Problem Set 2

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1. Suppose that $\pi \in \text{End}(V)$ satisfies $\pi^2 = \pi$. Prove that $V \cong \text{im}\pi \oplus \ker\pi$.

2. Suppose that $V$ is an $n$-dimensional vector space over the field $\mathbb{Z}/p$. Find the number of $k$-dimensional subspaces of $V$.

3. Suppose that $G$ is a finite group, that $g \in Z(G)$, and that $V$ is an irreducible representation over $\mathbb{C}$. Prove that $g$ acts on $V$ by multiplying by a scalar. Let $\chi_V(g)$ denote the scalar by which $g \in Z(G)$ acts on $V$, show that this “central character” gives a group homomorphism $Z(G) \to \mathbb{C}^\times$.

4. Let $Q$ be the 8-element quaternion group (it’s elements are $\pm 1$, $\pm i$, $\pm j$, and $\pm k$ with the usual quaternion rules for multiplication). Find an irreducible 2-dimensional representation of $Q$ and prove that it is irreducible. Find its central character.

5. Consider the 2-dimensional representation of $\mathbb{Z}/3$ over $\mathbb{R}$ given by

$$\rho(g^k) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^k.$$ 

Show that it is irreducible. Find a non-scalar 2-by-2 matrix which commutes with the action of every element of $\mathbb{Z}/3$.

6. Let $V$ be the 2-dimensional representation of $\mathbb{Z}/3$ over $\mathbb{R}$ defined in the last problem. Find an isomorphism between $\text{End}_G(V)$ and one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

7. Consider the 6 dimensional regular representation of the symmetric group $S_3$. Find all its irreducible subrepresentations.