Homework # 2: A Special Case of $L(1, \chi) \neq 0$.

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In this problem set you will work through Dirichlet's proof that $L(1, \chi) \neq 0$ for $\chi = \left(\frac{\cdot}{p}\right)$ where p is a prime number.

1. Use our formula for Γ to prove, $\Gamma(s) = n^s x^{n-1} \int_0^1 (\log(\frac{1}{x}))^{s-1} dx$. Conclude that,
$$L\left(s, \left(\frac{\cdot}{p}\right)\right) \Gamma(s) = -\int_0^1 \frac{f(x)}{x^p - 1} \left(\log\left(\frac{1}{x}\right)\right)^{s-1} dx,$$
for some polynomial f. Find f.

2. Let $\zeta_p = e^{\frac{2\pi i}{p}}$ be a primitive pth root of unity. Show that,
$$\frac{f(x)}{x^p - 1} = \sum_{a=0}^{p-1} \zeta_p^a f(\zeta_p^a) \frac{1}{x - \zeta_p^a}.$$
Therefore,
$$L\left(1, \left(\frac{\cdot}{p}\right)\right) = -\frac{1}{p} \sum_{a=0}^{p-1} \zeta_p^a f(\zeta_p^a) \int_0^1 \frac{dx}{x - \zeta_p^a}.$$

3. Compute $\int_0^1 \frac{dx}{x - \zeta_p^a}$. Plug this into your equation for $L\left(1, \left(\frac{\cdot}{p}\right)\right)$.

4. Notice that $\zeta_p^a f(\zeta_p^a) = \sum_{m=1}^{p-1} \left(\frac{m}{p}\right) \zeta_p^{am}$, which is called the Gauss sum, g_a. Show that $g_a \neq 0$. (In fact one can show, $g_a = \left(\frac{a}{p}\right) g_1$; and $g_1^2 = \left(\frac{-1}{p}\right) p$. To prove the latter statement, consider the sum $\sum_a g_a g_{-a}$ in two different ways.)

5. Prove
$$L\left(1, \left(\frac{\cdot}{p}\right)\right) = -\frac{g_1}{p} \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \left(\log\left(2 \sin \frac{a\pi}{p}\right) + \frac{\pi a i}{p}\right).$$

6. Suppose that $p \equiv 3 \pmod 4$. Show that $L\left(1, \left(\frac{\cdot}{p}\right)\right) \neq 0$.

7. Suppose $p \equiv 1 \pmod 4$. Show that
$$L\left(1, \left(\frac{\cdot}{p}\right)\right) = -\frac{g_1}{p} \log \frac{\prod_{a=\square} \sin \frac{a\pi}{p}}{\prod_{a\neq\square} \sin \frac{a\pi}{p}}.$$
Show
$$\frac{\prod_{a=\square} \sin \frac{a\pi}{p}}{\prod_{a\neq\square} \sin \frac{a\pi}{p}} = \frac{\prod_{a=\square}(1 - \zeta_p^a)}{\prod_{a\neq\square}(1 - \zeta_p^a)} = \frac{A}{B}.$$
Notice that A and B live in the field $\mathbb{Q}(\zeta_p)$. Using Galois theory, show that we must have $A = x + y\sqrt{p}$ and $B = x - y\sqrt{p}$ with x, y rational. Notice that since $AB = p$, $x^2 - py^2 = p$. Clearly this implies that $y \neq 0$. Conclude that $L\left(1, \left(\frac{\cdot}{p}\right)\right) \neq 0$. (Also notice that we can show the negative Pell's equation $a^2 - pb^2 = -1$ has an integer solution when p is a prime 1 modulo 4.)