Invariants of Knots w/ a Flat Connection

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Kashaev-Reshetikhin: math/0202211
math/0202212
math/0410182

Morrison-S.: Soon

R.-S.: Spring '09

$$\begin{tikzcd}[column sep=huge] \text{Quantum Topology} \arrow[r, bend left, "\text{Volume Conjecture}"] & \text{Hyperbolic Geometry} \arrow[l, bend left, "???"] \end{tikzcd}$$

Hyperbolic Quantum Invariants

- Kashaev-Reshetikhin (analogue: R.-Turaev)
- Baseilhac-Benedetti (analogue: Turaev-Viro)

Hyperbolic Structures $\longleftrightarrow \mathrm{Hom}(\pi_1(K'), SL_2)/SL_2$

Review: Reshetikhin-Turaev

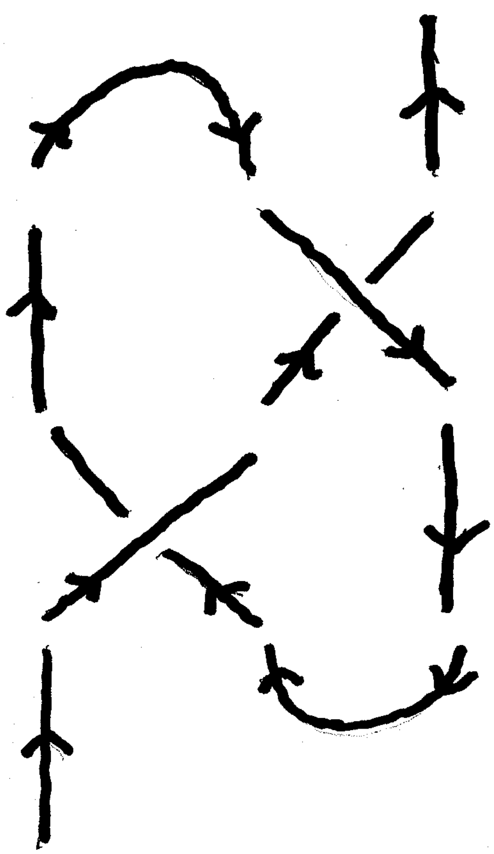
Fix: Quantum Group $U_q(\mathfrak{g})$

Representation V

Tangle $\rightsquigarrow$ Map of $\otimes$'s of V, V^*

Knot $\rightsquigarrow$ $\in \mathrm{End}(\mathbb{1}) = \mathbb{C}[q,q^{-1}]$

String Knot $\rightsquigarrow$ $\in \mathrm{End}(V) = \mathbb{C}[q,q^{-1}]$ (Schur)



$$\begin{array}{ll} V & \\ \uparrow & \mathrm{tr}\otimes\mathrm{id} \\ V\otimes V^*\otimes V & \\ \uparrow & \mathrm{id}\otimes\sigma_{V,V^*} \\ V\otimes V\otimes V^* & \\ \uparrow & \sigma_{V,V}\otimes\mathrm{id} \\ V\otimes V\otimes V^* & \\ \uparrow & \mathrm{id}\otimes\mathrm{coev} \\ V & \end{array}$$

Goal: Modify this for Kac-DeConcini $U_\zeta^{unr}(\underline{sl}_2)$

$\boxed{\text{Structure of } U_\zeta^{unr}(\underline{sl}_2)\text{:}}$

$$\begin{tikzcd}
\langle E^\ell, F^\ell, K^\ell \rangle = Z_0 \arrow[d, hook] & \mathrm{Spec}(Z_0) = PSL_2^* \\
\langle Z_0, C \rangle = Z \arrow[d, hook] & \mathrm{Spec}(Z) \arrow[u, "\substack{\text{Gen.} \\ \ell\text{-to-}1}"'] \\
U_\zeta^{unr}(\underline{sl}_2) & \text{rep-}U_\zeta^{unr}(\underline{sl}_2) \arrow[u, "\substack{\text{Gen.} \\ 1\text{-to-}1}"']
\end{tikzcd}$$

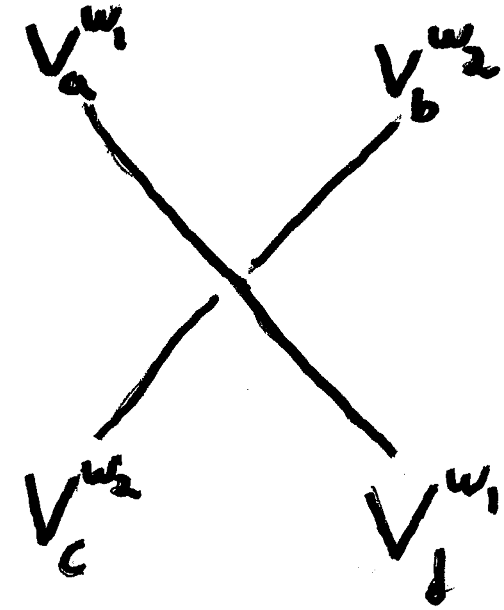
Where $PSL_2^* = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ c & a \end{pmatrix} \right\}$

The Braiding

$R \notin U_{\zeta}^{unr}(\underline{sl}_2)$

R is an <u>Outer</u> <u>Rational</u> automorphism

Induces maps of bimodules



Given by a "refactorization" formula of Weinstein–Xu

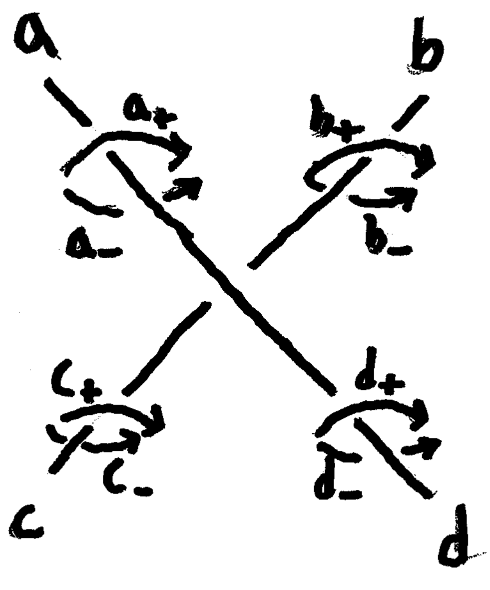
Refactorization

$$PSL_2^* \xrightarrow{\text{Varieties}} SL_2$$

$$(g_+, g_-) \longmapsto g_+ g_-^{-1}$$

Generically 1-to-1

Inverse called "factorization"



$$a_+ b_+ = c_+ d_+$$

$$a_- b_- = c_- d_-$$

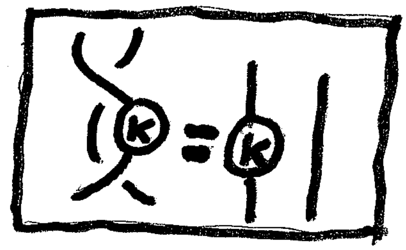
$$a_- b_+ = c_+ d_-$$

Agrees with braiding!

Correspondence between $\mathrm{Hom}(\pi_1(K'), SL_2)$ and $U_\zeta^{unr}(\mathfrak{sl}_2)$

$$\begin{array}{ccc} (U_\zeta^{unr}\text{-mod})\text{-labelings} & & \\ \updownarrow{\scriptstyle \text{birat.}} & & \\ \mathrm{Spec}(Z)\text{-labelings} & \subseteq & \mathrm{Hom}(\pi_1^{(\ell)}, SL_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(Z_0)\text{-labelings} & \subseteq & \mathrm{Hom}(\pi_1, SL_2) \end{array}$$

Invariant under conj.



KaRe: $\mathrm{Hom}(\pi_1^{(\ell)}(K'), SL_2)/\!/_{SL_2} \xrightarrow{\text{rat.}} \mathbb{C}$

Structure of $\mathrm{Hom}(\pi_1^{(\ell)}, SL_2)/\!/_{SL_2}$

$$
\begin{array}{ccc}
\mathrm{Hom}(\pi_1^{(\ell)}, SL_2)/\!/_{SL_2} & & \\
\cup\!| & & \\
\text{1-dim parts} & \xrightarrow{\text{peripheral}} & \mathbb{C}^* \times \mathbb{C}^* \\
\cup\!| & & \\
\text{Geometric part} & \xrightarrow{\text{periph.}} & \mathbb{C}^* \times \mathbb{C}^*
\end{array}
$$

"Geometric A_ℓ-curve" $\subseteq \mathbb{C}P^2$

"Geometric KaRe" is a rational function on it.

Example: The Trefoil

$$\pi_1(K') = \langle a, b \mid aba = bab \rangle$$

$$M = a \quad \underline{\text{and}} \quad L = a^{-3} b^2 a$$

A-poly: $(L-1)(1-LM^6)$

$\uparrow$ abelian $\quad \uparrow$ geom.

Geom: $a \mapsto \begin{pmatrix} M & \\ & M^{-1} \end{pmatrix}$

$$b \mapsto \begin{pmatrix} \frac{-M^{-2}}{M-M^{-1}} & 1 \\ -1-\frac{1}{(M-M^{-1})^2} & \frac{M^2}{M-M^{-1}} \end{pmatrix}$$

$$KaRe_3^{geom}(K) = 3\left(M^2 + 1 + M^{-2}\right)^2$$

A Mystery

Suppose $\mathrm{Hom}(\pi_1, SL_2)$

1) Doesn't factor through H_1

2) Meridian conjugate to $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

$\boxed{\text{Example: Fin. Vol. complete Hyp.}}$

Take $\ell = 3$.

$\mathrm{Hom}(\pi_1^{(3)}, SL_2) \rightsquigarrow$ Choose $\sqrt[3]{1}$

Mystery:

Choose $1 \rightsquigarrow$ Interesting

Choose ζ or $\zeta^2 \rightsquigarrow$ Zero

(checked for twist knots)