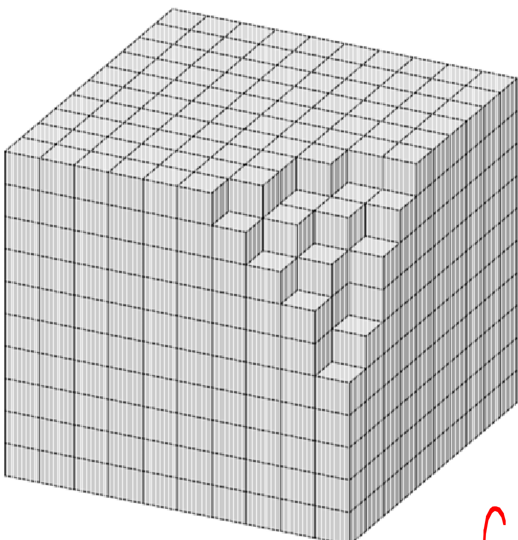
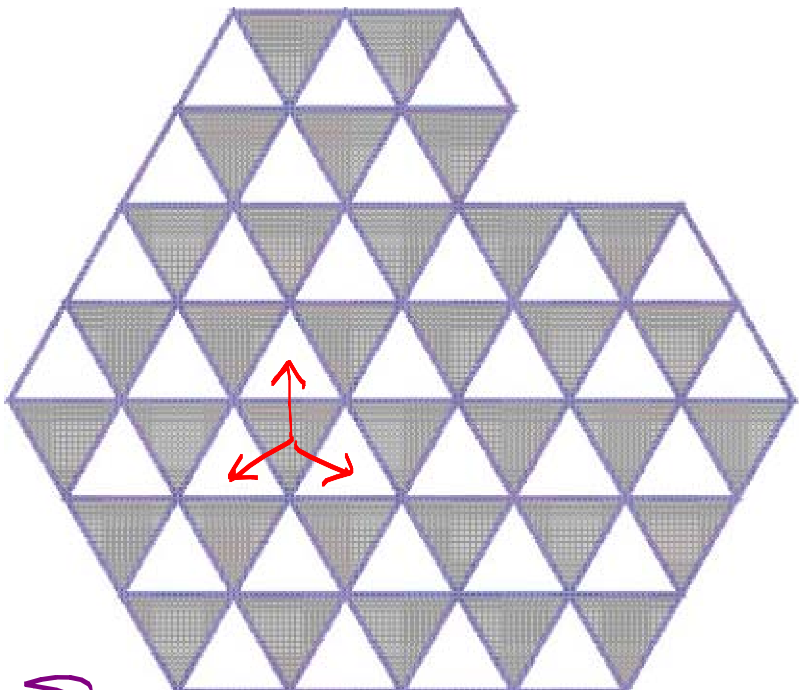
Noncommutative geometry and Painlevé equations

based on joint work with E. Rains and I. Krichever

Recap of Lecture 1



Stepped surfaces $\xrightarrow{\text{Kasteleyn theory}}$ discrete holomorphic functions



domain Ω

A function $\Psi(\blacktriangledown)$ is discrete holomorphic if

$$K\Psi = 0$$

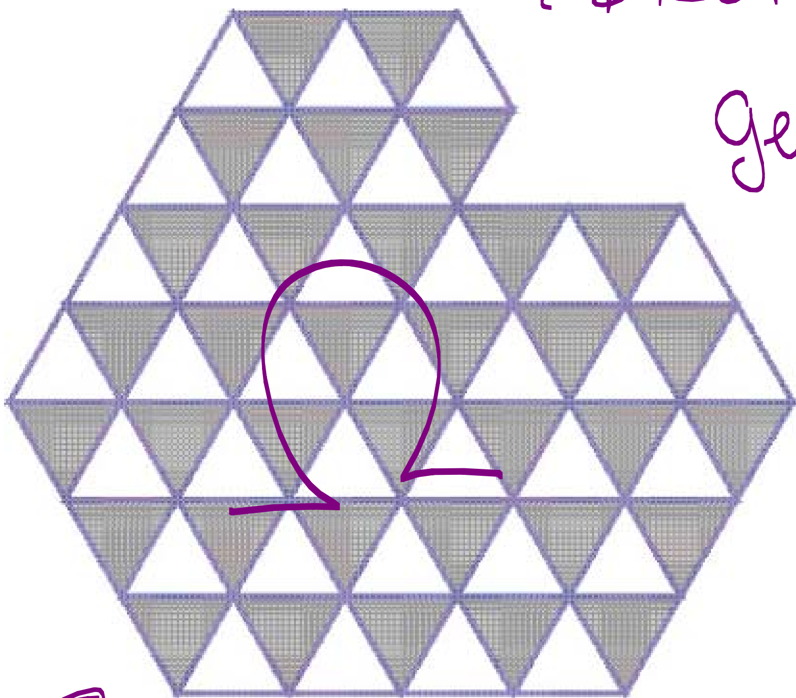
Kasteleyn operator

This has a natural q^{vol} deformation, when

K = right multiplication by $x_1 + x_2 + x_3$ in

$$A = \mathbb{C}\langle x_1, x_2, x_3 \rangle \Big/ x_j x_i = q_{ij} x_i x_j$$

where $q = q_{12} q_{23} q_{31}$



$3d$-gon $= \deg \Omega$

Discrete holo functions in Ω generate a left A-module $\widehat{Q}$ with

$$\text{rank} = 0 \quad !!!$$

$$\text{degree} = d$$

$$\chi = \text{index}(K)$$

this means, we have a map

moduli of modules

$$\text{domains } \Omega \longrightarrow \hat{Q} \in \mathrm{Hilb}(A;0,d,\chi)$$

$$\Omega = \mathbb{Z}^{3d} \cap \text{Cone}$$

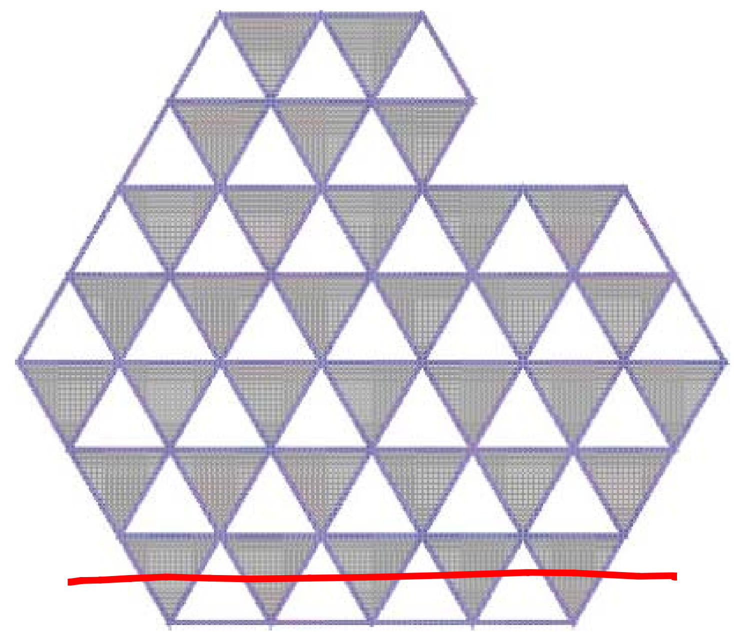
alg. variety of dimension d^2+1

<u>Q.</u> What is this map?

<u>A.</u> An orbit of $\mathbb{Z}^{3d}$-action, basically

Indeed, look at

$$\Omega' = \Omega \setminus \text{strip}$$



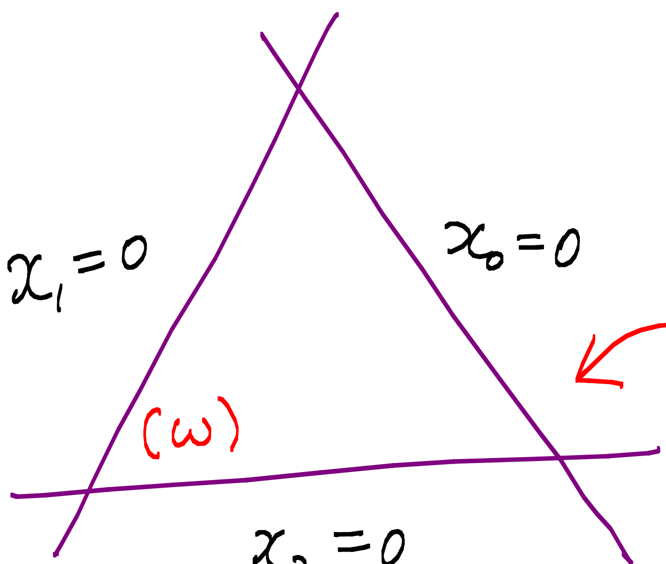
We have

$$0 \to \mathrm{Holo}(\Omega') \to \mathrm{Holo}(\Omega) \to \mathrm{Holo}(\text{strip}) \to 0$$

<u>point</u> module !!!

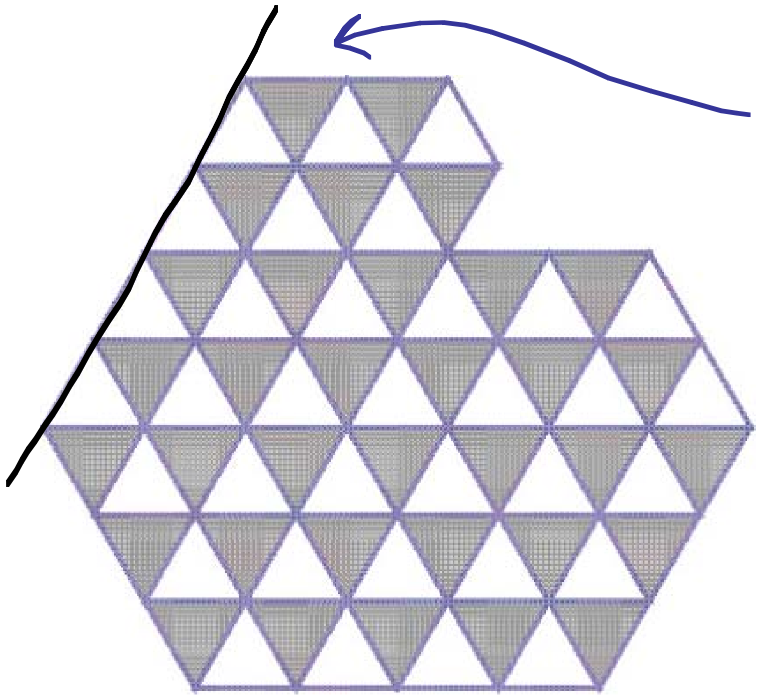
The algebra A is a quantization of $\mathbb{P}^2$ using

$$\omega = \frac{dx_0 \wedge dx_1}{x_0 x_1} + \text{cycl.}$$



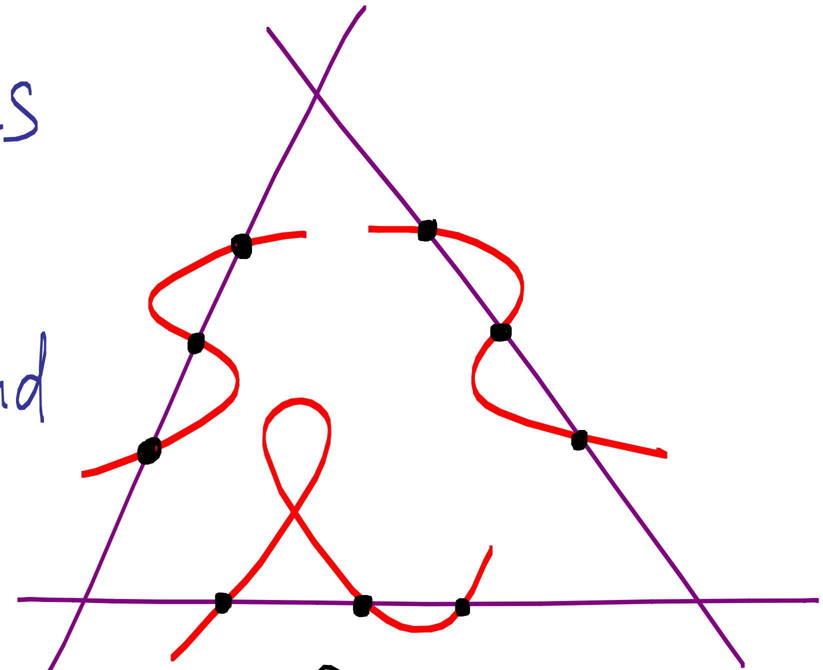
The poles of ω remain points in $\mathbb{P}^2_{\text{noncomm}}$, i.e

point modules $\mathcal{O}_p$ like $A \big/ (x_0, x_1 + \text{const}\, x_2)$



boundaries of Ω

Correspond to



point modules $\mathcal{O}_p$ that $\widehat{Q}$ meets

Moving the boundary means

$$0 \to \widehat{Q}' \to \widehat{Q} \to \mathcal{O}_p \to 0$$

noncomm. shift on the Jacobian

Theorem/Observation [Rains–O] Even for more general noncomm proj surfaces like Sklyanin algebra this defines an action

$$S(d \cdot c_1) \ltimes \mathbb{Z}^{d \cdot c_1} \circlearrowright \coprod_{\chi} \mathrm{Hilb}(0, d, \chi)_{\mathrm{par}}$$

3d for $\mathbb{P}^2$

by birational transformations

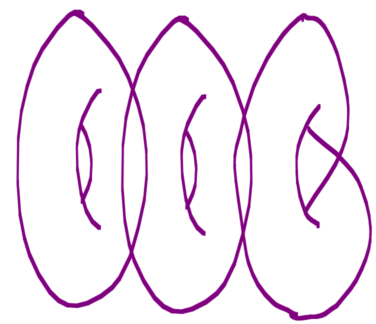
In the commutative case, obviously, the basic move

$$Q \mapsto \mathrm{Ker}\,(Q \to \mathcal{O}_p)$$

does not change

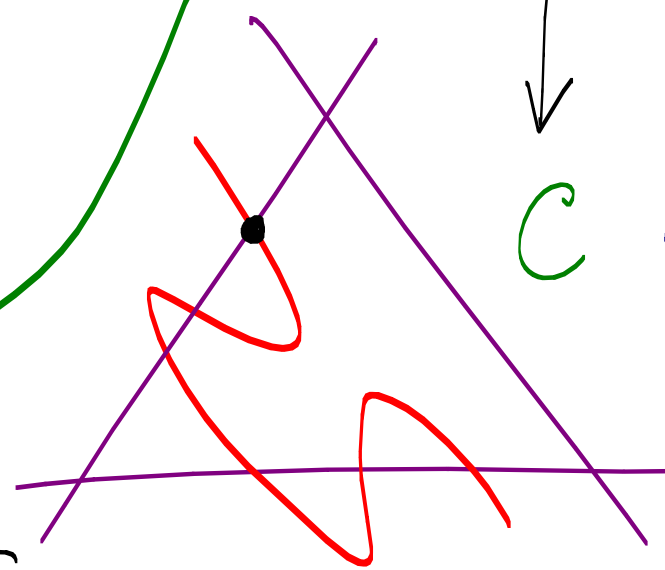
$$C = \mathrm{Supp}\, Q$$

only rotates

$\mathrm{Pic}\, C \subset$ 

$$C \in \mathbb{P}^{\binom{d+2}{2}-1}$$

curves of degree d

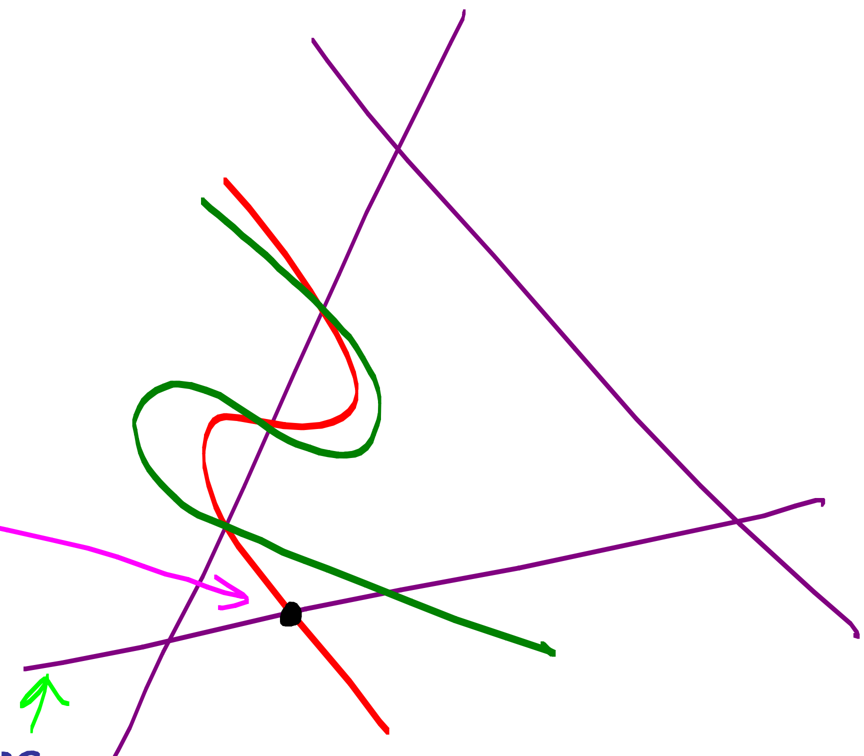


<u>Alg. integrable syst.</u>

There is no invariant fibration in the noncommutative case

$$0 \to \widehat{Q}' \to \widehat{Q} \to \mathcal{O}_p \to 0$$

even moves the point by a canonical automorphism of the anticanonical divisor



Analytic invariants may be constructed over $\mathbb{C}$

Some comments:

(1) For $d=3$ and Sklyanin algebra, get Sakai's elliptic Painlevé $\rightsquigarrow$ classical PI–VI

(2) An obvious generalization of affine Springer action

(3) should extend to a larger group in D^b

(4) ....

In math. physics, there are many instances of the reaction

Fermion + Moving bdry $\longrightarrow$ Painlevé

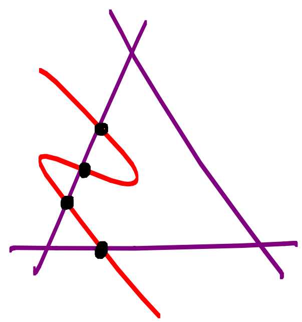
e.g.

- Spin-spin corr. in Ising [Barouch-McCoy-Tracy-Wu]
- largest eigenvalue in RMT [Tracy-Widom] [Adler-van Moerbeke]
- some discrete analogs of [Baik], [Borodin], ...

......

I believe our construction gives a uniform explanation

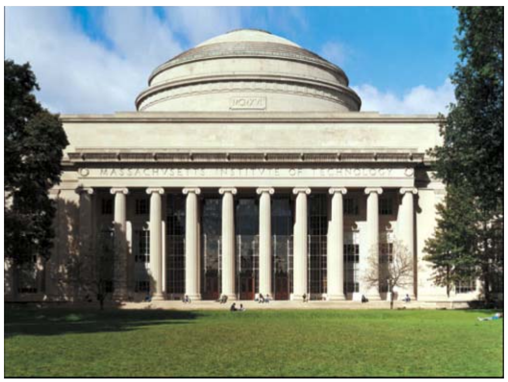
Another abstract comment, in the commutative world

curves meeting given points  = curves in the blowup of $\{\bullet\} \subset \mathbb{P}^2$

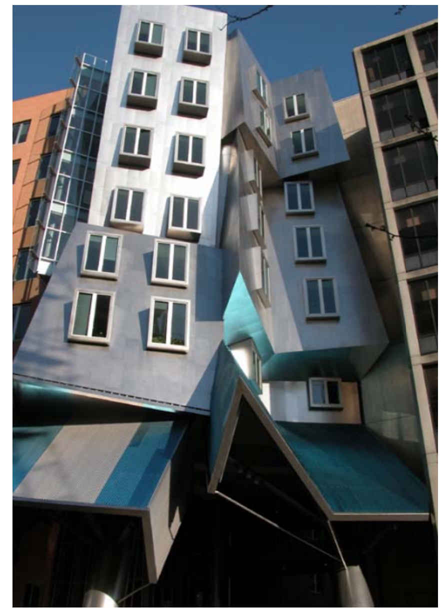
dynamics = $\bigotimes \mathcal{O}(\text{exceptional})$

after quantization, the blowup points move!

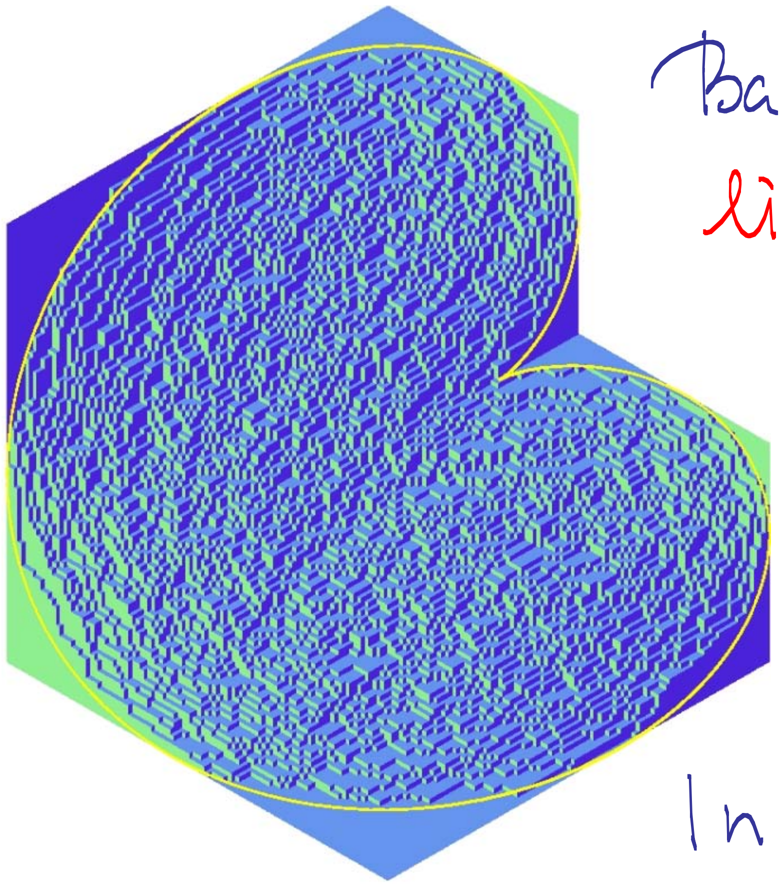
In the noncommutative world, tensor product warps the space



$$\xrightarrow{\otimes \mathcal{L}}$$



mathematically, $\mathcal{L}$ quantizes to a (A, A')- bimodule where A' is a related, but different algebra



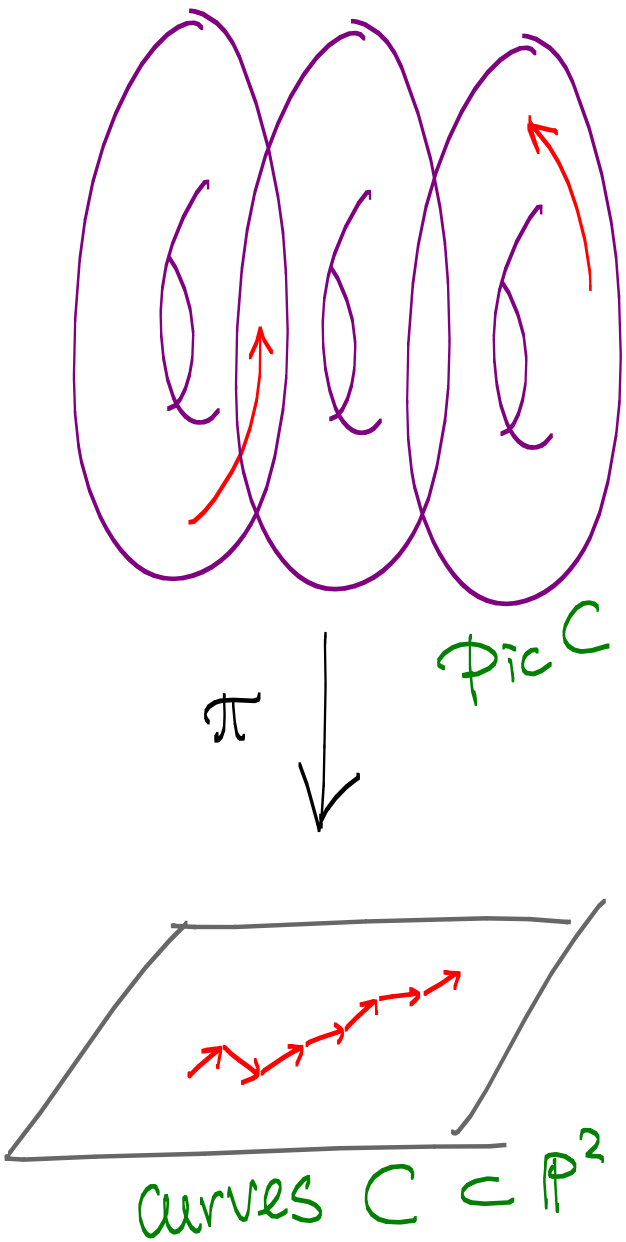
Back to Earth, there is a particular limit probabilists want take

$$\text{mesh} \to 0$$
$$\log q \to 0$$

at same rate

In our dynamics, $\hbar = \log q$ measures the deviation from integrability and

$\dfrac{1}{\text{mesh}} \sim \dfrac{1}{\hbar}$ is the number of iterations



This is a question of

Averaging perturbations

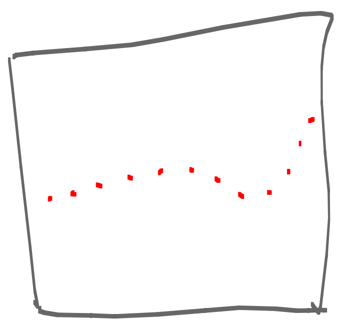
For times $\sim \dfrac{1}{\text{perturbation}}$ the projection on the base typically follows

$$\int_{\text{fiber}} \pi(\text{perturbation})$$

Theorem* [Krichever - O] As $\hbar \to 0$

Orbits of "Painlevé" $\longrightarrow$ Leaves of the foliation

Studied by Grushevsky-K



$$\mathrm{Re}\ \frac{1}{\hbar}\int \omega = \text{const}$$

$\in H_2(\mathbb{P}^3, \mathbb{C})$

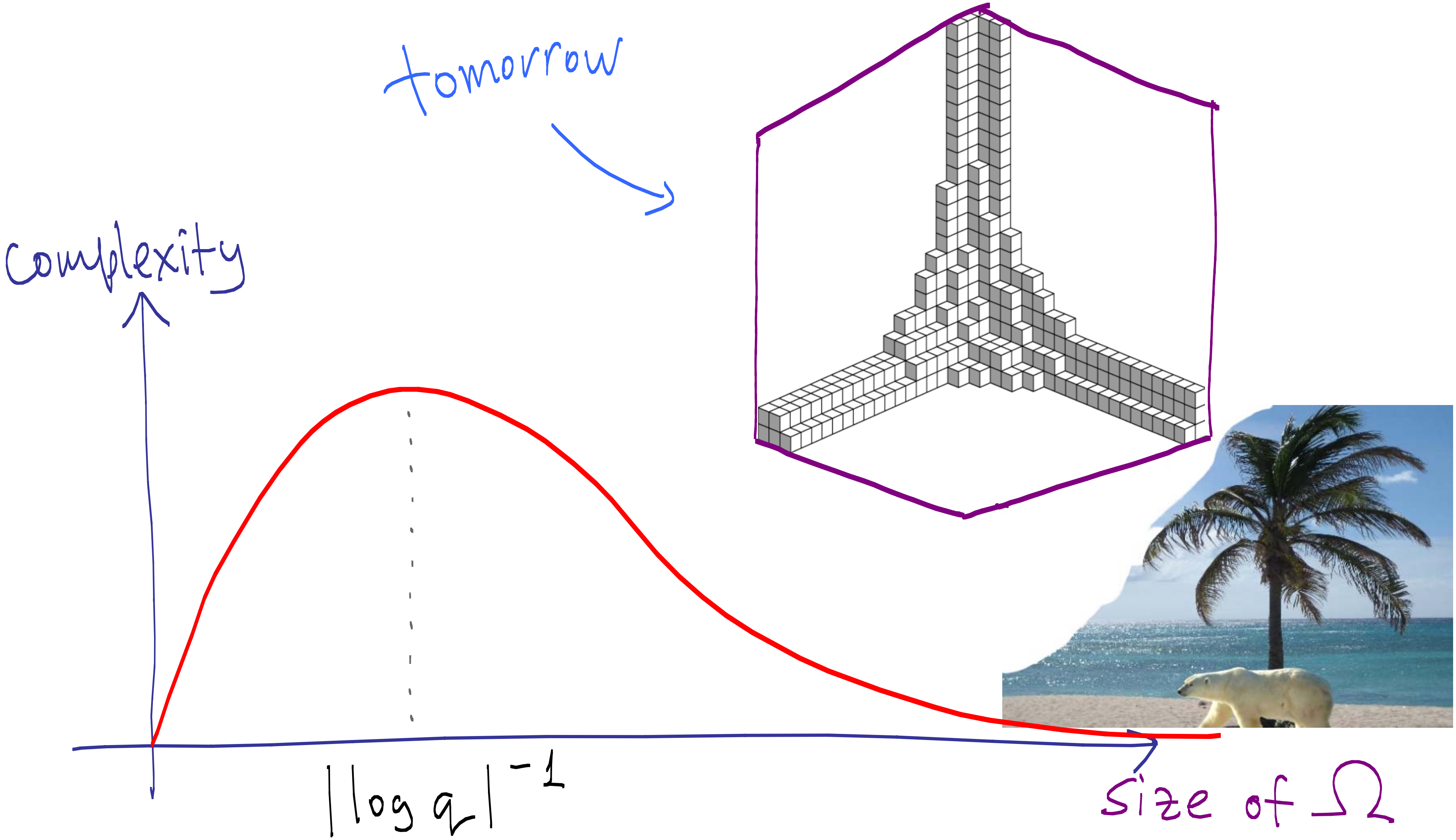
form that defines the quantization

<u>Corr.</u> As $q \to 1$, mesh $\to 0$

$$\operatorname{Supp} \hat{Q} \longrightarrow Q = (\text{frozen boundary})^{\vee}$$

Rational curves form a leaf of GK foliation

Can check e.g. in the tropical limit



Life is easy in the tropics because all is frozen