# Finite Quantum Groups

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# Outline

1. What is a “quantum group”?
2. Examples
3. Relationships to other subjects
4. Current research

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# Quantum Spaces

**Geometry can be encoded in algebra**

geometric space → algebra of functions

But not every algebra occurs as an algebra of functions on an honest space. For example, algebras of functions on honest spaces are:

- commutative
- nilpotent free

**Quantum spaces**

quantum space ← arbitrary algebra

(Examples: noncommutative geometry and schemes.)

# Quantum Groups

Let's play the same game with groups.

group ⟶ category of representations

Just as the collection of functions on a space has the structure of an algebra, the category of representations of a group has a certain structure called a "tensor category."

quantum group ⟵ arbitrary tensor category

**Warning:**

Most people use "quantum group" in a more specific context, namely for quantum analogues of Lie groups.

# Tensor categories

What structure does the category of representations $\mathrm{Rep}(G)$ have?

**Structures**

- Abelian category
- Tensor product of representations $V \otimes W$
- Dual representations $V^*$
- Trivial representation $\mathbf{1}$
- Associator $\omega_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$
- $u_X : \mathbf{1} \otimes X \to X$
- $ev_V : V^* \otimes V \to \mathbf{1}$
- etc.

These should satisfy some properties, for example you can associate $X \otimes Y \otimes Z \otimes W$ in different ways, you can look at $\mathbf{1} \otimes X \otimes Y$, etc.

# Finite quantum groups

Which tensor categories ($\otimes$, $*$, $\mathbf{1}$ as in last slide) look like the category of representations of a finite group?

**Definition**

*A fusion category is tensor category $\mathcal{C}$ which satisfies:*

- *All Hom spaces are finite dimensional.*
- *$\mathcal{C}$ is semisimple.*
- *$\mathcal{C}$ has finitely many simple objects, up to isomorphism.*

Fusion categories can be thought of as representations of finite quantum groups.

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# Basic examples

### Trivial example

The category of vector spaces $\mathrm{Vec}$. Only one simple object $\mathbf{1}$.

### Representations of $G$

The category of representations of a finite group $\mathrm{Rep}(G)$

### $G$-graded vector spaces

The category of $G$-graded vector spaces $\mathrm{Vec}_G$. Here the simple objects are indexed by elements of $G$ and the tensor product is given by the group structure.

# Twisting by 3-cocycles

One way that fusion categories get trickier is that a lot of information is encoded in the associator. For example, we can look at $G$-graded vector spaces with a nontrivial associator.

**Associator**

$$\omega_{\alpha,\beta,\gamma} : V_{\alpha\beta\gamma} = (V_\alpha \otimes V_\beta) \otimes V_\gamma \to V_\alpha \otimes (V_\beta \otimes V_\gamma) = V_{\alpha\beta\gamma}$$

assigns a scalar to every triple $(\alpha, \beta, \gamma)$.

- Compatibility $=$ $\omega$ is a 3-cocycle
- $\mathrm{Vec}(G, \omega)$ up to equivalence only depends on $\omega \in H^3(G, \mathbb{C}^*)$.

# Characterizing $\mathrm{Rep}(G)$

## Gelfand’s theorem

A $C^*$ algebra comes from an honest space iff it’s commutative.

## Other structures on $\mathrm{Rep}(G)$

- It is symmetric: $\sigma_{V,W} : V \otimes W \to W \otimes V$ and the $\sigma$ satisfy the relations of the symmetric group.
- It is pivotal $V \cong V^{**}$.
- It has a $C^*$ structure.

## Theorem (Deligne)

*A symmetric tensor category over $\mathbb{C}$ satisfying some mild growth conditions, is of the form $\mathrm{Rep}(G)$ for an algebraic <u>supergroup</u> $G$.*

Doplicher-Roberts have a similar result in the $C^*$ setting.

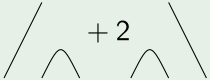
# Diagram categories

Where else do tensor categories come from?

## Diagram categories

Objects: Boundaries of diagrams
Morphisms: (Linear comb. of) diagrams with fixed boundaries
Composition: Vertical stacking
Tensor product: Horizontal disjoint union

## Temperley-Lieb



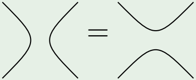
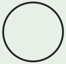
$$\in \{\mathrm{Hom}(\bullet^{\otimes 3} \to \bullet)\}$$

# Example

## $\mathrm{Rep}(\mathbb{Z}/2)$

There is a nice diagrammatic description of $\mathrm{Rep}(\mathbb{Z}/2)$, where we use $\bullet$ to denote the nontrivial irrep. Note that we have an isomorphism $\bullet \otimes \bullet \to \mathbf{1}$ which we can denote:

## Relations

- 
-  $= 1$

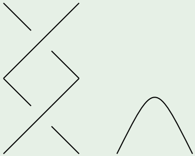
## Outline

# Knot polynomials and braidings

**Tensor category of tangles**

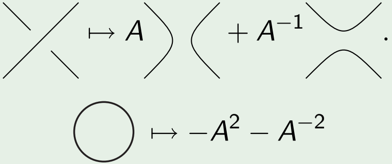
Objects: Finite collections of (oriented) points
Morphisms: (Oriented) tangles with fixed boundary
Composition: Vertical stacking
Tensor product: Horizontal disjoint union

**Example:**



# Knot polynomials

## Functor from framed tangles to Temperley Lieb



## Jones polynomial/Kauffman bracket

Consider a link, its image under this functor is a scalar. This gives an invariant of framed links.

# Braidings

## What does a crossing mean?

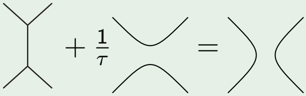
- Reinterpreting the last slide, we can say that you can interpret crossings inside the Temperley-Lieb category.
- Crossings give isomorphisms $S_{V,W} : V \otimes W \to W \otimes V$.
- This is similar to a symmetric structure, but it satisfies the relations of the braid group instead of the braid group. It's called a "braiding."

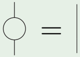
# An example

## The Fibonacci category

This fusion category has two simple objects $X$ and 1. Furthermore, $X \otimes X \cong X \oplus \mathbf{1}$.

## Relations

- 

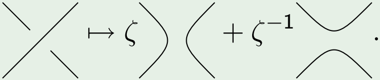
- 

-  $= \tau$

Where $\tau = \frac{1+\sqrt{5}}{2}$.

# Braiding on the Fibonacci category

## The braiding



Where $\zeta = e^{\frac{3}{10}2\pi i}$ is a certain 10th root of unity.

## Link invariant

Thus the Fibonacci fusion category gives the value of a certain specialization of the Jones polynomial.

## Aside:

Sometimes the same fusion category comes up in two different specializations of knot polynomials, this gives knot polynomial identities.

# Dimensions

Since every object in $\mathrm{Rep}(G)$ is a vector space, you can look at $\dim X$. But for a general fusion category the objects aren't vector spaces.

## Example: Fibonacci category

- $X \otimes X \cong X \oplus \mathbf{1}$
- $(\dim X)^2 = \dim X + 1$
- $\dim X = \frac{1 \pm \sqrt{5}}{2}$ (note that $\bigcirc = \frac{1+\sqrt{5}}{2}$)

## Frobenius-Perron dimension

There's a unique way to assign a positive real number to every object, called the Frobenius-Perron dimension.

# Aside: Physics

## Big Surprise

Fusion categories appear in real physical systems!

## Fractional Quantum Hall effect

- 2-dimensional systems.
- Gaps in electron fields behave like "particles."
- Low temp. & high magnetic field yield "topological phases."
- Particles correspond to objects, histories correspond to morphisms.

## Possible application:

Quantum computer which uses topology for error correction.

# Semisimple algebras

## Algebras without using elements

How much of algebra can we go through without ever referring to elements?

- $A$ is an object in $\mathrm{Vec}$.
- Multiplication is a morphism $A \otimes A \to A$
- The unit is a morphism $\mathbf{1} \to A$
- Associativity says that a certain diagram commutes.
- etc.

## Generalizing

We can talk about algebra objects in <u>any</u> tensor category.

# Example

## G-graded vector spaces

- Let $\mathcal{C} = \mathrm{Vec}_G$.
- Let $A$ be the object which is 1-dimensional in every grade.
- We can think of $A$ as the group algebra of $G$, as such it has a multiplication.
- This multiplication map plays well with gradings and so $A$ is an algebra object.

## This algebra is simple!

Note that there's no way to write this as a sum of smaller algebras. So semisimple algebras in other categories are much more interesting than those in $\mathrm{Vec}$. (At least over $\mathbb{C}$, over smaller fields you have field extensions and division rings.)

# Subfactors

## More examples of algebra objects

Suppose that $N \subset M$ is a finite index inclusion of $II_1$ factors, then:

- The category of finite index $N$-$N$ bimodules is a tensor category.
- $M$ is an algebra object inside that tensor category.
- In many cases, $M$ tensor generates a fusion category.

## Subfactors and quantum group actions

How do we understand subfactors?

- Understand all quantum groups.
- For each quantum group understand all ways that it can act on a given factor.

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# Topics of research

### My main topics of research

- Classification of small fusion categories and subfactors (see the rest of this talk)
- Arithmetic of fusion categories (see my talk tomorrow)
- 3-dimensional topology of fusion categories
- Families of tensor categories

### Two main cases

It turns out that fusion categories have a very different feel depending on whether:

- Dimensions of objects are rational integers
- Dimensions of objects are irrational

# Different ways of measuring size

## What does it mean to be small?

- Small rank (number of simple objects).
- Small global dimension (sums of the squares of the dimensions).
- One object of small dimension.

## Small rank

- If $\mathcal{C}$ has rank 1, then it's $\mathrm{Vec}$.
- If $\mathcal{C}$ has rank 2, then it's $\mathrm{Rep}(\mathbb{Z}/2)$ or the Fibonacci category (each comes in two flavors) [Ostrik02].
- There are only finitely many integral dimensional fusion categories of fixed rank [Etingof].
- This question in open in the irrational case.

# Categories of small global dimension

## Rational integer case

- Complete classification if $\dim \mathcal{C} = p$ or $\dim \mathcal{C} = p^2$. Well understood if $\dim \mathcal{C} = p^k$. [Ostrik-Nikshych-Etingof02]
- Complete classification if $\dim \mathcal{C} = pq$ [E.-Gelaki-O.03].
- Complete classification if $\dim \mathcal{C} = pq^2$ [Jordan-Larson09]
- Complete classification if $\dim \mathcal{C} = pqr$ [ENO08]

## Goal

Can you describe all integral fusion categories just using group theoretic data?

## Irrational case

Currently totally open.

# Subfactors of small index

These correspond correspond to small algebra objects in unitary tensor categories (called their “standard invariant”).

**Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)**

*The standard invariants with index less than 4 are given by an ADE classification:* $A_n, D_{2n}, E_6, \overline{E_6}, E_8, \overline{E_8}$

**Theorem**

*The standard invariants with index exactly 4 can be classified via group theory.*

# Index between 4 and 5

For index above 4, Temperley-Lieb gives a standard invariant for every value of the index. But once you ignore these, the possibilities are again quantized.

**Theorem (Haagerup94, Goodman-de la Harpe-J.89, Asaeda-H., Bisch98, A.-Yasuda09, Bigelow-Morrison-Peters-S.09, Calegari-M.-S.10, M.-S., M.-P.-Penneys-S., Izumi-J.-M.-S., P.-Tener)**

*There are exactly ten standard invariants other than Temperley-Lieb with index between* 4 *and* 5*.*

# Small objects in fusion categories

What if we just want to find a small object?

**Theorem**

*Let $X$ be an object in a fusion category whose FP dimension satisfies* $2 < \dim(X) \leq 76/33 = 2.30\ldots$ *then* $\dim(X)$ *is one of:*

$$\begin{aligned} \frac{\sqrt{7}+\sqrt{3}}{2} &= 2.188901059\ldots \\ \sqrt{5} &= 2.236067977\ldots \\ 1+2\cos(2\pi/7) &= 2.246979603\ldots \\ \frac{1+\sqrt{5}}{\sqrt{2}} = 2\cos(\pi/20)+2\cos(9\pi/20) &= 2.288245611\ldots \\ \frac{1+\sqrt{13}}{2} &= 2.302775637\ldots \end{aligned}$$

# What's the structure of these small fusion categories?

**Theorem (Morrison-S.10)**

*The Haagerup fusion category cannot be completely defined over a cyclotomic field.*

**Theorem (Grossman-S.11)**

*Classification of all "subgroups" of the Haagerup subfactor.*

Bigelow can construct explicit bases for the Hom spaces for most of these exceptional small fusion categories.