1. Which of the following are true and which are false (2 pts. each):

(a) If $G$ is any group (finite or infinite) and $V$ is an irreducible representation, then $V$ is indecomposable.

**Answer:** True, if $V$ has no nontrivial proper subrepresentations then it certainly can’t be the direct sum of two nontrivial proper subreps.

(b) If $G$ is any group (finite or infinite) and $V$ is an indecomposable representation, then $V$ is irreducible.

**Answer:** False, if $G$ is the integers, and $V$ is the 2 dimensional representation where the generator of $\mathbb{Z}$ acts by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $V$ is indecomposable (because it only has one nontrivial proper subrep and so can’t be the direct sum of nontrivial proper subreps) but $V$ is not irreducible (because it has a nontrivial proper subrep, namely the 1-dimensional subspace of eigenvectors).

(c) If $G$ is a finite group and $V$ is an irreducible representation, then $V$ is indecomposable.

**Answer:** True, by part (a).

(d) If $G$ is a finite group and $V$ is an indecomposable representation, then $V$ is irreducible.

**Answer:** True, this is a big theorem that we proved twice.
2. Consider the group $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. Let’s write this group multiplicatively, so a typical element is $g^x h^y$ where $x \in \mathbb{Z}/4$ and $y \in \mathbb{Z}/4$ and multiplication is $(g^x h^y)(g^{x'} h^{y'}) = g^{x+x'} h^{y+y'}$. A representation of $G$ is determined by the actions of $g$ and $h$, but not every way of assigning maps to $g$ and $h$ individually extends to a valid representation of $G$. Which of the following define valid representations of $G$? (2 pts. each):

**Answer:** In order to check whether such an assignment gives a representation we need to check that this really gives an action of the group, that is we need to make sure that all the relations in $G$ hold for the corresponding linear maps. In this case we need to check that $g^4$ acts by $1$, that $h^4$ acts by $1$, and that the actions of $g$ and $h$ commute.

(a) The vector space is $\mathbb{C}$, and $g$ acts by (1) while $h$ acts by $(i)$.

**Answer:** Yes. Check that $1^4 = 1$, that $i^4 = 1$, and that the two matrices commute.

(b) The vector space is $\mathbb{C}^2$, and $g$ acts by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ while $h$ acts by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

**Answer:** No, these two matrices do not commute.

(c) The vector space is $\mathbb{C}^2$, and $g$ acts by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ while $h$ acts by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

**Answer:** Yes. Both matrices square to the identity, so their fourth powers are also the identity, furthermore any matrix commutes with itself.

(d) The vector space is $\mathbb{C}^2$, and $g$ acts by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ while $h$ acts by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

**Answer:** No. The first matrix raised to the fourth is $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, not the identity.

3. Suppose that $G$ is a group and that $V$ and $W$ are representations of $G$, what does it mean to say that a linear map $f : V \to W$ is a map of representations? (3 pts.)

**Answer:** That $f$ commutes with the action of $g$, namely $gf(v) = f(gv)$.

4. If $V$ is a vector space what’s the definition of the dual space $V^*$? (3 pts.)

**Answer:** $V^*$ is the space of all linear maps from $V$ to $\mathbb{C}$.

5. If $V$ and $W$ are vector spaces and $A$ is a linear map $V \to W$, what’s the definition of $A^* : W^* \to V^*$? (3 pts.)

**Answer:** If $f \in W^*$ then $A^*$ is given by $(A^* f)(v) = f(Av)$.

6. If $V$ is a representation of $G$ what is the definition of the dual representation $V^{**}$? (3 pts.)

**Answer:** $g$ acts on $V^*$ by $(g^*)^{-1}$. That is $(gf)(v) = f(g^{-1}v)$. 


7. Let $P_7$ be the 7-dimensional permutation representation of the group $S_7$. Compute the value of the character $\chi_{P_7}((123)(456))$. (3 pts.)

**Answer:** $\chi_{P_7}((123)(456)) = 1$. Compute the trace in the obvious basis. Since $\sigma e_i = e_{\sigma i}$ the only elements on the diagonal of the matrix correspond to points fixed under the action of $\sigma$. Hence the trace of the action of $\sigma$ is just the number of fixed points of $\sigma$. In this particular case there's one fixed point, namely 7.

8. Let $A_4$ be the alternating group on 4 letters. I’ve written in part of its character table, complete the rest and give a short explanation of your reasoning. (9 pts.)

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<td>[e]</td>
<td>[(12)(34)]</td>
<td>[(123)]</td>
<td>[(132)]</td>
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<td>-1</td>
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**Answer:** To find $\chi_3$ either take the tensor square of $\chi_2$ or take the dual of $\chi_2$. Then you can compute $\chi_4$ using character orthogonality.

Another valid technique is to find $\chi_4$ by starting with the permutation representation and subtracting off a copy of the trivial, then you can find $\chi_3$ either as above or by character orthogonality.

9. Let $V$ be a representation of $G$ and let $\mathbf{1}$ be the trivial representation of $G$, prove that $\mathbf{1} \otimes V \cong V$ as representations of $G$. (Hint: first show that there’s a unique map of vector spaces $\mathbb{C} \otimes V \rightarrow V$ sending $1 \otimes v \rightarrow v$, and then check that this is a map of representations.)

**Answer:** Consider the map $\mathbb{C} \times V \rightarrow V$ given by $(c, v) \mapsto cv$. This map is bilinear because $V$ is a vector space. Thus, by the universal property of tensor product, there exists a unique map $\mathbb{C} \otimes V \rightarrow V$ sending $c \otimes v \mapsto cv$. Now we need to check that this gives a map of representations. Well, on the left hand side $g$ sends $\sum c_i \otimes v_i$ to $\sum gc_i \otimes gv_i = \sum c_i \otimes gv_i$ (since we’re looking at the trivial representation). On the right hand side $g$ sends $\sum c_i v_i$ to $\sum g(c_i v_i) = \sum c_i g(v_i)$ by linearity of the action.

10. If $G$ is a finite group and $V$ is a representation of $G$, prove that $\pi : \frac{1}{|G|} \sum_{g \in G} g : V \rightarrow V$ is projection onto the $G$-invariant subspace $V^G$.

**Answer:** First we claim that the image of $\pi$ is contained in $V^G$ and that $\pi$ restricted to $V^G$ is the identity.

**Answer:** First we claim that the image of $\pi$ is contained in $V^G$. To see this note that $\sigma \frac{1}{|G|} \sum_{g \in G} gv = \frac{1}{|G|} \sum_{g \in G} \sigma gv = \frac{1}{|G|} \sum_{g \in G} gv$ (since as $g$ runs over all of $g$ so does $\sigma g$ since multiplication by $\sigma$ is invertible).
Second we claim that $\pi$ restricted to $V^G$ is the identity. To see this just note that $\frac{1}{\# G} \sum_{g \in G} gv = \frac{1}{\# G} \sum_{g \in G} v = v$ for $v \in V^G$.

Third we claim that the image of $\pi$ is exactly $V^G$, to see this note that the image certainly contains $V^G$ since it’s the identity on $V^G$.

Finally, we note that $\pi$ is a projection since $\pi^2 v = \pi(\pi v) = \pi v$ where the second equality holds because $\pi v \in V^G$ and $\pi$ acts by the identity on $V^G$.

11. If $V$ is an irreducible representation of $G$ and $f : V \to V$ is a map of representations, show that there exists some scalar $\lambda$ such that $f(v) = \lambda v$ for all $v \in V$.

**Answer:** Consider the action of $f$ on $V$, since we’re over $\mathbb{C}$ there’s an eigenvector for $f$, and hence there’s a nontrivial eigenspace $V_\lambda$ for the action of $f$. We claim that this eigenspace is all of $V$, by irreducibility it suffices to show that it is a subrepresentation. So we compute: $f(gv) = gf(v) = g\lambda v = \lambda gv$, hence if $v \in V_\lambda$ so is $gv$. Thus $V_\lambda = V$ and $f$ acts on $V$ by the scalar $\lambda$. 