there exists some \( \omega \in H^{pr} \) whose horizontal and vertical cylinders are exactly the curves of \( C \).

**Proof.** Let \( \omega \in H \) be the differential constructed in Lemma 7.16. If \( \text{gcd}(\kappa) \) is even, then by Theorem 7.5 the entire preimage of \( H \) in \( \Omega M_{g}^{lab}(\kappa) \) is \( H^{pr} \) and so any prong marking of \( \omega \) lives in \( H^{pr} \). If \( \text{gcd}(\kappa) \) is odd, then choose an arbitrary prong marking \( \omega^{pr} \) of \( \omega \) and let \( p_{i} \) be some zero of odd order (corresponding to a prong point of even order).

Now by Corollary 7.6 since \( T_{p_{i}} \notin \mathcal{P}R' \) the differentials \( \omega^{pr} \) and \( T_{p_{i}}(\omega^{pr}) \) must lie in different components of \( \Omega M_{g}^{pr}(\kappa) \). Since there are only two components (by Theorem 7.5), one of \( \{T_{p_{i}}(\omega^{pr}), \omega^{pr}\} \) must be in \( H^{pr} \). □

It is now completely straightforward to deduce that the monodromy group of a component \( H^{pr} \) of \( \Omega M_{g}^{pr}(\kappa) \) is exactly the stabilizer of the relative framing induced on the real oriented blow-up.
Proof of Theorem 7.13. Let \((X, \omega) \in H_{pr}\) be the prototype surface built in Lemma 7.17 above. Then by Theorem 7.13, Lemma 7.15, and Corollary 7.11, we have

\[
\text{Mod}_{g,n}^\phi = \langle T_c \mid c \in \mathcal{C} \rangle \subseteq G_{pr} \subseteq \text{Mod}_{g,n}^\phi
\]

completing the proof of Theorem 7.13.

8. Further corollaries

8.1. Classification of components. The monodromy computations in Theorems A 7.13 and 7.14 lead to the following classification of the non-hyperelliptic components of strata of marked differentials (c.f. Theorem A of [CS19]):

Corollary 8.1. There is a bijection between the non-hyperelliptic components of \(\Omega_T^\text{lab}_g(\kappa)\) and the isotopy classes of absolute framings of \(\Sigma^n_g\) with signature \(-1 - \kappa\).

If \(\gcd(\kappa)\) is odd then the permutation action of \(\text{Mod}_{g,n}^\phi\) is transitive, while if \(\gcd(\kappa)\) is even there are two orbits, classified by Arf invariant.

Corollary 8.2. There is a bijection between the non-hyperelliptic components of \(\Omega_T^\text{pr}_g(\kappa)\) and the (relative) isotopy classes of relative framings of \(\Sigma_{g,n}\) with signature \(-1 - \kappa\).

The action of \(\text{Mod}_{g,n}^\phi\) is transitive if \(\gcd(\kappa)\) is odd and has two orbits if \(\gcd(\kappa)\) is even, classified by the Arf invariant of the absolute framing. The action of \(\text{Mod}_{g,n}\) has two orbits no matter the parity of \(\gcd(\kappa)\), classified by the generalized Arf invariant.

The proofs of both corollaries are simply a consequence of the orbit–stabilizer theorem applied to the \(\text{PMod}_{g,n}^\phi\) action on the set of components of \(\Omega_T^\text{lab}_g(\kappa)\), respectively the \(\text{Mod}_{g,n}\) action on the components of \(\Omega_T^\text{pr}_g(\kappa)\), together with the classification of orbits (Theorems 7.2 and 7.5, respectively).

8.2. Cylinder shears and fundamental groups of strata. For a component \(H\) of a general stratum \(\Omega_M^\kappa\), no explicit set of generators for \(\pi_1^\text{orb}(H)\) is known. Cylinder shears play a role analogous to Dehn twists in the theory of the mapping class group, and it is natural to wonder about the extent to which shears generate \(\pi_1^\text{orb}(H)\). If the partition \(\kappa\) contains any repeated elements, the corresponding zeros can be exchanged, but this certainly cannot be accomplished using shears. Thus one must first pass to a “labeled stratum,” i.e., a component \(H_{\text{lab}}\) of the cover \(\Omega_M^\text{lab}_g(\kappa)\). Even here, the work of Boissy (in the guise of Corollary 7.6) implies that \(\pi_1^\text{orb}(H_{\text{lab}})\) is never generated by shears alone, since shears map trivially onto the prong rotation group \(PR\). However, prong rotation is not detected by the monodromy representation \(\rho : \pi_1^\text{orb}(H_{\text{lab}}) \to \text{PMod}_g^\phi[\bar{\phi}]\), only by the refinement whose target is the pronged mapping class group.

As a corollary of our monodromy computations and Corollary 6.12, we find that the arithmetic of the partition \(\kappa\) provides an obstruction for the subgroup of \(\pi_1^\text{orb}(H_{\text{lab}})\) generated by cylinder shears to generate the monodromy group in \(\text{PMod}_g^\phi[\bar{\phi}]\). Thus, the prong rotation group “leaves a trace” in the group \(\text{PMod}_g^\phi[\bar{\phi}]\), even though there is no way of measuring prong rotation in \(\text{PMod}_g^\phi[\bar{\phi}]\) directly.
Corollary 8.3. Let $\kappa$ be a partition of $2g-2$ for $g \geq 5$ and let $H_{lab}$ be a non-hyperelliptic component of the stratum $\Omega M_g^{lab}(\kappa)$. Write $\kappa = (\eta_1,\ldots,\eta_p,v_1,\ldots,v_q)$ where $\eta_i$ are even, $v_j$ are odd, and $p+q = n$. If $q > 2$, or if the elements of
\[
\left\{ \eta_1 + 1, \ldots, \eta_p + 1, \frac{v_1 + 1}{2}, \ldots, \frac{v_q + 1}{2} \right\}
\]
are not pairwise coprime, then the subgroup of $\pi_1^{orb}(H_{lab})$ generated by cylinder shears does not surject onto $G_{lab}^\kappa$.

Proof. Choose an arbitrary component $H_{pr}$ of the preimage of $H_{lab}$ in $\Omega M_g^{pr}(\kappa)$ and let $\phi$ denote the induced relative framing. Let $C$ denote the subgroup of $\pi_1^{orb}(H_{lab})$ generated by cylinder shears (or rather, by elements which are conjugate to cylinder shears by some path connecting a basepoint $(X,\omega)$ to the surface $(Y,\omega)$ realizing the relevant cylinder). As cylinder shears preserve prong–markings, we see that $C \leq \pi_1^{orb}(H_{pr})$.

Recall that $\rho_{pr} : \pi_1^{orb}(H_{pr}) \to \text{Mod}_{g,n}$ denotes the monodromy representation of $H_{pr}$ with image $G_{pr}^\kappa$, and that $G_{lab}^\kappa$ denotes the image of the monodromy representation of $H_{lab}$ into $\text{PMod}_{g,n}^\kappa$. Now Theorem A finds that $G_{lab}^\kappa = \text{PMod}_{g}^{\kappa}[\overline{\phi}]$, and Theorem 7.13 finds that $\rho_{pr}(C) \leq G_{pr}^\kappa = \text{Mod}_{g,n}[\phi]$. By Corollary 6.12 under our current hypotheses, the map $\text{Mod}_{g,n}[\phi] \to \text{PMod}_{g}^{\kappa}[\overline{\phi}]$ is never surjective. Therefore, the image of $C$ in $G_{lab}^\kappa$ is a strict subgroup. □

8.3. Change–of–coordinates for saddle connections. In this section, we use the machinery of prong–markings and the framed change–of–coordinates principle to prove Corollary 1.4 (realization of arcs as saddles). As in Corollary 1.3, the idea is to use the framed change–of–coordinates principle to take a given arc to some target arc which is realized as a saddle connection. However, unlike cylinders, saddle connections do not share a common winding number. The main difficulty in the proof of Corollary 1.4 is therefore to construct a sufficiently large set of saddle connections to play the role of target arcs (Lemma 8.5).

We begin by clarifying some conventions with regards to arcs on surfaces with boundary versus surfaces with marked points. Recall that if $(\Sigma_{g,n},\phi)$ is a relatively framed surface, then we have fixed once and for all a legal basepoint on each boundary component and we only consider arcs ending on this prescribed basepoint. When $(\Sigma_{g,n},\phi)$ arises from the blow-up of a prong–marked abelian differential, this is equivalent to stipulating that arcs must leave and enter the zeros with prescribed tangent directions.

Upon capping each boundary component of $\Sigma_{g,n}$ with a punctured disk (equivalently, forgetting the prong structure coming from the differential), each (relative) isotopy class of arc $a$ on $\Sigma_{g,n}$ projects to an (absolute) isotopy class of arc on $\Sigma_{g}^n$, which we will denote by $\pi(a)$. Observe that the map $\pi$ is not injective; elements of its fibers are related by Dehn twists about the endpoint(s) of the arc.

Saddle connections on one–cylinder differentials. In order to exhibit the desired collection of saddles, it will be convenient to use different model surfaces than the ones introduced in Section 7.4.

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8In fact, the proof of Theorem 7.13 shows that the image of $C$ under the monodromy map is all of $\text{Mod}_{g,n}[\phi]$. 
To that end, recall that an abelian differential is called Jenkins–Strebel if it is completely horizontally periodic, i.e., if it can be written as the union of closed horizontal cylinders. We will in particular be interested in those which can be obtained by identifying boundary edges of a single cylinder, called one–cylinder Jenkins–Strebel differentials.

The existence result we will use is the following; see [Zor08, Section 2] for an explicit construction.

**Proposition 8.4.** There exists a one–cylinder Jenkins–Strebel differential \((Y, \eta)\) in every (nonempty) component of every stratum \(\Omega M_g(\kappa)\).

As in the proof of Lemma 7.17, we can also upgrade these \(\eta\) to yield one–cylinder Jenkins–Strebel differentials in each component of \(\Omega M_{g,\text{lab}}(\kappa)\) and \(\Omega M_{g,\text{pr}}(\kappa)\) by labeling zeros and prongs, respectively.

Using these new model differentials, we may now exhibit saddle connections which have (a preimage under \(\pi\) which has) arbitrary winding number.

![Figure 16. A one–cylinder Jenkins–Strebel differential in \(\Omega M_3(3, 1)\) and saddle connections on it. On the left the arcs have been realized geodesically; on the right they have been realized with prescribed tangential data.](image)

**Lemma 8.5.** Let \(H_{\text{pr}}\) be a component of \(\Omega M_{g,\text{pr}}(\kappa)\) and let \(p, q\) be two distinct zeros. Then for every \(s \in \mathbb{Z} + \frac{1}{2}\), there is a differential \((Y, \eta) \in H_{\text{pr}}\) and nonseparating arc \(a\) on \(Y^*\) from \(\Delta_p\) to \(\Delta_q\) (based at the legal basepoints prescribed by the prong–marking) such that

1. \(\phi(a) = s\) for the relative framing \(\phi\) induced by \(1/\eta\) and
2. the geodesic representative of \(\pi(a)\) on \((Y, Zeros(\eta))\) is a saddle connection.

**Proof.** As twisting around \(\Delta_q\) changes \(\phi(a)\) by \(\pm \phi(\Delta_q)\), it suffices to prove that there exist such arcs for each residue class mod \(\phi(\Delta_q)\).

Let \((Y, \eta)\) be a one–cylinder Jenkins–Strebel differential in \(H_{\text{pr}}\) (which exists by Proposition 8.4). Now by definition, we can write \(Y = C/\sim\), where \(C = S^1 \times [0, 1]\) is a closed cylinder and \(\sim\) identifies segments of the “top boundary” \(S^1 \times \{1\}\) with segments of its “bottom boundary” \(S^1 \times \{0\}\). Let \(Q : C \to Y\) denote the quotient map.

We observe that our choice of prong at \(p\) determines a unique half–separatrix on \((Y, \eta)\) and hence a pair of half–segments in \(\partial C\). In particular, the prong–marking determines a unique point \(\tilde{p} \in Q^{-1}(p) \cap (S^1 \times \{0\})\).

Now consider the set \(A\) of all arcs in \(C\) which start at \(\tilde{p}\) and end at a point of \(Q^{-1}(q) \cap S^1 \times \{1\}\); up to Dehn twisting along the core curve of \(C\), there are exactly \(\phi(\Delta_q)\) such arcs (see Figure 16).
Moreover, since the arcs of $\mathcal{A}$ are each realized as straight line segments on $C$, the arcs of $\pi(Q(\mathcal{A}))$ are all realized as saddle connections on $(Y, \eta)$. Isotoping these arcs to leave $p$ and enter $q$ with the prescribed tangential data, we may measure the winding numbers of $a \in Q(\mathcal{A})$. Careful inspection of Figure 16 shows that

$$\{\phi(a) \mod \phi(\Delta_q) : a \in Q(\mathcal{A})\} = \left\{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2\phi(\Delta_q) - 1}{2}\right\},$$

finishing the proof of the lemma.

Now that we have a sufficiently large collection of target saddle connections, we may apply the framed change–of–coordinates principle to deduce Corollary 1.4.

Proof of Corollary 1.4. Let $\bar{a}$ be an arc on $(X, \omega)$ with endpoints $p$ and $q$. Choose an arbitrary prong–marking of $(X, \omega)$ and an arc $a$ on $X^*$ such that $\pi(a) = \bar{a}$. Let $(Y, \eta)$ be the 1–cylinder Jenkins–Strebel differential from Lemma 8.5 in the same component of $\Omega \mathcal{M}^pr_g(\kappa)$ as $(X, \omega)$ and let $\phi$ denote the induced relative framing on $Y^*$.

Choose a path $\alpha$ connecting $(X, \omega)$ to $(Y, \eta)$ and let $\alpha_*(a)$ denote the parallel transport of $a$ along $\alpha$ (equivalently, lift $\alpha$ to a path in $\Omega \mathcal{M}^pr_g(\kappa)$ and use the marking maps). Then by Lemma 8.5 there is an arc $b$ on $Y^*$ with $\phi(b) = \phi(\alpha_*(a))$ and so that $\pi(b)$ is a saddle connection on $(Y, \eta)$.

Now by the framed change–of–coordinates principle (Proposition 2.14), there is an element of $\text{Mod}_{g,n}[\phi]$ taking $\alpha_*(a)$ to $b$. By Theorem 7.13, this element can be represented by loop $\beta$ in $\mathcal{H}^pr$. The concatenated path $\alpha \cdot \beta$ therefore takes $a$ on $X^*$ to $b$ on $Y^*$, and so its projection to $\Omega \mathcal{M}^pr_g(\kappa)$ is the desired path.

References


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