How to recognize $U_q(\mathfrak{g}_2)$ or S_t.

Noah Snyder
http://math.columbia.edu/∼nsnyder

Columbia University

Quantum Invariants of 3-manifolds and Modular Categories, Macalester, April 2010

Outline

Motivating question

Question:

What is the "simplest" modular category which doesn't come from a quantum group?

"simplest?"

We could mean a lot of different things by simplest:

- Small rank.
- Small global dimension.
- An object of small dimension.
- An object whose skein theory is relatively simple, i.e. $X^{\otimes 2}$ is not too complicated.

In this talk we will consider the last notion of simple.

Recognition theorems

Theorem (Kauffman, Kazhdan, Jones, Tuba, Wenzl...)

Suppose that X is simple object in a unimodal ribbon category $\mathcal{C}$.

if $X \otimes X \cong$	then $\mathcal{C}$ takes a braided functor from
A	cyclic group category
$1 \oplus A$	Temperley-Lieb category
$A \oplus B$	HOMFLY skein category
$1 \oplus A \oplus B$	Kauffman or Dubrovnik skein categories

Furthermore, the eigenvalues of the braiding and the dimension of X determine the parameters in the right column.

For all the arguments in one place see Knot polynomial identities and quantum group coincidences (Morrison-Peters-Snyder).

Why might you care about recognition theorems?

Corollary

If X is an object in a unimodal ribbon category

if $X \otimes X \cong$	*the knot invariant is a specialization of*
A	$(\pm 1)^{\# L}$
$1 \oplus A$	*the Jones polynomial*
$A \oplus B$	*the HOMFLY polynomial*
$1 \oplus A \oplus B$	*Kauffman or Dubrovnik polynomials*

Corollary

The pseudo-unitary modular categories which are tensor generated by an object X which breaks up as above can be classified.

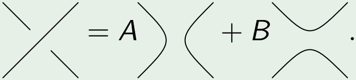
Corollary

Wenzl-style Grothendieck reconstruction theorems.

How would you prove such a result?

Proof.

These results follow from standard skein theory arguments. For example suppose $X \otimes X \cong 1 \oplus A$. Since End $(X \otimes X)$ is 2-dimensional there must be a linear dependence of the form



Following Kauffman, rotate this equation, glue them together and apply Reidemeister 2 to see that $B = A^{-1}$ and $A^2 + A^{-2} = \dim V$. Hence there's a braided functor from Temperley-Lieb.

The other cases are only a little more complicated. □

Outline

What about $U_q(\mathfrak{g}_2)$

How does $X \otimes X$ break up?

If X is the fundamental representation of $U_q(\mathfrak{g}_2)$ then

$$X \otimes X \cong 1 \oplus X \oplus A \oplus B.$$

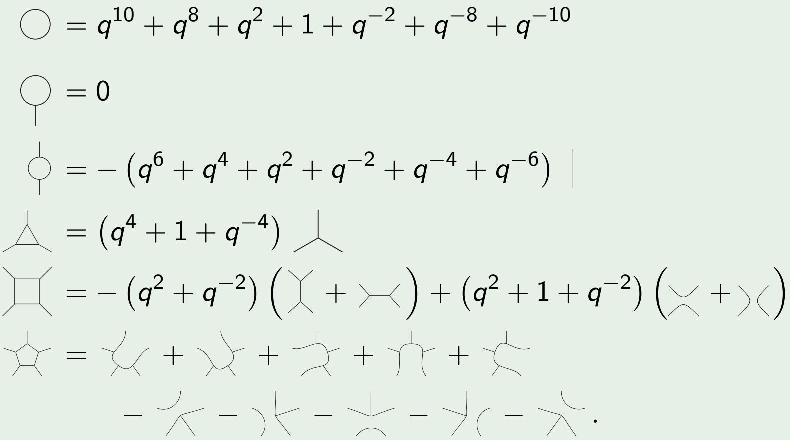
Is $U_q(\mathfrak{g}_2)$ the only such braided category?

No! The same decomposition holds for the fundamental representation of the symmetric group S_n, or even for Deligne's S_t where t is not an integer. Are there any others?

The G_2 spider

Theorem (Kuperberg)

For q generic, $U_q(\mathfrak{g}_2)$ has the following presentation,



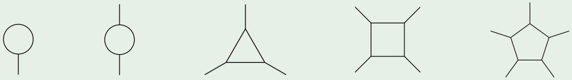
Kuperberg's recognition theorem

Definition

- *A trivalent vertex is a rotationally symmetric map* $X \otimes X \to X$.
- *A tree is a trivalent graph without cycles (but allowing disjoint components).*

Theorem (Kuperberg)

Suppose we have a trivalent vertex in a semisimple pivotal tensor category $\mathcal{C}$*, such that the graphs*



are each equal to linear combinations of trees. Then there is a pivotal functor $Rep(U_q(\mathfrak{g}_2)) \to \mathcal{C}$ *for some* $q \in \mathbb{C}$.

What if $\mathcal{C}$ is braided?

Our assumption and goal

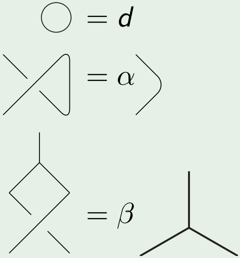
Assume that $\mathcal{C}$ is braided and that $X \otimes X \cong 1 \oplus X \oplus A \oplus B$. (For simplicity, we'll also occasionally assume that $\mathcal{C}$ has no zero-dimensional objects although this is probably not necessary. We want to show that $\mathcal{C}$ takes a functor from $U_q(\mathfrak{g}_2) - mod$ or $S_t - mod$.

We will split into three cases:

- The $U_q(\mathfrak{g}_2)$ case: The four trees with 4 boundary points are linearly independent
- The $U_q(\mathrm{SO}_3)$ case: The trees span a 3-dimensional space which contains the crossing.
- The S_t case: The trees together with the crossing span the 4-box space.

Some conventions

We fix the following conventions:



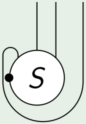
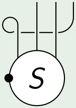
Trivalent vertex

Theorem

Under our assumptions, the (unique up to scaling) nonzero map $X \otimes X \to X$ is a trivalent vertex.

Proof.

Notice that,

$\zeta S =$  $=$  $= (\alpha\beta^2)^{-1} S.$

Similarly, by pulling the string over, we see $\zeta = \alpha\beta^2$. Hence ζ is a cube root that is its own inverse, and hence is 1. □

The $U_q(\mathfrak{g}_2)$ case

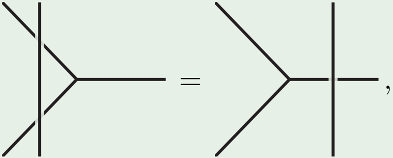
Small polygon bursting

In the $U_q(\mathfrak{g}_2)$ case, a counting argument shows that bigons, triangles, and squares burst. So all that remains is pentagons.

Theorem (Pentagon bursting)

In the $U_q(\mathfrak{g}_2)$ case there is always a pentagon-bursting relation.

Proof.



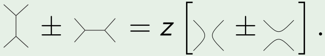
Expand the crossings and there will be exactly one pentagon. □

Outline

The $U_q(\mathrm{SO}_3)$ case

$I = H$ relation

We must have a relation between the four trees with two boundary components. This must be of the form:



Lemma

The sign above is negative.

Proof.

Evaluate the triangular prism directly and using the $I = H$ relation. This forces d to be the golden ratio which contradicts the dimensions are nonzero assumption. □

This category must be $\mathcal{D}_6$

Theorem

In the $U_q(\mathfrak{so}_3)$ case the category tensor generated by X must be $\mathcal{D}_6$

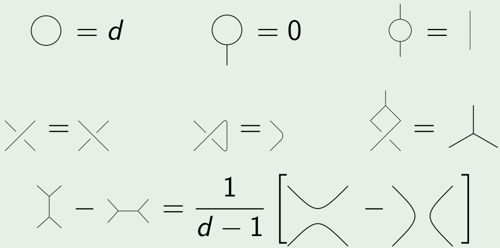
Proof.

The subcategory generated by the trivalent vertex is $U_q(\mathfrak{so}_3)$. Since the crossing is in the span of the trees, it follows that the braiding is the usual braiding on $U_q(\mathfrak{so}_3)$. So we're looking for a braided "quotient" of $U_q(\mathfrak{so}_3)$ gotten by adding a new 4-box. Since boxes must pull through twists we see that q is a particular root of unity. No zero dimensional objects then tells you that we're actually looking for a "quotient" of the semisimplified category. This must then be a de-equivariantization, and the only one that works is $\mathcal{D}_6$. □

What is S_t?

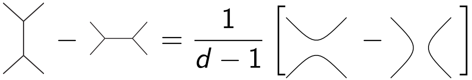
Definition

S_{d+1} is generated by a trivalent vertex and a crossing satisfying the following relations.

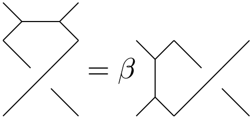


The S_t case

As before we must have



Using this relation we simplify both sides of the following equation:

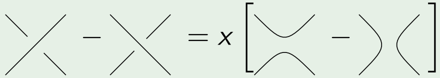


The resulting equality implies that in the S_t-case the vertex twist factor $\beta = 1$, and hence the twist factor $\alpha = 1$.

Rotational eigenvalues

There must be some linear dependence between the crossings and the trees. Since $\mathbb{Z}/2$ acts on this space by rotation, this relation has rotational eigenvalue ± 1.

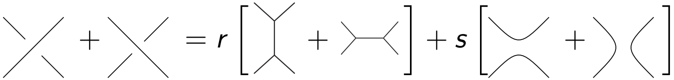
The -1 eigenvalue case



Using the fact that $\beta = 1$ we see that $x = 0$. Hence $\mathcal{C}$ takes a functor from $S_t - mod$.

Eigenvalue $+1$

If the rotational eigenvalue of the relation is $+1$ then the relation is of the following form:



Using the fact that $\alpha = \beta = 1$ we get two equations. Simplifying the tetrahedron gives another equation. However, these three equations in three unknowns have a family of solutions!

$$r = \frac{d(d-1)}{d^2 - d - 1} \quad \text{and} \quad s = \frac{d-2}{d^2 - d - 1}.$$

Question

Are these relations consistent? Is this a new braided tensor category?

Summary

If $\mathcal{C}$ is a ribbon category with no zero-dimensional objects which is generated by X such that

$$X \otimes X \cong 1 \oplus X \oplus A \oplus B$$

with A, B, and X distinct simple objects, then either:

- $\mathcal{C} \cong U_q(\mathfrak{g}_2) - mod$, semisimplified if q is a root of unity. (Assuming spider = algebraic at roots of unity.)
- $\mathcal{D}_6$.
- $\mathcal{C}$ is a "4-transitive subgroup of S_t." When n is an integer these are S_n, A_n, and the Mathieu groups (warning: uses classification of finite simple groups!).
- Possibly some other examples coming from the exceptional family above (if it exists).