1 Introduction

Given an algebraic curve X one can consider the group of automorphisms $\operatorname{Aut}(X)$. One natural question to ask is which groups can come up this way. It turns out that in most situations (so long as the genus is at least 2), the group of automorphisms turns out to be finite. Furthermore, in characteristic zero, there are methods for determining precisely which finite groups can act on a curve of given genus g. In this paper, we give a survey of the important results concerning finite automorphism groups of curves and give some indication of the direction of current research. In particular, we will give an in depth treatment of the method of turning questions about automorphisms of curves over the complex numbers into questions about presentations of finite groups by generators and certain relations, discuss briefly the extension of some of these results to characteristic p, and give a complete classification of all finite groups which act on hyperelliptic curves of any genus and thus give a much quicker classification of all finite groups which act on curves of genus 2.

2 Hurwitz's Results

For the time being, we will delay any discussion of the full group of automorphisms, and ask two questions: Given a genus g which finite groups G can act as a subgroup of the group of automorphisms on a curve of genus g? Conversely, given a group G for which genera g do there exists curves of genus g on which G act as a group of automorphisms? Although these results are usually stated only for curves of genus $g \geq 2$ where the automorphism group is finite, we will try as much as possible to state results which hold uniformly for finite groups acting faithfully on a curve of any genus.

First we recall that the category of curves over the complex numbers is equivalent to two other categories: Riemann surfaces and fields of transcendence degree 1 over $\mathbb{C}$. Thus automorphisms of curves are equivalent to automorphisms of Riemann surfaces or automorphisms of finite field extensions of $\mathbb{C}(x)$ fixing $\mathbb{C}$. The last of these suggests another closely related question, what groups come up as Galois groups $\operatorname{Gal}(K/\mathbb{C}(x))$? This is equivalent to asking about automorphisms of curves or Riemann surfaces which preserve a fixed map to $\mathbb{P}^1$. Clearly any such group acts as a group of automorphisms on that Riemann surface. On the other hand, if G is a finite group acting on X, then X/G is some curve, and we can take any map from $X/G \to \mathbb{P}^1$ and get that G acts as a subgroup of a Galois group $\operatorname{Gal}(K/\mathbb{C}(x))$. Thus, to determine which finite groups appear as groups of automorphisms of curves, it is sufficient to answer the inverse Galois problem for $\mathbb{C}(x)$.

Theorem 2.1. *Any finite group G (of size N) is the Galois group $Gal(K/\mathbb{C}(x))$ for some Galois field extension. Furthermore, such extensions correspond precisely to presentations of G as generated by elements $g_1, \ldots, g_n$ of order $a_1, \ldots, a_n$ subject to the relations $g_1 g_2 \cdots g_n = 1$, and the numbers $a_1, \ldots a_n$ give the degrees of ramification for* *each of the precisely n points which ramify in this extension. Finally, the genus of K is*

$$g = 1 + \frac{N}{2}\left(-2 + \sum_{i=1}^{n} 1 - \frac{1}{a_i}\right).$$

Proof. First we reduce this to a problem in topology. Recall that a Galois extension $K/\mathbb{C}(x)$ of degree d with Galois group G corresponds to a Galois branched N-sheeted covering map of some Riemann surface X to $\mathbb{P}^1$ with group of deck transformations $\mathrm{Deck}(X/\mathbb{P}^1) = G$. Since any covering map from a topological space to a Riemann surface gives a unique complex structure on the covering space which preserves the map, this is in turn equivalent to an N-sheeted covering map of the topological space S^2 minus n points which is Galois in the sense that it acts transitively on fibers. By algebraic topology, these coverings correspond precisely to finite quotients of $\pi_1(S^2 - \{p_1, \ldots, p_n\})$. However, $\pi_1(S^2 - \{p_1, \ldots, p_n\})$ is exactly the group generated by $g_1, \ldots g_n$ subject to the relation $g_1 \cdots g_n = 1$. (For proofs of these results on Riemann surfaces, cf. [**?**, p. 10-39].)

Suppose that we realize G as a quotient of $\pi_1(S^2 - \{p_1, \ldots, p_n\})$ where g_i has order a_i. The normal subgroup which we are quotienting by is the fundamental group of the the covering space. Hence, if one wraps once around a point q_i in $f^{-1}(p_i)$, the image wraps around p_i a_i times. Thus, near the point q_i there are a_i different sheets. Therefore, the degree of ramification of p_i is a_i.

To find the genus of K we compute the Euler characteristics of the surface X. Consider $\mathbb{P}^1$ as the union of two polygons whose vertices are the points p_i and pull these back to a polygonal tiling of X. The Euler characteristic $\chi(X) = V - E + F$ where V is the number of vertices, E the number of edges, and F the number of faces. Since $X/\mathbb{P}^1$ is an N sheeted covering, clearly $F = 2N$. Next notice that each edge is contained in precisely two faces, and each face contains exactly n edges, so $E = nF/2 = Nn$. Lastly notice that, since the extension is Galois, the number of points above p_i is precisely N/a_i. Thus the total number of vertices is $\sum_i N/a_i$. Plugging this into the formula we get

$$2(1-g) = \chi(X) = \sum_i N/a_i - Nn + 2N = N(2\sum_i -1 + N/a_i).$$

Solving for g gives the last part of this result. □

The final computation of the genus from the ramification points is, of course, a special case of the celebrated Riemann-Hurwitz formula, which was proved by Hurwitz in his paper on automorphisms of Riemann surfaces [**Hu**].

The above method can be easily generated to automorphism groups of X whose quotient X/G has genus g'. Again we reduce the question to a problem in topology, namely, which finite groups appear as quotients of the fundamental group of a surface of genus g' minus n points. Now this fundamental group is generated by $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots \alpha_{g'}, \beta_{g'}, \gamma_1 \ldots \gamma_n$ subject to the relation $[\alpha_1, \beta_1] \cdots [\alpha_{g'}, \beta_{g'}]\gamma_1 \cdots \gamma_n = 1$. Again if we are looking at a quotient group of this where the image of γ_i has order a_i, then a_i is the order of ramification of the point p_i. This suggests the following definition.

Definition 2.2. *A generating vector $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots \alpha_{g'}, \beta_{g'}, \gamma_1 \ldots \gamma_n)$ for a group G is a collection of elements which generate G and satisfy the relation, $[\alpha_1, \beta_1] \cdots [\alpha_{g'}, \beta_{g'}]\gamma_1 \cdots \gamma_n =$*

1 *and each* g_i *has order* a_i. *A group with such a generating vector is called of* type $(g'; a_1, \dots a_n)$. *If* $g' = 0$ *we will leave it out. If a certain number is repeated several time among the a's, say a group of type* $(3, 3, 3)$ *we shall write this shorthand as* (3^3).

In this new language notice that the dihedral group with $2n$ elements, D_{2n}, is of type $(2, 2, n)$.

Finally, by the Riemann-Hurwitz formula, the genus of X satisfies

$$2(g-1) = N \cdot 2(g'-1) + \sum_i \frac{N}{a_i} 1 - a_i.$$

In conclusion,

Theorem 2.3. *If* G *is a finite group of size* N *which acts faithfully on a Riemann surface* X *of genus* $g \neq 1$, *then* $N = r(g-1)$ *where* r *is a rational number of the form*

$$2/\left(2(g'-1) + \sum_i 1 - \frac{1}{a_i}\right)$$

where g' *is a nonnegative integer, and all the* a_i *are positive integers. Furthermore,* G *acts on some Riemann surface of* g *precisely when* G *is a group of type* $(g'; a_1, \dots a_n)$ *satisfying the above equation.*

The spectrum of rational numbers of the form $2/(2(g'-1) + \sum_i 1 - \frac{1}{a_i})$ in the range 16 to 84 looks like:

Notice that large values are particularly sparse. In fact, if we want a positive large number of the form $2/(2(g'-1) + \sum_i 1 - \frac{1}{a_i})$ we need $g' = 0$ and $(-2 + \sum_i 1 - \frac{1}{a_i})$ to be very close to 0 while remaining positive. If $n \geq 4$, then the smallest possible value is $-2 + 1/2 + 1/2 + 1/2 + 2/3$ which gives $r = 6$. If $n = 3$ then we have zero for the triples $(2, 3, 6)$, $(2, 4, 4)$, and $(3, 3, 3)$. Thus the largest possible r must come from one of $(2, 3, 7)$, $(2, 4, 5)$, or $(3, 3, 4)$. A simple computation gives $r = 84$ in the first case and smaller values in the latter cases.

Theorem 2.4. *If* X *is a Riemann surface of genus* $g \geq 2$ *and* G *is a finite group of automorphisms acting faithfully on* X, *then* $|G| \leq 84(g-1)$ *with equality if and only if* $Aut(X)$ *is group of type* $(2, 3, 7)$. *(Such a group is called a Hurwitz group.)*

Notice that the above theorem is an oversimplification of the actual situation. It is not simply that $84(g-1)$ is the upper bound, if $N < 84(g-1)$ th next largest possibility is $48(g-1)$ and so on through the spectrum given in the above diagram.

As a further warning notice that although any Hurwitz group is generated by an element of order 2 and an element of order 3, it is perfectly possible for such groups to be terribly large. As we shall discuss in the last section there are several infinite families of Hurwitz groups, and even the Monster group is a Hurwitz group.

All of the above results were originally proved by Hurwitz [**Hu**]. For an classical presentation of these results in English see [**Bu**, Ch. XVIII].

3 Realizing Automorphism Groups as Quotients of Fuchsian Groups

Suppose that X is a Riemann surface on which a finite group G acts. X has a covering map to X/G branched at n points. This map comes from a covering map from the universal cover of $X/G - \{p_1, \ldots, p_n\}$ which we will call Y. Y is a simply connected Riemann surface. By the Riemann mapping theorem, any such surface is either the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or the upper halfplane $\mathbb{H}$ (cf. [**?**, p. 210]). Furthermore, it is easy to see that Y is $\mathbb{P}^1$ precisely when $X/G = \mathbb{P}^1$ and there are no branch points, that Y is $\mathbb{C}$ precisely when $X/G = \mathbb{P}_1$ and there is one or two branch points or when X/G is a torus and there are no branch points, and that Y must be $\mathbb{H}$ in all other cases.

First we deal with the exceptional cases. With the Riemann-Hurwitz formula it is easy to find which groups can occur in these special cases. Any un-branched cover of the Riemann sphere must be the trivial covering of the Riemann sphere by itself. There are no covers of the Riemann sphere branched over precisely one point. If the cover of the Riemann sphere is branched over precisely two points, by the arguments in the last section the Galois group of this covering (which is G) must be generated by two elements whose product is the identity. Thus, G would have to be cyclic. Finally, if X/G is a torus and X is an un-branched cover X must also be a torus. Then, by the above arguments, G would have to be a quotient of the fundamental group of the torus, that is it must be abelian and generated by at most two elements.

Outside of these special cases all other groups of automorphisms arrive from covering maps from the upper halfplane $\mathbb{H}$. So we have, for some discrete subgroups Γ and Γ' of $\mathrm{PSL}_2(\mathbb{R})$, $\mathbb{H}/\Gamma = X$ and $\mathbb{H}/\Gamma' = X/G$. Therefore, $G = \Gamma/\Gamma'$. Discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ with compact quotients are called Fuchsian groups. Thus we've seen that with very few particular exceptions, any automorphism group of any curve must be a quotient of Fuchsian groups.

For example, notice that $\mathrm{PSL}_2(\mathbb{Z})$ can be generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which have orders ∞ and 2 and whose product has order 3. Therefore, if we let G be the smallest normal subgroup containing $\begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}$, then any Hurwitz group is some quotient of $\mathrm{PSL}_2(\mathbb{Z})/G$. The most obvious normal subgroup containing G is the modular group $\Gamma(7)$ consisting of matrices congruent to the identity modulo 7. The quotient $\mathrm{PSL}_2(\mathbb{Z})/\Gamma(7) \cong \mathrm{PSL}_2(\mathbb{F}_7)$ is therefore a Hurwitz group. It has 168 elements and it can be realized as the automorphism group of $\mathbb{H}/\Gamma(7)$ which is the modular curve $X(7)$. Using techniques from modular forms one can see that this curve is the famous Klein quartic $x^3y + y^3z + z^3x = 0$.

4 Classifying All Finite Groups Acting on Curves of Genus 0

Theorem 4.1. *If G acts faithfully on a curve of genus 0, then G is isomorphic to one of: a cyclic group C_k, a dihedral group D_{2k} with $2k$ elements, A_4, S_4, or A_5.*

Proof. Since X has genus 0, by Riemann-Hurwitz, so must X/G. By earlier results, any such G must be a group of type $(a_1, \ldots a_n)$ where $\frac{-2}{|G|} = -2 + \sum_i 1 - \frac{1}{a_i}$. Clearly if $n \geq 4$ the right hand side is nonnegative. Take $(g_1, \ldots, g_n)$ a generating vector for G of this type.

If $n = 2$ the only possibilities are the cyclic groups (since G is generated by g_1), which do in fact occur as groups of type (n, n). If $n = 3$ the right hand side is 0 for the triples $(2, 3, 6)$, $(2, 4, 4)$, and $(3, 3, 3)$. Thus it is negative only for triples of the form $(2, 2, k)$ where $N = 2k$ or one of the triples $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$ where N is 12, 24, or 60 respectively.

If G with $2k$ elements is generated by g_1 and g_3 of orders 2 and k respectively, then $\langle g_3 \rangle$ has index 2 in G and thus is normal. So G is a semidirect product of a normal C_k acted on by C_2. Thus it must either be dihedral or abelian. If it were abelian than $g_2 = g_1^{-1} g_3^{-1}$ would have order $\operatorname{lcm}[2, k]$ not 2. Hence $G \cong D_{2k}$.

Now suppose G has 12 elements corresponding to the triple $(2, 3, 3)$. Suppose the Sylow 3-subgroup were normal. Then modding out by it we would get a group of order 4 which was generated by elements whose product was the identity and whose orders divided $(2, 3, 3)$. This is clearly impossible. Thus, by Sylow's theorem, there must be 4 Sylow 3-subgroups. G acts transitively on these by conjugation. Thus we have a map from G into S_4. The only transitive subgroups of S_4 are S_4, A_4, the subgroup generated by a 4-cycle, and the subgroup consisting of the identity and all the elements of order 2. The image cannot be S_4 since that is too large. If the image of G in S_4 were the last of these subgroups then its kernel would be a normal subgroup with 3 elements, which we know does not exist. If its image were the subgroup generated by a 4 cycle, then the cyclic group of order 4 would have to be generated by elements of orders dividing $(2, 3, 3)$ which is impossible. Therefore, we have an isomorphism $G \cong A_4$ by its action on Sylow 3-subgroups.

Similarly suppose G has 24 elements corresponding to the triple $(2, 3, 4)$. Again suppose the Sylow 3-subgroup were normal. Then there would be a group of 8 elements corresponding to a triple dividing $(2, 3, 4)$. Since 3 and 8 are relatively prime we would thus have to have that this group was cyclic and generated by an element of order dividing 4. Therefore there must be 4 Sylow 3-subgroups. Again G acts transitively on these elements giving a map from G to S_4. Again the image must be A_4 or S_4. If it were A_4 then G would have a normal subgroup with 2 elements. When we modded out by this group we would get that A_4 corresponded to something dividing $(2, 3, 4)$. But there are no elements of order 4 and so it would have to be generated by something dividing $(2, 3, 2)$ which would make it a quotient of D_6 which is too small. Therefore $G \cong S_4$.

Finally suppose that G has 60 elements corresponding to the triple $(2, 3, 5)$. I claim that G is simple. If this were not the case, then its quotient would correspond to a triple dividing $(2, 3, 5)$, but there are no such smaller groups (for example, because we've already classified all groups acting on a curve of genus 0). Now notice that G has either 1, 3, or 5 Sylow 2-groups on which it acts transitively. Since G is simple the first this subgroup cannot be normal. Thus we have a map without kernel from G to S_3 or S_5 with no kernel. Since G has 60 elements we must have $G \cong A_5$. $\square$

5 A Few Examples

When we look at covering maps corresponding to realizing G as a group of type (a_1, a_2, a_3) (which are the only cases where particularly large automorphism group relative to the genus) one can draw particularly nice pictures which realize the extension $X/\mathbb{P}^1$. Since the automorphism group of $\mathbb{P}^1$ acts 3-transitively, we can assume without loss of generality that the three branch points occur equally spaced on a great circle. Thus we can think of the projective line minus three points as two triangles, one light and one dark, as in the following picture:

In this context, X can be thought of as an equal number of light and dark triangles, alternating color, with labelled vertices. Furthermore, each element generator of the group of deck transformations corresponds to rotating through one light and one dark triangle around a vertex with that label. Thus each element of the group will correspond to one light triangle, and the rules for multiplication will tell us how to glue the edges together.

For example, consider the group S_3 which is a group of type $(2, 2, 3)$ with generating vector $((12), (13), (123))$. The picture of the Riemann surface with this presentation of this group as its group of automorphisms is (with edges on the outside connected by the arrows):

Any semi-direct product of two cyclic groups will have a particularly simple picture, like the one above. For example, consider $C_3 \rtimes C_4$ with generating vector $(g_1, g_2, g_1 g_2^3)$ of type $(3, 4, 4)$. This has the following picture (with outside edges glued to the other edge with the same capital letter, the orientation is determined by the labels of the vertices matching up):

The quaternions have generating vector $(i, j, -k)$ of type $(4, 4, 4)$. This results in the following genus 2 surface:

Burnside, [**Bu**, p. 396], gives the following picture of this surface as a closed genus 2 surface in 3-space:

Finally, the minimal Hurwitz group $\mathrm{PSL}_2(\mathbb{F}_7)$ has the following picture coming from a tiling of the hyperbolic plane by regular 7-gons (this picture is taken from [**G**, p.115]):

6 Aut(X) is Finite for $g \geq 2$.

Now we outline the proof that Aut(X) is Finite for $g \geq 2$. The original proof is due to Hurwitz and also appears in [**Hu**]. Our methods are taken from several exercises in [**Hart**, p.337+348].

The basic idea is that if we look at the canonical embedding of a curve in $\mathbb{P}^n$ for some n, we get a precise handle on the group of automorphisms, because it must be some subgroup of Aut($\mathbb{P}^n$). An element of Aut($\mathbb{P}^{g-1}$) is determined by where it sends $g+1$ points provided that they do not lie in a hyperplane. Thus, if we could find at least $g+1$ special points which do not lie on a hyperplane and which must be permuted by Aut(X), then we could conclude that Aut(X) is finite.

However, this approach will not work if X is hyperelliptic. But a similar approach will.

Proposition 6.1. *If X is hyperelliptic, then Aut(X) is finite.*

Proof. Since X is hyperelliptic it has a unique degree two map $f : X \to \mathbb{P}^1$ which is ramified above $2g+2$ points. Any automorphism of X must preserve the map f and permute the ramification points. So any automorphism is determined by its action on the fibers of f, that is it either preserves these two points or switches them. Since X minus the ramification points is still connected it must either act trivially on *all* the fibers or it must act non-trivially on *all* the fibers. Thus, modulo this cannonical involution, automorphisms of X are given by automorphisms of $\mathbb{P}^1$ which permute $2g+2 \geq 3$ points. Since an automorphism of $\mathbb{P}^1$ is determined by what it does to 3 points, there can only be finitely many such automorphisms. Thus Aut(X) is finite. □

Notice that the above proof shows something stronger, which we shall need later. If X is a hyperelliptic curve, then Aut(X) has a normal subgroup N with two elements, and Aut(X)/N is the Galois group of the map of Riemann surfaces $\mathbb{P}^1 \to \mathbb{P}^1$ which permutes some $2g+2$ points. Hence Aut(X)/N is one of the groups of genus 0, C_n, D_{2n}, A_4, S_4, or A_5. For fixed g some of these are impossible, for example C_n can only occur if $2g+2 > n$. Furthermore, if Aut(X) has a central involution z, then there is a degree 2 map $X \to X/\langle z \rangle$, if this latter space is $\mathbb{P}^1$, then conversely X must be hyperelliptic.

Now if X is not hyperelliptic we can consider the cannonical embedding and look at the hyperosculation points on X. These points are called Weierstrass points and there are $(g-1)^2 g + gd$ of them, where g is the genus and d is the degree of the cannonical embedding. Since $(g-1)^2 g + gd > d$ they cannot all lie on a single hyperplane. Since $(g-1)^2 g + gd > g-1$ the images of these points are preserved by automorphisms. Therefore we conclude,

Proposition 6.2. *If X is not hyperelliptic, elliptic, or rational, then Aut(X) is finite.*

(For the details of the definition of hyperosculation points and the method for counting them, see [**Hart**, p.337].)

It should be noted at this point that a generic curve of genus $g = 2$ has Aut(x) $= C_2$ and a generic curve of genus $g > 2$ has no nontrivial automorphisms [**Ba**].

7 Characteristic $p > 0$

All of the above results due to Hurwitz use the theory of Riemann surfaces and hence only work over the complex numbers (although by the Lefshetz principle they also work over any field of characteristic 0). The first clear direction to generalize the results of Hurwitz's paper is to try to adapt his results to algebraic curves in characteristic $p > 0$. The fact $\mathrm{Aut}(X)$ is finite, carries over easily into characteristic p. There are several proofs of this result, including [**I-T**].

However, Hurwitz's bound $|\mathrm{Aut}(X)| \leq 84(g-1)$ does not carry over exactly to characteristic p. The first result, due to Roquette [**R**], is that Hurwitz's bound works so long as $p > g + 1$ with a single exception $y^2 = x^p - x$ in characteristic $p = 2g + 1$ where $|\mathrm{Aut}(X)| = 2p(p^2 - 1) = 8(2g^3 + 3g^2 + g)$.

When p is small Singh [**Si**] was still able to get a bound

$$|\mathrm{Aut}(X)| \leq \frac{4pg}{p-1}\left(\frac{2g}{p-1} + 1\right)\left(\frac{4pg^2}{(p-1)^2} + 1\right).$$

There seem to be very few results on explicitly constructing curves with a particular group as an automorphism group in characteristic p. The methods in the complex case don't seem to apply.

8 Hurwitz Groups

Much of the modern research concerning automorphisms of curves is in finding Hurwitz groups. The first example is $\mathrm{PSL}_2(\mathbb{F}_7)$ for $g = 3$. Wiman proved in [**Wim**] that there are no other Hurwitz groups for $g \leq 6$. For $g = 7$ there is such a group, namely $\mathrm{PSL}_2(\mathbb{F}_8)$. This is discussed in [**MB2**] where Macbeath actually finds explicit polynomials defining the curve of genus 7 with this automorphism group. At the time, and to my knowledge currently, there are no other curves with maximal automorphism group whose explicit definition in terms of polynomials is known.

The first big breakthrough since Hurwitz was Macbeath's [**MB1**] in which he finds infinitely many Hurwitz groups. In particular, given one Hurwitz group he constructs an extension of C_n^{2g} by G which is also a Hurwitz group. In particular, using Klein's curve, there is a curve with maximal automorphism group of genus $2n^6 + 1$ for any integer n.

Since 2, 3, and 7 are distinct primes, any quotient of a Hurwitz group is trivial or another Hurwitz group. Hence a logical place to look for small Hurwitz groups is among the simple groups. There has been a lot of progress in this area, for example:

1. [**MB5**] $\mathrm{PSL}_2(\mathbb{F}_q)$ is a Hurwitz group exactly when $q = p$ and $q \equiv \pm 1 \pmod 7$ or when $q = p^3$ and $p \equiv \pm 2,\ \pm 3 \pmod 7$.
2. [**Coh**] $\mathrm{PSL}_3(\mathbb{F}_q)$ is not a Hurwitz group unless $q = 2$.
3. [**Con1**] A_n is a Hurwitz group for $n \geq 168$. Furthermore, for $n < 168$ Conder determined precisely which A_n are Hurwitz groups. The smallest is A_{15}.

4. [**Sa**] [**Sa**] The Ree groups $G_2^2(3^p)$ are all Hurwitz groups.

5. [**C-W-W**] [**Wil1**] [**Wil2**] [**Wil3**] The following 12 of the 26 sporadic groups are Hurwitz: $J_1, J_2, He, Co_3, Fi_{22}, Ly, J_4, Ru, HN, Th, Fi'_{24}, M$. The rest are not.

6. [**?**] [**?**] For $n \geq 287$ and for any prime power q, $\mathrm{SL}_n(q)$ (and thus its simple quotients) is Hurwitz. For $n \geq 371$ and for any prime power q, $\mathrm{Sp}_{2n}(q)$, $\Omega^-_{2n}(q)$, and $\mathrm{SU}_{2n}(q)$ (and thus all their simple quotients) are Hurwitz.

This information has been taken from Macbeath's survey article [**MB4**] and a comment in [**Con3**]. For another survey article see [**Con2**].

One fascinating recent result about Hurwitz groups is an estimate on their distribution. In [**La**] Larsen proves that the genera for which there exist Hurwitz groups are distributed roughly like the cubes in the sense that the sum $\sum_g g^{-s}$ converges exactly when $s > 1/3$.

An active area for further research is finding for each infinite family of simple groups, which are Hurwitz groups.

9 Several Other Directions of Recent Research

One can define $N(g)$ to be size of the largest group which can act on a curve of genus g. For example, $N(2) = 48$, but $N(3) = 168$. Hurwitz's result shows that $N(g) \leq 84(g-1)$ and Macbeath's results show that this bound is achieved infinitely often. However, one can ask the opposite question, how small can $N(g)$ be? Accola [**Ac**] and MacLachlan [**ML2**] showed independently that $N(g) \geq 8(g+1)$ and that this bound is achieved infinitely often. To see that $N(G) \geq 8(g+1)$ one can simply look at the group generated by g_1, and g_2 subject to the relations $g_1^4 = g_2^{2(g+1)} = (g_1 g_2)^2 = (g_1^{-1} g_2)^2 = 1$ of order $8(g+1)$. Clearly this group is of type $(4, 2g+2, 2)$ and thus acts on a curve of genus g' where $2(g'-1)/8(g+1) = 1 - \frac{1}{4} - \frac{1}{2(g+1)} - \frac{1}{2}$. Clearing denominators we see that, $(g'-1) = 4(g+1) - (g+1) - 2 - 2(g+1) = g-1$. Hence this group acts on a curve of genus g. Showing that $N(G) = 8(g+1)$ infinitely often is quite a bit more difficult.

One other direction of research is to find out which groups can act as the full automorphism groups of a curve of genus g. The key techniques here are to find Fuchsian groups Γ_1 and Γ_2 such that Γ_2 is a maximal normal subgroup of Γ_1. Several sources for results in this direction are [**B-C**] and [**V-M**].

One can then turn to computing how large the genus can be for groups of a certain special sort. This has been done for cyclic groups [**Harv**], soluble groups ([**MB4**] cites Accola for this, but does not provide a reference), supersoluble groups [**G-M**], and nilpotent groups [**Z**].

Rather than looking simply for the minimal genus on which a group acts one can try to find all possible g for which a given group G can act on some surface of genus g. For example, suppose $G = C_p$ the cyclic group with p elements, p a prime. Any generating vector consists solely of elements of order p. Clearly C_p can be realized as a group of type $(g'; p^k)$ for any g' and any $k \geq 2$, and in no other ways. Hence C_p can only act on curves of genus g where $2(g-1)/p = 2(g'-1) + k - k/p$. Hence,

$g = pg' + (k-1)(p-1)/2$. Since p and $(p-1)/2$ are relatively prime, we see that C_p acts on all curves of genus at least $p(p-1)/2$. [**Harv**] and [**Ll**] give a similar classification for any cyclic group. A particularly fascinating result due to Kulkarni [**K**] shows that for any group the spectrum of possible genera are eventually periodic with some period dividing the order of the group.

10 Classifying All Groups Acting on Curves of Fixed Small Genus g

Finally we consider the question of finding all groups which can act on a curve of fixed small genus g. We have already answered this question for genus 0. [**Bu**] already has this full solution for genus $g = 0$ and $g = 1$. Macbeath (in [**MB4**]) explains that this classification problem has important applications to the study of Teichmüller spaces: "For every g we have a *Teichmüller space* $\mathbb{T}_g$ of "marked Riemann surfaces" analogous to the modular figure in genus 1. Topologically $\mathbb{T}_g$ is a Euclidean space of $6g-6$ real dimensions. The *mapping class group* M_g is a discontinuous group acting on $\mathbb{T}_g$. The quotient spaces $\mathbb{T}_g/M_g$ is the space R_g of all closed Riemann surfaces of genus g. The quotient mapping $\mathbb{T}_g \to R_g$ is a branched covering and the points where ramification occurs are the Riemann surfaces with nontrivial automorphisms. To understand this situation it is necessary to get some understanding of the whole set of groups involved as well as the dimensions of the subspaces of $\mathbf{T}_g$ consisting of the fixed point sets for each group.

The classification of all groups acting on curves of genus 2 was done in [**Wim**]. This classification for genus 3 was done by Maclachlan in his thesis which is unpublished. Broughton extends these results in [**Bro**] to classifying not just the possible groups, but also the possible actions up to topological equivalence on curves of genus 2 and 3. In [**MB3**] Macbeath explains an algorithm which answers this question for a small genus provided that one knows all finite simple groups of order not exceeding $84(g-1)$. In [**Bre**], the author claims to have found the list of all such groups for $g \leq 48$, however, according the the mathscinet review, "the author claims that the database of automorphism groups of Riemann surfaces of genera ranging in $2 \leq g \leq 48$ is available (the reader actually can find in the book only information about the number of such actions) from him, but no email address or concrete website is announced in the book;"

The general method in Broughton's work is to begin by finding all possible cyclic groups which can act on a curve of genus 2 using the Riemann-Hurwitz formula. From this list and the Riemann-Hurwitz formula it is not difficult to find all possible types and sizes of groups which can act on a curve of genus 2. He then begins with the smallest groups and proceeds through the larger ones using induction together with the fact that quotients and subgroups of any later group must already be on his list. These methods are purely group theoretic. With a simple geometric observation we can cut down significantly the amount of work needed to make this list. Recall that any curve of genus 2 is hyperelliptic. Also, the full automorphism groups of a hyperelliptic curve is precisely all possible central extension of C_2 by a group of genus 0.

Theorem 10.1. *If X is a hyperelliptic curve, then $Aut(X)$ must be one of the following groups (n is always an odd number):*

1. C_{2n}
2. $C_{2^{k+1}\cdot n}$ *or* $C_2 \times C_{2^k \cdot n}$ *for* $k \geq 1$.
3. $C_n \rtimes (C_2 \times C_2 \times C_2)$ *where the generators of the* C_2*'s act by multiplication by* 1, 1, *and* -1 *respectively.*
4. $C_n \rtimes Q_8$ *where* i *and* j *act by multiplication by* -1.
5. $C_n \rtimes D_8$ *where a simple rotation acts by* -1 *and the quotient group of reflections acts trivially.*
6. $C_n \rtimes D_8$ *where a simple rotation acts by* 1 *and the quotient group of reflections acts non-trivially.*
7. $C_n \rtimes (C_4 \times C_2)$ *where the generator of* C_4 *acts by* -1 *and the generator of* C_2 *acts by* 1.
8. $C_n \rtimes (C_4 \times C_2)$ *where the generator of* C_4 *acts by* 1 *and the generator of* C_2 *acts by* -1.
9. $D_{2^{k+1}n} \times C_2$ *for* $k \geq 2$.
10. $D_{2^{k+2}n}$ *for* $k \geq 2$.
11. $C_{2^{k+1}n} \rtimes C_2$ *where the generator of* C_2 *acts by multiplication by* $2^k - 1$ *on* $C_{2^{k+1}}$ *and by* -1 *on* C_n *for* $k \geq 2$.
12. $C_{2^k n} \rtimes C_4$ *where the generator of* C_4 *acts by multiplication by* -1 *for* $k \geq 2$.
13. $C_n \rtimes A_{2^{k+2}}$ *where* $A_{2^{k+2}}$ *is the group generated by* i *and* j *where* $i^{2^k} = j^2 = -1$ *and* -1 *is central of order* 2 *and* $ji = i^{-1}j$ *and* i *acts on* C_n *trivially and* j *acts by multiplication by* -1 *for* $k \geq 2$.
14. $C_n \rtimes B_{2^{k+2}}$ *where* $A_{2^{k+2}}$ *is the group generated by* i *and* j *where* $i^{2^k} = j^2 = -1$ *and* -1 *is central of order* 2 *and* $ji = -i^{-1}j$ *and* i *acts on* C_n *trivially and* j *acts by multiplication by* -1 *for* $k \geq 2$.
15. $SL_2(\mathbb{F}_3)$ *or* $PSL_2(\mathbb{F}_3) \times C_2 \cong A_4 \times C_2$.
16. $SL_2(\mathbb{Z}/4) \cong GL_2(\mathbb{F}_3)$ *or* $PSL_2(\mathbb{Z}/4) \times C_2 \cong PGL_2(\mathbb{F}_3) \times C_2 \cong S_4 \times C_2$.
17. $SL_2(\mathbb{F}_5)$ *or* $PSL_2(\mathbb{F}_5) \times C_2 \cong A_5 \times C_2$.

Proof. This theorem is simply a lot of somewhat tedious group theory. We need to classify all central extensions of C_2 by cyclic groups, dihedral groups, or one of the three groups of symmetries of the platonic solids. We call two such extensions $0 \to C_2 \to G \to G' \to 0$ equivalent, when there is a map from G to G making the two exact sequences commute. Thus the number of equivalent extensions with the same group G is the the number of automorphisms of G' divided by the size of the image of the automorphisms in G.

First we notice that, so long as our group has a normal subgroup of size relatively prime to 2 (which happens outside of the the three exceptional cases) we can reduce to finding the central extensions by C_2 of the Sylow 2-subgroup. To see this suppose $N \lhd G$ with G/N a 2-group and N a group of order prime to 2 and let G' be any group with a central subgroup $\langle z\rangle$ of order 2 such that $G'/\langle z\rangle = G$. Since everything in G' is a product of z and elements in G' and z is central, N must be normal in all of G'. Furthermore, since the G'_2, the Sylow 2-group of G', is disjoint from N, G' is a semidirect product $N\rangle G'_2$. G'_2 is clearly a central extension of C_2 by G_2. Furthermore, N must act trivially on z and we know its action on $G_2/\langle z\rangle = G'_2$. Hence, once we know all of the central extensions of C_2 by G'_2 we can find all of the central extensions of C_2 by G by simply taking the only possible semi-direct product. Notice that although each equivalence class of extensions of G'_2 gives a unique equivalence class of extensions of G'_2, two different extensions which give isomorphic groups at the level of G'_2 might not give isomorphic groups at the level of G', since the semi-direct product structure will be different.

Thus we can restrict to looking at cyclic groups of order a power of two, dihedral groups of order a power of two, and the three exceptional cases. There are two different ways to proceed at this point. One way is by brute force. For example, any central extension of C_2 by $D_{2^{k+1}}$ must have 2^{k+1} elements and must be in one of the following three cases: the generators of D_{2^k} still have orders 2^{k+1} and 2, the generators of D_{2^k} have orders 2^{k+2} and 2, the generators of D_{2^k} still have orders 2^{k+1} and 4, or the generators of D_{2^k} still have orders 2^{k+2} and 4. A bit of work shows that these cases correspond to cases 9, cases 10 and 11, case 12, and cases 13 and 14, respectively. Some tedious group theory along the above lines easily allows one to answer all but the exceptional cases. A_4 and S_4 have a normal $C_2 \times C_2$ and this allows one to see that their only C_2 extensions are cases 15 and 16 respectively. Finding all C_2 extensions of A_5 by hand is significantly more difficult.

On the other hand, we can simply use cohomology. The C_2 central extensions of a group G' up to equivalence are given by the elements of the second cohomology group $H^2(G', \mathbb{Z}/2)$. Thus if we compute these cohomology groups for the groups G' in question and then write down a large enough list of extensions we know that they must be all of them. The only book which actually explains how to compute such cohomologies is [**Ad**], which I requested a week ago and the library still has not returned to me. I will include this method as soon as possible. □

Now we can apply this last theorem to the special case of curves of genus 2. Since such a curve has 6 ramification points, the genus 0 quotient group must be C_n or D_2n for $n \leq 6$ or A_4 or S_4 (but not S_5 because the icosahedron has too many vertices). Using the above classification this gives us a finite list of candidates for possible G. Furthermore, any such G could only have a few possible types. If some point ramifies of degree k in X, then it ramifies of degree k or $k/2$ in X/C_2. Since we know the ramification information for the genus zero curve X/C_2, there are only finitely many candidates for the possible types of each G, and these can be easily checked.

In particular, the only possibilities for the full automorphism group of a curve of genus 2 are listed below:

G'	type of G'	size of G	type of G	G
C_1	(1)	2	(2^6)	C_2
C_2	(2^2)	4	(2^5)	$C_2 \times C_2$
C_2	(2^2)	4	$(2^2, 4^2)$	C_4
C_3	(3^2)	6	$(3, 6^2)$	C_6
C_3	(3^2)	6	$(2^3, 6)$	C_6
C_3	(3^2)	6	$(2^2, 3^2)$	C_6
C_4	(4^2)	8	$(2, 8^2)$	C_8
C_4	(4^2)	8	(4^3)	None
C_5	(5^2)	10	$(2, 5, 10)$	C_{10}
C_6	(6^2)	12	$(2, 6^2)$	$C_6 \times C_2$
$D_4 \cong C_2 \times C_2$	(2^3)	8	(4^3)	Q_8
$D_4 \cong C_2 \times C_2$	(2^3)	8	$(2^3, 4)$	D_8
D_6	$(2^2, 3)$	12	$(2, 6^2)$	None
D_6	$(2^2, 3)$	12	$(3, 4^2)$	$C_3 \rtimes C_4$
D_6	$(2^2, 3)$	12	$(2^3, 3)$	D_12
D_8	$(2^2, 4)$	16	$(2, 4, 8)$	$C_8 \rtimes C_2$ acting by multiplication by 3.
D_12	$(2^2, 6)$	24	$(2, 4, 6)$	$C_3 \rtimes (C_2 \times C_2 \times C_2)$ by -1, 1 and 1
A_4	$(2, 3, 3)$	24	$(3, 3, 4)$	$\mathrm{SL}_2(\mathbb{F}_3)$
S_4	$(2, 3, 4)$	48	$(2, 3, 8)$	$\mathrm{GL}_2(\mathbb{F}_3)$

This list agrees with the list in Table 4 of [**Bro**] if one keeps in mind that we have written down just the groups which can appear as the full group of automorphisms, not just some subgroup of that. For example, if C_{10} can be the full group of automorphisms, certainly C_5 can act as a group of automorphisms.

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