Lecture # 4: The Analytic Continuation and Functional Equation of Riemann's Zeta Function.

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As we mentioned before, by getting more information about the ζ-function we can recover more information about prime numbers. In this lecture we will explain how to extend the ζ-function to the entire complex plane, look at some of its basic properties there, and discuss how Riemann outlined using these properties to get good information about the prime numbers.

1 Continuing ζ to the Line $\mathrm{Re}(s) = 0$.

Theorem 1.1. *The ζ-function can be meromorphically continued to the right halfplane $Re(s) > 0$ with a simple pole of order 1 at $s = 1$ and no others.*

Proof. Let $\zeta_2(s) = -1^{-s} + 2^{-3} - 3^{-s} + 4^{-s} - \ldots$. By our earlier result, this converges for $\mathrm{Re}(s) > 0$. Notice that

$$\zeta(s) + \zeta_2(s) = 2\sum_{2|n} n^{-s} = 2\frac{\frac{1}{1-2^{-s}} - 1}{\frac{1}{1-2^{-s}}}\zeta(s).$$

Therefore, $\zeta(s) + \zeta_2(s) = 2^{1-s}\zeta(s)$. Hence, $\zeta(s) = \frac{1}{2^{1-s}-1}\zeta_2(s)$. The righthand side makes sense for any $\mathrm{Re}(s) > 0$ except for possible simple poles at $s = 1 + 2\pi i \log_2(n)$ for $n \in \mathbb{Z}$.

But we can go through a similar argument for 3. That is, let $\zeta_3(s) = -1^{-s} - 2^{-s} + 2 \cdot 3^{-s} - 4^{-s} - \ldots$. Again this converges for $\mathrm{Re}(s) > 0$. Furthermore,

$$\zeta(s) + \zeta_3(s) = 3\frac{\frac{1}{1-3^{-s}} - 1}{\frac{1}{1-3^{-s}}}\zeta(s).$$

Therefore, $\zeta(s) = \frac{1}{3^{1-s}-1}\zeta_3(s)$. This expression makes sense for any $\mathrm{Re}(s) > 0$ except for possible simple poles at $s = 1 + 2\pi i \log_3(n)$ for $n \in \mathbb{Z}$.

Combining these two results, we've shown that $\zeta(s)$ can be continued to $\mathrm{Re}(s) > 0$ with poles only at numbers both of the form $s = 1 + 2\pi i \log_3(n)$ and $s = 1 + 2\pi i \log_2(m)$. Thus we need to find any integers m and n such that $\log_3(n) = \log_2(m)$. That is to say, $2^n = 3^m$. By unique factorization, this only happens when $n = m = 0$. Thus the only pole is at $s = 1$. $\square$

There is another proof of this result.

Proof. Notice that $\zeta(s) = \int_{1^-}^{\infty} x^{-s} d\lfloor x \rfloor$. Integrating by parts, we see $\zeta(s) = s\int_{1^-}^{\infty} x^{-s-1}\lfloor x \rfloor dx$. Notice that $\lfloor x \rfloor = x - \{x\}$ where $\{x\}$ denotes the fractional part. Therefore,

$$\zeta(s) = s\int_{1^-}^{\infty} x^{-s}dx - s\int_{1^-}^{\infty} x^{-s-1}\lfloor x \rfloor dx = -1 + \frac{1}{1-s} - s\int_{1^-}^{\infty} x^{-s-1}\lfloor x \rfloor dx.$$

Notice that this last integral is bounded by $\int_{1^-}^{\infty} x^{-s-1}dx$, and so converges for any $\mathrm{Re}(s) > 0$. Thus we have a formula which agrees with ζ and makes sense on the halfplane $\mathrm{Re}(s) > 0$ except for a simple pole at $s = 1$. $\square$

2 Continuing Beyond Re$(s) = 0$ and the Functional Equation

Theorem 2.1 (Functional Equation of the ζ-function). *The ζ-function can be meromorphically continued to the entire complex plane with a single pole at $s = 1$. Furthermore, this function satisfies the functional equation*

$$\Gamma(s)\pi^{-s}\zeta(2s) = \Gamma\left(\frac{1}{2} - s\right)\pi^{-\frac{1}{2}+s}\zeta(1-2s).$$

Proof. Take the definition of the Gamma function and make the change of variables $x \to n^2\pi x$ to get

$$n^{-2s}\pi^{-s}\Gamma(s) = \int_0^\infty e^{-n^2\pi x}x^s\frac{dx}{x}.$$

Again we sum over all n and use uniform convergence to interchange sum and integral to get

$$\zeta(2s)\pi^{-s}\Gamma(s) = \int_0^\infty \frac{1}{2}(\theta(ix) - 1)x^s\frac{dx}{x}.$$

Now we can apply the functional equation of the θ function to the right hand side:

$$\begin{aligned}
\zeta(2s)\pi^{-s}\Gamma(s) &= \int_0^\infty \frac{1}{2}(\theta(ix) - 1)x^s\frac{dx}{x} = \int_0^1 \frac{1}{2}(\theta(ix) - 1)x^s\frac{dx}{x} + \int_1^\infty \frac{1}{2}(\theta(ix) - 1)x^s\frac{dx}{x} \\
&= \int_1^\infty \frac{1}{2}(\theta(-1/ix) - 1)x^{-s}\frac{dx}{x} + \int_1^\infty \frac{1}{2}(\theta(ix) - 1)x^s\frac{dx}{x} \\
&= \int_1^\infty \frac{1}{2}(x^{\frac{1}{2}}\theta(ix) - 1)x^{-s}\frac{dx}{x} + \int_1^\infty \frac{1}{2}(\theta(ix) - 1)x^s\frac{dx}{x} \\
&= \int_1^\infty \frac{1}{2}(\theta(ix) - 1)(x^{\frac{1}{2}-s} + x^s)\frac{dx}{x} + \int_1^\infty \frac{1}{2}(-x^{\frac{1}{2}-s} - x^{-s})\frac{dx}{x} \\
&= -\frac{1}{2(\frac{1}{2} - s)} - \frac{1}{2s} + \int_1^\infty \frac{1}{2}(\theta(ix) - 1)(x^{\frac{1}{2}-s} + x^s)\frac{dx}{x}.
\end{aligned}$$

Notice that the right hand side is defined and analytic for all of $\mathbb{C}$ except for simple poles at $s = 0$ and $s = \frac{1}{2}$. Thus we have given another analytic continuation of ζ to the complex plane except for $s = 1$ (the pole at 0 coming from the Γ factor). But more importantly, this formula is clearly symmetric under the change of variables $s \to \frac{1}{2} - s$ and our theorem is proved. Making the change of variables back from $2s \to s$, we see that the completed ζ-function $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is symmetric about the line Re$(s) = 1/2$ and only has poles at $s = 0$ and $s = 1$. □

This proof is Riemann's second proof of the analytic continuation and functional equation of the ζ-function. Week 3's homework will work through his first proof.

Definition 2.2. *Let* $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Notice that $\xi(s)$ is holomorphic on the entire complex plane, furthermore, it satisfies the functional equation $\xi(s) = \xi(1-2)$. Often it will be more useful to consider this function than the original ζ-function.

3 The Zeroes of the Zeta Function

Proposition 3.1. $\zeta(s) \neq 0$ *for any Re*$(s) > 1$.

Proof. Since Re$(s) > 1$ we can use the Euler factorization $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$. We need only show that $\log\zeta(s)$ is finite. But, $\log\zeta(s) = \sum_p \sum_{k=1}^\infty \frac{1}{k}p^{-ks}$, and this last sum is clearly bounded. □

Therefore, by the functional equation, the only zeroes of the ζ function with $\mathrm{Re}(s) < 0$ are those which come from the poles of the Γ function. Thus outside the strip $0 \leq \mathrm{Re}(s) \leq 1$ the only zeroes of the ζ-function lie at $s = 2, 4, \ldots$ and these are simple zeroes of order 1. These are called the "trivial zeroes."

Of the remaining zeroes, Riemann remarked, that it was likely that they all lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This is the celebrated Riemann Hypothesis, which remains open to this day (with a million dollar bounty on its head).

4 A Brief Tangent: Möbius Inversion

Definition 4.1. *If f and g are functions from the natural numbers to say $\mathbb{C}$, let $f \star g(n) = \sum_{d|n} f(d)g(\frac{n}{d})$.*

Recall that $f(s,a)f(s,b) = f(s, a \star b)$. Thus $\star$ must be associative and commutative. Furthermore, since the function 1 is the identity under multiplication, the sequence $\varepsilon(n)$ which is 1 if $n = 1$ and zero otherwise is the multiplicative identity. One can easily show by hand that a sequence has a $\star$ inverse exactly when $a_1 \neq 0$. In particular the function $1(n) = 1$ has a star inverse which we will call μ. Notice that the property defining μ is that $\sum_{d|n} \mu(d) = \varepsilon(n)$. By hand one can compute that $\mu(n) = (-1)^{\text{number of prime factors}}$ if n is square free, and $\mu(n) = 0$ otherwise.

Theorem 4.2 (Möbius Inversion). *If $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(n/d)g(d)$.*

Proof. We are given that $g \star 1 = f$. Therefore, $g \star 1 \star \mu = f \star \mu$. But $1 \star \mu = \varepsilon$ the identity, thus $g = f \star \mu$ which is exactly what we are trying to prove. □

There are several other versions of Möbius inversion which we will be using.

Corollary 4.3. *If $f(x) = \sum_{n=1}^{\infty} g(x/n)$ then $g(x) = \sum_{n=1}^{\infty} \mu(n) f(x/n)$.*

Proof. First plug in the formula for $f(x)$ to see,

$$\sum_{n=1}^{\infty} \mu(n) f(x/n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(n) g(x/mn).$$

Now let $\ell = mn$ and $d = m$. Thus,

$$\sum_{n=1}^{\infty} \mu(n) f(x/n) = \sum_{\ell=1}^{\infty} \sum_{d|\ell} \mu(d) g(x/\ell) = \sum_{\ell=1}^{\infty} g(x/\ell) \sum_{d|\ell} \mu(d) = g(x).$$

□

5 Riemann's Argument

Riemann used his analytically continued ζ-function to sketch an argument which would give an actual formula for $\pi(x)$ and suggest how to prove the prime number theorem. This argument is highly unrigorous at points, but it is crucial to understanding the development of the rest of the theory.

Notice that $\log \zeta(s) = \sum_p \sum_n \frac{1}{n} p^{-ns}$ for $\mathrm{Re}(s) > 1$. Letting $J(x)$ be the number of prime powers less than x, notice that $\log \zeta(s) = \int_0^{\infty} x^{-s} dJ(x)$ again for $\mathrm{Re}(s) > 1$. Now use integration by parts to get

$$\log \zeta(s) = s \int_0^{\infty} J(x) x^{-s-1} dx.$$

Now this is a Mellin transform, so, assuming some technical results, we should be able to use Melin inversion. Thus,

$$J(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log \zeta(s)}{s} x^s ds.$$

This converges when $\sigma > 1$.

Thus in order to find a formula for $J(x)$ we need only get a better formula for $\log\zeta(s)$.

Riemann claimed that $\xi(s) = \xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right)$, where the product is taken over all roots of the ξ function (that is, over all nontrivial zeroes of the ζ-function). This product does not converge absolutely, and we should pair any terms with $im(\rho)$ positive with a corresponding term with negative imaginary part to get a convergent product. The proof of this product formula basically depends on getting nice bounds on the growth of the number of zeroes.

Now we notice that,

$$\zeta(s) = 2\frac{1}{s(s-1)}\pi^{s/2}\frac{1}{\Gamma(s/2)}\xi(s) = 2\frac{1}{s(s-1)}\pi^{s/2}\frac{1}{\Gamma(s/2)}\xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right).$$

Therefore,

$$\log\zeta(s) = \log 2 - \log s - \log(s-1) + \frac{s}{2}\log\pi - \log\Gamma(s/2) + \log\xi(0) + \sum_\rho \log\left(1-\frac{s}{\rho}\right).$$

We want to substitute this into our integral formula and evaluate termwise, however doing so would lead to divergent integrals (for example in the $\frac{s}{2}\log\pi$ term). Thus Riemann first integrated by parts to get,

$$J(x) = -\frac{1}{2\pi i}\cdot\frac{1}{\log x}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{d}{ds}\left(\frac{\log\zeta(s)}{s}\right)x^s ds.$$

Now we can substitute our formula for $\zeta(s)$ and evaluate term by term. With a good bit of work, Riemann evaluated these integrals and got the formula,

$$J(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) + \int_x^\infty \frac{1}{t(t^2-1)\log t}dt - \log 2.$$

Notice that $J(x) = \sum_{n=1}^\infty \frac{1}{n}\pi(x^{\frac{1}{n}})$. We can invert this formula to get, $\pi(x) = \sum_{n=1}^\infty \mu(n)\frac{1}{n}J(x^{1/n})$.

This gives us a formula for $\pi(x)$. Its dominant term is $\sum_{n=1}^\infty \mu(n)\frac{1}{n}\mathrm{Li}(x^{1/n})$. This would show the prime number theorem if we could actually prove that this term was dominant. The key to proving this is to show that the $\sum_\rho \mathrm{Li}(x^\rho)$ terms are each smaller, that is to say we need to show that $\mathrm{Re}(\rho) < 1$.