Based on joint work with E. Rains and D. Krichever

Noncommutative geometry

And Painlevé equations
Stepped surfaces

Holomorphic functions

Discrete Kasteleyn theory

End of lecture 1
$K$astekov operators

\[ K \eta = 0 \]

Domain

A function $f(z)$ is discrete holomorphic if

\[ \Delta f(z) = 0 \]
$\mathcal{L} = q_1 q_2 q_3$

$x^3 = 0 \iff x = 0$

$A = \langle x_1, x_2, x_3 \rangle$

$K = \text{right multiplication by } \chi_1 + x_2 + x_3$

This has a natural vector deformation, when
\[ \text{index}(k) = \chi \]

\[ \gamma \Rightarrow \text{degree} \]

\[ \text{rank } k = 0 \]

\[ \exists \text{ a left } A\text{-module} \text{ with discrete holomorphic functions in } U \]
This means, we have a map \( \phi \in \text{Hom}(A', \mathcal{O}) \) of modules over \( A' \).
What is this map $\phi$?

An orbit of $\mathbb{Z}_3$-action.

basically
We have

Indeed, look at

\[ \bigcup
\]
Points in $\mathbb{P}^2$ non-collinear, i.e., non-incident.

The poles of $\mathcal{O}$ remain.

$$y = \frac{x_0^1}{x_0^0 x} + \text{const.}$$

The algebra $A$ is a quantization of $\mathbb{P}^2$ using Point modules $\mathcal{O}_P$. Like A/ (x_0, x_1 + \text{const.} x_2)$. 
Moving the boundary means point modules $\mathcal{G}$ that $\mathcal{G}$ meets. Correspond to boundaries of $\mathcal{G}$.
by fractional transformations

\[ S(p, q, r) \times \mathbb{Z} \rightsquigarrow \prod \mathbb{H}(0, d, x) \]

This defines an action of \( S(p, q, r) \times \mathbb{Z} \) on \( \mathbb{H}(0, d, x) \). Thus defines an action on \( \mathbb{H}(0, d, x) \). This action is transitive on \( \mathbb{H}(0, d, x) \).

Even for more general noncommutative surfaces like Sl Kathryn Theorem / Observation
Alg. integrable syst.

only rotates

$C = \text{Supp} \mathcal{C}$

does not change

$\mathcal{O} \xrightarrow{i} \ker(\mathcal{O})$

$\text{Pic} C = \mathcal{O}$

In the commutative case, obviously,

Degree of curves of

$P \left( \frac{d+2}{2} \right) - 1$
Another invariant may be constructed over \( \mathbb{C} \).

By a canonical transformation, even moves the point.

\[
\begin{align*}
0 & \leftrightarrow 0 \\
\circ & \leftrightarrow \circ \\
\circ & \leftrightarrow \circ \\
\circ & \leftrightarrow 0
\end{align*}
\]

There is no invariant fibration in the noncomultiplicative case.
(4) Stand in DP

(3) Should attach to a larger
affine Springer action

(2) An obvious generalization of
Sakai's affine Poincaré
classical PI-DI

(1) For $d=3$ and Sklyanin algebra, get

Some comments:
I believe our construction gives a uniform explanation.

... some discrete analogs of [Roel], [J. Pollock], ...

[Tracy Wisdom] [Adler-von Heerbreke]

[Barouch-McCoy-Trahey-WU]

Ferrim + Moving body → Pain

In math, physics, there are many instances of the reaction
Another abstract comment, in the cohomological world.

\[ \text{Given points} \quad \Rightarrow \quad \text{Curves meeting} \]

\[ \{ \bullet \} \quad \text{following in the} \quad \text{Curves in the} \quad \mathbb{P}^2 \]

\[ \mathcal{O} \otimes \mathcal{O} \quad = \quad \text{Dynamics} \]
When $A$ is a graded, but not different algebra, mathematically $A$ quantifies $A$ (a, $A$)-bimodule, in the noncommutative world, tensor product warps the space.
In our dynamics, \( t = \log \alpha \) at the same rate.

As it approaches zero, the mesh is a prerunner back to Earth, then is a prerunner mesh zero.

\[
\frac{t}{1} \sim \frac{\text{mesh}}{1}
\]
Fiber $\int \nabla (\text{permutation})$

follows

Projection on the base typically

For times $\tau$ the permutation

Averaging permutations

This is a question of

Curves $C = \mathbb{P}^2$
\[ E \in H^2(\mathbb{R}, C) \]

\[ \text{leaves of foliation} \]

\[ \text{Orbits of } \Gamma \]

\[ \text{Theorem: \quad Krichever - } [2] \]

\[ \theta \to 0 \]

\[ \text{as } \theta \to 0 \]
Can check e.g. in the tropicalgrunt.

Rational curves form a leaf of GK foliation.

\[ \text{Supp} \, \mathcal{O} \leftarrow \mathcal{O} = \text{frozen boundary} \]

As \( q \to 1 \), mesh \( \to 0 \)

Corr.
Life is easy in the tropics because all is frozen.