Problem Set 1

Noah Snyder

Due on Sept. 15th

1. The center of a group $G$ is denoted $Z(G)$, it is defined by:

$$Z(G) = \{ g \in G : gh = hg \text{ for any } h \in G \}.$$  

Prove that $Z(G)$ is a normal subgroup of $G$.

2. Suppose that $p$ is a prime number and that $G$ is a finite group whose order is a power of $p$ (in other words, suppose that $G$ is a $p$-group), and suppose that $G$ acts on a finite set $S$. Let $\Sigma \subset S$ denote the subset of points fixed by every element of $G$. Show that $\#\Sigma \equiv \#S \mod p$.

3. If $G$ is a $p$-group show that $Z(G)$ is nontrivial. (Hint: find an action of $G$ on itself whose fixed points are $Z(G)$.)

4. Suppose that the quotient group $G/Z(G)$ is cyclic, prove that $G/Z(G) = 1$.

5. The commutator subgroup of $G$ is denoted $[G,G]$ and is generated by elements of the form $xyx^{-1}y^{-1}$ (that is $[G,G]$ consists of all products of elements of that form). Show that $[G,G]$ is a normal subgroup.


7. Suppose that $G$ is a non-abelian group of order $p^3$ for some prime $p$. Find all 1-dimensional representations of $G$. (Hint: show that $[G,G] = Z(G)$ and that that subgroup has order $p$.)

8. Consider the set $V$ of all functions from the set $\mathbb{Z}/p$ to $\mathbb{C}$. Prove that this forms a vector space, and find the dimension of this vector space.

9. Let $V$ be the vector space defined in the last problem. We define an inner product on $V$ by

$$\langle f_1, f_2 \rangle = \frac{1}{p} \sum_{g \in \mathbb{Z}/p} f_1(g)\overline{f_2(g)}.$$  

Show that with respect to this inner product the 1-dimensional representations of $\mathbb{Z}/p$ are an orthonormal basis.