Lectures # 5 and 6: The Prime Number Theorem.

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1 Riemann's Argument

Riemann used his analytically continued ζ-function to sketch an argument which would give an actual formula for $\pi(x)$ and suggest how to prove the prime number theorem. This argument is highly unrigorous at points, but it is crucial to understanding the development of the rest of the theory.

Notice that $\log \zeta(s) = \sum_p \sum_n \frac{1}{n} p^{-ns}$ for $\mathrm{Re}(s) > 1$. Letting $J(x) = \sum_{p^k \le x} \frac{1}{k}$, notice that $\log \zeta(s) = \int_0^\infty x^{-s} dJ(x)$ again for $\mathrm{Re}(s) > 1$. Now use integration by parts to get

$$\log \zeta(s) = s \int_0^\infty J(x) x^{-s-1} dx.$$

Now this is a Mellin transform, so, assuming some technical results, we should be able to use Melin inversion. Thus,

$$J(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log \zeta(s)}{s} x^s ds.$$

This converges when $\sigma > 1$.

Thus in order to find a formula for $J(x)$ we need only get a better formula for $\log \zeta(s)$.

Riemann claimed that $\xi(s) = \xi(0) \prod_\rho \left(1 - \frac{s}{\rho}\right)$, where the product is taken over all roots of the ξ function (that is, over all nontrivial zeroes of the ζ-function). This product does not converge absolutely, and we should pair any terms with $im(\rho)$ positive with a corresponding term with negative imaginary part to get a convergent product. The proof of this product formula basically depends on getting nice bounds on the growth of the number of zeroes.

Now we notice that,

$$\zeta(s) = 2 \frac{1}{s(s-1)} \pi^{s/2} \frac{1}{\Gamma(s/2)} \xi(s) = 2 \frac{1}{s(s-1)} \pi^{s/2} \frac{1}{\Gamma(s/2)} \xi(0) \prod_\rho \left(1 - \frac{s}{\rho}\right).$$

Therefore,

$$\log \zeta(s) = \log 2 - \log s - \log(s-1) + \frac{s}{2} \log \pi - \log \Gamma(s/2) + \log \xi(0) + \sum_\rho \log \left(1 - \frac{s}{\rho}\right).$$

We want to substitute this into our integral formula and evaluate termwise, however doing so would lead to divergent integrals (for example in the $\frac{s}{2} \log \pi$ term). Thus Riemann first integrated by parts to get,

$$J(x) = -\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{d}{ds} \left(\frac{\log \zeta(s)}{s} \right) x^s ds.$$

Now we can substitute our formula for $\zeta(s)$ and evaluate term by term. With a good bit of work, Riemann evaluated these integrals and got the formula,

$$J(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) + \int_x^\infty \frac{1}{t(t^2-1)\log t} dt - \log 2.$$

Notice that $J(x) = \sum_{n=1}^{\infty} \frac{1}{n}\pi(x^{\frac{1}{n}})$. We can invert this formula to get, $\pi(x) = \sum_{n=1}^{\infty} \mu(n)\frac{1}{n}J(x^{1/n})$.

This gives us a formula for $\pi(x)$. Its dominant term is $\sum_{n=1}^{\infty} \mu(n)\frac{1}{n}\mathrm{Li}(x^{1/n})$. This would show the prime number theorem if we could actually prove that this term was dominant. The key to proving this is to show that the $\sum_{\rho} \mathrm{Li}(x^{\rho})$ terms are each smaller, that is to say we need to show that $\mathrm{Re}(\rho) < 1$.

2 Chebyshev's Functions

Before Riemann's work the only significant progress towards the prime number theorem was made by Chebyshev who proved that, for sufficiently large x and some constants $c_1 < 1 < c_2$, $c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}$. To prove this he introduced two functions which are crucial in later proofs of prime number theory. Recall that we conjecture that the chances that a number n is prime is roughly $\frac{1}{\log n}$. Thus, if we counted each prime as $\log p$ instead of as 1, then we would get a better behaved function.

Definition 2.1. *Let $\theta(n) = \sum_{p \leq n} \log p$ (where, as usual, at jumps we define the function to be halfway in between the two values).*

As we've seen from Riemann's argument it is often simpler to count prime powers instead of primes.

Definition 2.2. *Let $\psi(n) = \sum_{p^k \leq n} \log p$.*

There is another way of writing ψ in terms of Von Mangoldt's Λ function.

Definition 2.3. *Let*

$$\Lambda(n) = \begin{cases} \log n & \textit{if } n \textit{ is a prime power} \\ 0 & \textit{else} \end{cases}.$$

Clearly $\psi(x) = \sum_{x \leq n} \Lambda(n)$.

First we notice that one can express each of the functions ψ, θ, π, and J in terms of any of the others.

Proposition 2.4.

$$J(x) = \sum_{n=1}^{\infty} \frac{1}{n}\pi(x^{1/n}).$$

$$\pi(x) = \sum_{n=1}^{\infty} \mu(n)\frac{1}{n}\pi(x^{1/n}).$$

$$\psi(x) = \sum_{n=1}^{\infty} \theta(x^{1/n}).$$

$$\theta(x) = \sum_{n=1}^{\infty} \mu(n)\psi(x^{1/n}).$$

Proof. We've already shown the first two, and the proof of the second two are exactly the same. □

Proposition 2.5.

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_0^{\infty} \frac{\theta(t)}{t(\log t)^2}dt.$$

$$J(x) = \frac{\psi(x)}{\log x} + \int_0^{\infty} \frac{\psi(t)}{t(\log t)^2}dt.$$

$$\psi(x) = J(x)\log x - \int_0^{x} \frac{J(t)}{t}dt.$$

$$\theta(x) = \pi(x)\log x - \int_0^{x} \frac{\pi(t)}{t}.$$

Proof. Notice that $\pi(x) = \int_0^x \frac{1}{\log t} d\theta(t)$. The theorem follows from integration by parts. Similarly $J(x) = \int_0^x \frac{1}{\log t} d\psi(t)$, and we integrate by parts again. Conversely, $\theta(x) = \int_0^x \log t d\pi(t)$ and $\psi(x) = \int_0^x \log t dJ(t)$. Integrating these by parts gives the second two equations. ☐

Since θ and ψ are trivially $O(x \log x)$ and π and J are trivially $O(x)$ we can rewrite these equations in terms of error estimates. The long and short of all of this is that to prove the prime number theorem it is enough to prove any of $\pi \sim \text{Li}(x)$, $J(x) \sim \text{Li}(x)$, $\theta(x) \sim x$, or $\psi(x) \sim x$. Furthermore, given any explicit error terms in the above approximations we can find explicit error terms for all of the other approximations. As it turns out ψ is the easiest function to deal with.

Proposition 2.6. *For any n, $\sum_{d|n} \Lambda(d) = \log n$.*

Proof. Notice that $n = \prod_{p|n} p^k$ where k is the largest number such that $p^k|n$. Thus, $n = \prod p^k|np$. Taking the logarithm shows that $\log n = \sum_{d|n} \Lambda(d)$.

There is another way of looking at this identity. In shorthand this proposition claims $\Lambda \star 1 = \log n$. Thus it is equivalent to some identity involving Dirichlet series. Notice that

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = \sum_p \sum_{m=1}^{\infty} (\log p) p^{-ms} = -\frac{\zeta'(s)}{\zeta(s)}.$$

Also

$$\sum_{n=1}^{\infty} \log n = \zeta'(s).$$

Therefore, $f(s, \Lambda)\zeta(s) = f(s, \log n)$ exactly as we had hoped to show. ☐

Before leaving this last proof we notice that one of the equations can be rewritten

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^{\infty} x^{-s} d\psi(x).$$

3 Chebyshev's Theorem

Theorem 3.1. *For sufficiently large x and some constants $c_1 < 1 < c_2$, $c_1 \leq \frac{\psi(x)}{x} \leq c_2$.*

Proof. Chebyshev noticed that if we sum $\sum_{d|n} \Lambda(d) = \log n$ over all $n \leq x$, then

$$T(x) = \sum_{m \leq x} \Lambda(m) \left\lfloor \frac{x}{m} \right\rfloor = \sum_{n \leq x} \log n = \log \lfloor x \rfloor !.$$

By Stirling's formula

$$T(x) = \log \lfloor x \rfloor ! = x \log x - x + O(\log x).$$

Notice that

$$T(x) = \sum_{m \leq x} \sum_{n \leq \frac{x}{m}} \Lambda(m) = \sum_{n \leq x} \sum_{m \leq \frac{x}{n}} \Lambda(m) = \sum_{n \leq x} \psi\left(\frac{x}{n}\right).$$

Therefore, by Möbius inversion,

$$\psi(x) = \sum_{n=1}^{\infty} \mu(n) T\left(\frac{x}{n}\right).$$

This suggests that finite expressions which have several terms from $\sum_{n=1}^{\infty} \mu(n) T\left(\frac{x}{n}\right)$ will give good approximations to ψ. But we also want good cancellations when we plug in the approximation from Stirling's formula. For example, it would be informative to look at expressions of the following form:

$$T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{2}\right),$$

$$T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{3}\right) + T\left(\frac{x}{6}\right),$$
$$T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{3}\right) - T\left(\frac{x}{5}\right) + T\left(\frac{x}{30}\right), \text{ etc.}$$

We will look at the first expression $T(x) - 2T\left(\frac{x}{2}\right)$. Chebyshev looked at the third expression and was able to get constants c_1 and c_2 closer to 1. Notice,

$$T(x) - 2T\left(\frac{x}{2}\right) = \sum_{m \le x} \Lambda(m)\left(\left\lfloor \frac{x}{m}\right\rfloor - \left\lfloor \frac{x}{2m}\right\rfloor\right).$$

The lefthand side is $x \log 2 + O(\log x)$. The righthand side is

$$\sum_{m \le x} \Lambda(m)\left(\left\lfloor \frac{x}{m}\right\rfloor - \left\lfloor \frac{x}{2m}\right\rfloor\right) \le \sum_{m \le x} \Lambda(m) = \psi(x).$$

Therefore, for large x and any constant $\varepsilon > 0$,

$$\log 2 - \varepsilon \le \frac{\psi(x)}{x}.$$

In particular, we can take $c_1 = .69$.

Similarly, the righthand side is

$$\sum_{m \le x} \Lambda(m)\left(\left\lfloor \frac{x}{m}\right\rfloor - \left\lfloor \frac{x}{2m}\right\rfloor\right) \ge \sum_{\frac{1}{2}x \le m \le x} \Lambda(m) = \psi(x) - \psi\left(\frac{x}{2}\right).$$

Therefore, $\psi(x) - \psi\left(\frac{x}{2}\right) \le x \log 2 + O(\log x)$. Summing these estimates yields,

$$\psi(x) \le x \cdot 2\log 2 + O(\log^2 x).$$

In particular, we can take $c_2 = 1.38$. □

By our previous results relating ψ and π, we also get that

$$.79 \cdot \frac{x}{\log x} \le \pi(x) \le 1.38 \cdot \frac{x}{\log x}.$$

4 Reducing the Prime Number Theorem to Facts About $\zeta(s)$.

Recall that

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty x^{-s} d\psi(x).$$

Integrate by parts to see that

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_0^\infty \psi(x) x^{-s-1} dx.$$

Our general method of attack is to rewrite this as a Mellin transform and then use Mellin inversion to retrieve $\psi(x)$ in terms of $\zeta(s)$. However, to make certain integrals behave well later on, we first make a slight change. Integrates by parts again to notice that

$$-\frac{\zeta'(s)}{\zeta(s)} = s^2 \int_0^\infty \left(\int_0^x \frac{\psi(t)}{t} dt\right) x^{-s-1} dx.$$

Definition 4.1. *Let* $\phi(x) = \int_0^x \frac{\psi(t)}{t} dt$.

Therefore we have

$$-\frac{\zeta'(s)}{\zeta(s)} = s^2 \int_0^\infty \phi(x) x^{-s-1} dx.$$

To write this as a Mellin transform we make the change of variables $s \mapsto 1-s$. Therefore,

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} \frac{1}{(1-s)^2} = \int_0^\infty \frac{\phi(x)}{x} x^{-s} \frac{dx}{x}.$$

In order to apply Mellin inversion we must check to see that the technical conditions of that theorem are satisfied. Notice that since $\psi(x) = O(x \log x)$ we have $\frac{\phi(x)}{x} = O(\log x)$. Therefore the integrand in the Mellin transform converges absolutely for $\Re(s) < 0$. Also,

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq \sum_{n=1}^\infty (\log n) n^{-\sigma}.$$

Thus, for any positive ε, in the region $\mathrm{Re}(s) \geq 1 + \varepsilon$, the function $\frac{\zeta'(s)}{\zeta(s)}$ is bounded by an absolute constant. Therefore, the integral

$$\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta'(1-s)}{\zeta(1-s)} \frac{1}{(1-s)^2} dx$$

converges absolutely for any $\sigma < 0$. Therefore the conditions of Mellin inversion are satisfied and,

$$\frac{\phi(x)}{x} = -\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta'(1-s)}{\zeta(1-s)} \frac{1}{(1-s)^2} x^{-s} ds,$$

for any $\mathrm{Re}(s) < 0$.

Now we can change variables back $s \mapsto 1-s$ and multiply both sides by x to get,

Proposition 4.2. *For any s with $Re(s) > 1$ the following integral converges absolutely and*

$$\phi(x) = -\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta'(s)}{\zeta(s)} \frac{1}{s^2} x^s dx.$$

□

Notice that thus far we could have gone through the argument with $\psi(x)$ instead of $\phi(x)$ and the resulting formula would have a $1/s$ instead of $1/s^2$.

Our argument from here on in consists of several parts. First we will assume that there are no zeroes of the ζ function on the line $\mathrm{Re}(s) = 1$. We will prove this in the next section. Thus the only pole of the integrand in the halfplane $\mathrm{Re}(s) \geq 0$ is $s = 1$. We can subtract off this pole to get a term which contributes the dominant term x. The remaining integral we can move all the way to the line $\mathrm{Re}(s) = 1$. Then we will get an explicit bound on this integral. This will give us an approximation for $\phi(x)$. Finally we will need to extract an estimate for $\psi(x)$ from our knowledge concerning $\psi(x)$.

So notice that

$$\phi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{s-1} \frac{1}{s^2} x^s dx - \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \right) \frac{1}{s^2} x^s dx.$$

The first integral can be written as the limit of an integral about the rectangle with corners $1 + 1/T \pm iT$ and $-T \pm iT$. The integrals along all but the right side die very quickly. Thus our integral is the sum of the residues to the left of $\mathrm{Re}(s) = 2$. The only poles are at $s = 0$ and $s = 1$. To this end expand $x^s = e^{s \log x} = 1 + s \log x + s^2 \log^2 x + \ldots$. Thus the residue at $s = 0$ is $-\log x$. At $s = 1$ the residue is x. Therefore this integral contributes the term $x - \log x$.

(The notes that I am basing this on say that this integral is $x - \log x - 1$. I cannot find out where the -1 comes from, but I do not trust my ability to do complex analysis very well, and so that is probably right. Nonetheless since we are only interested in approximation the -1 will not matter.)

Therefore, given our assumption that $\zeta(1 + it) \neq 0$, we have proved:

Proposition 4.3.

$$\phi(x) = x - \log x - \frac{x}{2\pi i}\int_{-\infty}^{i\infty}\left(\frac{\zeta'(1+it)}{\zeta(1+it)} + \frac{1}{it}\right)\frac{1}{(1+it)^2}e^{it\log x}dx.$$

□

In order to estimate this last integral we will need a few estimates on the size of $\zeta(s)$ and $\zeta'(s)$. These will be proved in the next section. Thus we will make the following assumptions:

Proposition 4.4. *Letting $s = \sigma + it$ as usual, we have the bound $\zeta^(k)(s) = O(\log^k t)$ in the region $\sigma > 1 - \frac{1}{\log t}$ and $t > 2$. Also we have $\frac{1}{\zeta(s)} = O(\log^7 t)$ in the region $\sigma \geq 1$ and $t > 2$.*

Proposition 4.5. *For any integer k, $\phi(x) = x + O\left(\frac{x}{(\log x)^k}\right)$.*

Proof. Let

$$f(t) = \frac{1}{2\pi i}\left(\frac{\zeta'(1+it)}{\zeta(1+it)} + \frac{1}{it}\right)\frac{1}{(1+it)^2}.$$

Recall that $\phi(x) = \int_{\mathbb{R}} f(t)e^{it\log x}$. Since the second term is rapidly oscillating, if we can get a decent bound on $f(t)$ we should get a very good bound on $\phi(x)$. From our estimates concerning ζ and its derivatives,

$$f^(k)(t) = O\left(\frac{\log t}{(1+t)^2}\right).$$

Therefore for each k there is a constant $C(k)$ with

$$\int_{\mathbb{R}} |f^(k)(t)|dt \leq C(k).$$

Now we integrate by parts k times to see that,

$$\int_{\mathbb{R}} f(t)e^{it\log x}dt = \frac{1}{(-i\log x)^k}\int_{\mathbb{R}} f^(k)(t)e^{it\log x}.$$

Therefore,

$$\left|\int_{\mathbb{R}} f(t)e^{it\log x}\right| \leq \frac{C(k)}{\log^k x}.$$

Combining this with our earlier results yields our required results. □

Notice that had we attempted to run through the above argument with ψ the final integral would not have converged absolutely. One would still expect the oscillatory term to cancel things out, but proving this would be more difficult.

All that remains to do (other than the analytic results put off till next section) is to turn this estimate for ϕ into an estimate for ψ. It is perhaps surprising that one can do this, since we are essentially differentiating an approximation. But since ψ behaves so nicely we can in fact do this.

Theorem 4.6. *For any integer k, $\psi(x) = x + O(\frac{x}{\log^{k/2} x})$.*

Proof. Suppose the $\varepsilon(x)$ is any function satisfying $0 < \varepsilon(x) \leq \frac{x}{2}$. Let $g_k(x) = \frac{x}{\log^k x}$. We have proved that for all sufficiently large x and some constant C,

$$x - Cg_k(x) \leq \phi(x) \leq x - Cg_k(x).$$

Therefore, since $g_k(2x) \leq g_k(x)$,

$$\phi(x + \varepsilon(x)) - \phi(x) \leq \varepsilon(x) + Cg_k(x + \varepsilon(x)) + Cg_k(x) \leq \varepsilon(x) + 3Cg_k(x).$$

On the other hand, since ψ is an increasing function,

$$\phi(x+\varepsilon(x))-\phi(x)=\int_x^{x+\varepsilon(x)}\frac{\psi(t)}{t}dt\geq\psi(x)\frac{\varepsilon(x)}{x+\varepsilon(x)}.$$

Combining these two equations shows that

$$\psi(x)\leq x+\varepsilon(x)+3Cg_k(x)\frac{x+\varepsilon(x)}{\varepsilon(x)}\leq x+\varepsilon(x)+\frac{6Cxg_k(x)}{\varepsilon(x)}.$$

Considering $\phi(x)-\phi(x-\varepsilon(x))$ in the same way yields

$$\phi(x)\geq x-\frac{2Cxg_k(x)}{\varepsilon(x)}.$$

Now we can choose $\varepsilon(x)$ in such a way to minimize the error term. The best such choice is $\varepsilon(x)=c\sqrt{xg_k(x)}$ where we choose c small enough so that we still have $\varepsilon(x)\leq\frac{x}{2}$. Plugging this expression into our previous results yields the theorem. ☐

This is equivalent to the prime number theorem. Plugging our estimate for ψ into our previous relations,

$$\pi(x)=\frac{x}{\log x}+\int_2^x\frac{1}{\log^2 t}dt+O\left(\frac{x}{\log^k x}\right).$$

However, by integration by parts, $\mathrm{Li}(x)=\frac{x}{\log x}+\int_2^x\frac{1}{\log^2 t}dt+O(1)$. Therefore, we have

$$\pi(x)=\mathrm{Li}(x)+O\left(\frac{x}{\log^k x}\right).$$

Notice that the approximation $\pi(x)=\frac{x}{\log x}$ only holds, a priori, up to $O\left(\frac{x}{\log^2 x}\right)$.

5 Some Facts About $\zeta(s)$.

Proposition 5.1. *For any real t, $\zeta(1+it)\neq 0$.*

Proof. Throughout this proof any time we use the symbol c it means a particular constant which may change from equation to equation.

Recall that

$$\log\zeta(s)\geq\sum_p p^{-s}+c.$$

Therefore,

$$\mathrm{Re}\zeta(s)\geq\sum_p\frac{\cos t\log p}{p^\sigma}+c.$$

If $s=1+it$ were a zero of the zeta function, then $\lim_{\sigma\to 1^+}\log\zeta(\sigma+it)=-\infty$. Therefore,

$$\lim_{\sigma\to 1^+}\frac{\cos t\log p}{p^\sigma}=-\infty.$$

This implies that the vast majority of numbers $\cos t\log p$ are near -1. Therefore, nearly all the numbers $\log p$ would lie near the points of the arithmetic progression $(2n+1)t^{-1}\pi$. This is impossible because this regularity would suggest that $\cos(2t\log p)$ were nearly 1 for the vast majority of primes. This in turn suggests that $\zeta(s)$ has a pole at $s=1+2it$.

Now we make this argument rigorous. Suppose $\zeta(s)$ had a zero at $s=1+it$, then $\zeta(s)/(s-1-it)$ would be analytic near $s=1+it$. In particular, taking the real part of log of $\zeta(s)/(s-1-it)$, we see that

$$\sum_p\frac{\cos(t\log p)}{p^\sigma}<\log(\sigma-1)+c.$$

Let $\delta > 0$ be some small positive number. Let S_1 be the sum of $p^{-\sigma}$ over all primes which satisfy $|(2n+1)\pi - t\log p| < \delta$ for some integer n, and let S_2 be the sum over primes which do not satisfy this condition. For terms in the second sum $\cos(t\log p) > -\cos\delta$. Therefore,

$$-S_1 - (\cos\delta)S_2 < \log(\sigma - 1) + K.$$

On the other hand, since there is a simple pole at 1, we have $S_1 + S_2 < -\log(\sigma-1) + c$. Therefore,

$$-S_1 - (\cos\delta)S_2 < -S_1 - S_2 + c.$$

Therefore,

$$S_2 < \frac{c}{1-\cos\delta}.$$

However, since $1+2\pi it$ is not a pole of $\zeta(s)$, the real part of $\log\zeta(s)$ is bounded above near $s = 1+2it$. Therefore

$$\sum_p \frac{\cos 2t\log p}{p^\sigma} < c.$$

Again we can split this sum up over the two sets of primes. For primes of the first type $\cos(2t\log p) > \cos 2\delta > 0$. Therefore,

$$S_1\cos 2\delta - S_2 < c.$$

Therefore,

$$S_1 < \frac{c}{(1-\cos\delta)\cos 2\delta}.$$

Hence, for some constant depending on δ, $S_1 + S_2 < C(\delta)$. Letting σ approach 1 makes the lefthand side blow up which is a contradiction.

For a more clever but perhaps less informative proof that a zero at $\zeta(1+it)$ would force a pole at $\zeta(1+2it)$ look at the proof of this result on one of the next few copied pages. □

The proofs of the following two results are on the next few photocopied pages.

Proposition 5.2. *Letting* $s = \sigma + it$ *as usual, we have the bound* $\zeta^{(}k)(s) = O(\log^k t)$ *in the region* $\sigma > 1 - \frac{1}{\log t}$ *and* $t > 2$.

Proposition 5.3. *Letting* $s = \sigma + it$ *as usual, we have the bound* $\frac{1}{\zeta(s)} = O(\log^7 t)$ *in the region* $\sigma \geq 1$ *and* $t > 2$.