Higher-order Signature Cocycles for Subgroups of the Mapping Class Group

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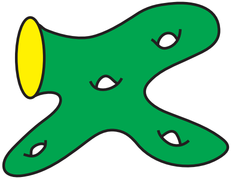
Outline

- Notation
- History
- Group cohomology and quasimorphisms
- Construction of invariants
- Calculations and theorems

- $\Sigma = \Sigma_{g,1} =$



- $\mathcal{M} = \pi_0 \operatorname{Diff}^+(\Sigma, \partial\Sigma)$
- $F = \pi_1 \Sigma = \langle x_1, \dots, x_{2g} \rangle$
- $H \triangleleft F$ a *characteristic subgroup*, i.e. every automorphism of F fixes H setwise

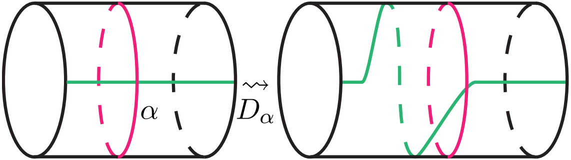
Example

H is a term of the *lower central series of* F.

- $F_1 = F$
- $F_2 = [F, F]$
- $F_3 = [F, F_2]$
- $F_{n+1} = [F, F_n]$

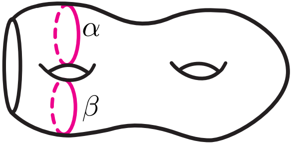
Given H, let $J(H)$ be the subgroup of $\mathcal{M}$ consisting of mapping classes inducing the identity modulo H.
$[f] \in J(H)$ iff $f_* \equiv \mathrm{id} : F/H \to F/H$.

- If $H = F_2$, then $J(H) = \mathcal{I}$ is the *Torelli group*.
- If $H = F_3$, then $J(H) = \mathcal{K}$ is the *Johnson group*.
- $\mathcal{K} < \mathcal{I} < \mathcal{M}$.



Facts

- $\mathcal{I}$ is generated by bounding pair maps



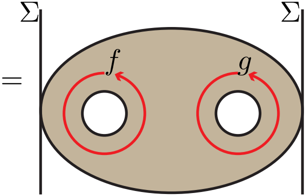
$D_\alpha \circ D_\beta^{-1}$ is a bounding pair map

- $\mathcal{K}$ is generated by Dehn twists along separating curves
- $\mathcal{I}$ is torsion free
- If Σ were closed, $\mathcal{I}$ would be finitely generated. Finitely presented?
- What about $\mathcal{K}$?
- In our case $\Sigma = \Sigma_{g,1}$, what are $\mathcal{I}$, $\mathcal{K}$?

Motivation

- Genus one, closed.
- *Meyer 2-cocycle*: $\sigma_M : \mathcal{M} \times \mathcal{M} \to \mathbb{Z}$, $\sigma_M(f,g) = \operatorname{sign}(W_{f,g})$, where

$W_{f,g} =$



- There is a Meyer function $\rho_M : \mathcal{M} \to \mathbb{Q}$ with $\delta\rho_M = \sigma_M$.

Properties:

M1: ρ_M is a quasimorphism

M2: $\rho_M|_{\mathcal{I}}$ is a homomorphism

M3: σ_M vanishes on $\mathcal{I} \times \mathcal{I}$

M4: σ_M is a bounded cocycle

M5: $[\sigma_M] \in \ker\left(H_b^2(\mathcal{M};\mathbb{Z}) \to H_b^2(\mathcal{M};\mathbb{Q})\right)$

Let G be a group.

Definition

A function $\rho : G \to \mathbb{R}$ is a *quasimorphism* if there exists a constant D_ρ (called the defect) such that for all $f, g \in G$, $|\rho(f) + \rho(g) - \rho(fg)| \leq D_\rho$.

- A homomorphism is a quasimorphism with $D = 0$.
- Let $\widehat{Q}(G)$ denote the vector space of all quasimorphisms, modulo bounded functions.

- A *p-cochain* on a group G is a function $\rho : G \times \cdots \times G \to \mathbb{R}$.
- Let $C^p(G;\mathbb{R})$ be the vector space of p-cochains.
- There is a *coboundary operator* $\delta : C^p \to C^{p+1}$.
- $(\delta\rho^1)(f,g) = \rho(f) + \rho(g) - \rho(fg)$
- $(\delta\rho^2)(f,g,h) = \rho(g,h) - \rho(fg,h) + \rho(f,gh) - \rho(f,g)$
- Note: $\rho : G \to \mathbb{R}$ is a 1-cocycle iff ρ is a homomorphism, so $H^1(G;\mathbb{R}) = \mathrm{Hom}(G,\mathbb{R})$.
- Higher cohomology groups have nice interpretations, too.

We will also consider *bounded cohomology*. The cochains must have bounded image in $\mathbb{R}$.

Key exact sequence:

$$0 \to H^1(G;\mathbb{R}) \to \widehat{Q}(G) \xrightarrow{\delta} H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$$

(Bestvina-Fujiwara) If $G < \mathcal{M}$ is not virtually abelian, then $\dim(\widehat{Q}(G)) = \dim(H_b^2(G)) = \infty$.

Today:

- Define quasimorphisms $\rho_\psi : J(H) \to \mathbb{R}$.
- Define bounded 2-cocycles $\sigma_\psi : J(H) \times J(H) \to \mathbb{Z}$ or $\mathbb{R}$.
- These will be analogous to Meyer's cocycle and function.

Here, ψ is a unitary representation.

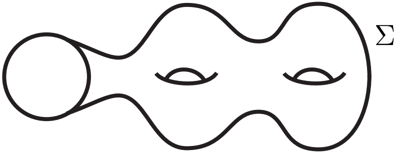
- Finite dimensional case: $\psi : F/H \to U(n)$
- Infinite dimensional case: $\psi : F/H \to U(\ell^{(2)}(F/H))$

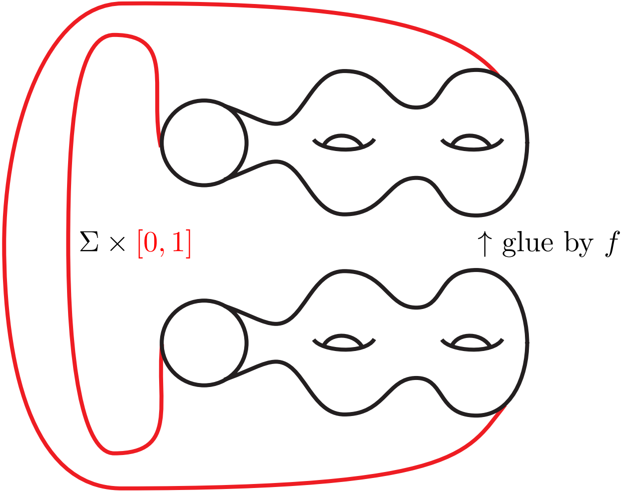
Example

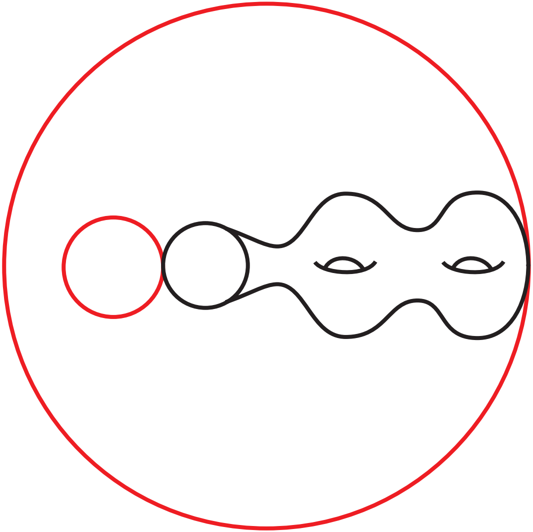
If $H = F_2$, then $F/H = \mathbb{Z}^{2g}$. Choose $\omega \in S^1 = U(1)$ and pick a basis for $\mathbb{Z}^{2g}$. Let $\psi : F/H \to U(1)$ be the representation that maps each basis element to ω.

Step 1:

Associate to $f \in J(H)$ a 3-manifold N_f and a unitary representation $\widetilde{\psi} : \pi_1(N_f) \to U(n)$

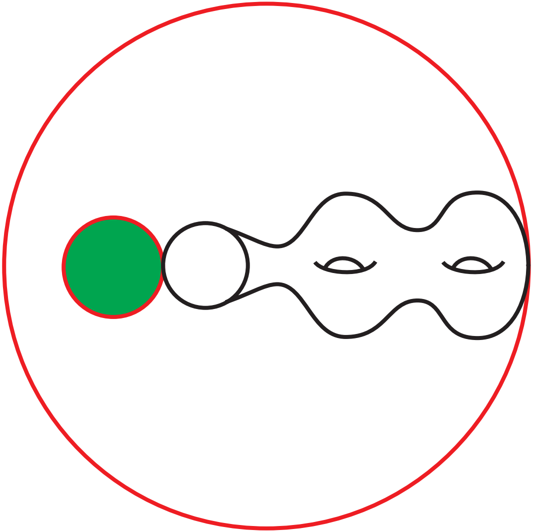



$\Sigma \times [0,1]$
$\uparrow$ glue by f



$\Sigma \times_f S^1$

boundary is $\partial S \times S^1$



Let N_f be the closed 3-manifold by longitudinal Dehn filling on each boundary component.

$$\begin{aligned}\pi_1(N_f) &= \langle x_1, \ldots, x_{2g}, t : \quad t x_i t^{-1} = f_*(x_i)\rangle / \langle t\rangle \\ &= \langle x_1, \ldots, x_{2g} : \quad x_i = f_*(x_i)\rangle\end{aligned}$$

$$\begin{aligned}\pi_1(N_f)/H &= \langle x_1, \ldots, x_{2g} : \quad x_i = f_*(x_i)\rangle / H \\ &= F/H\end{aligned}$$

Take $\widetilde{\psi} : \pi_1(N_f) \twoheadrightarrow \pi_1(N_f)/H = F/H \xrightarrow{\psi} U(n)$.

Step 2:

Define $\rho_\psi(f)$ by

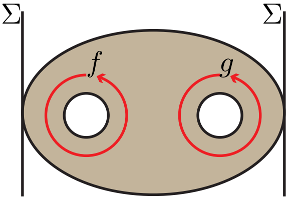
$$\rho_\psi(f) = \rho(N_f, \widetilde{\psi}) = \begin{cases} \text{the Atiyah-Patodi-Singer invariant} \\ \quad\text{(difference of } \eta\text{-invariants),} \\ \quad\text{in the finite dimensional case} \\ \text{the Cheeger-Gromov von Neumann } \rho\text{-invariant,} \\ \quad\text{in the infinite dimensional case} \end{cases}$$

Example

Let M_K denote zero-surgery on K, and $\widetilde{\psi} : \pi_1(M_K) \twoheadrightarrow \mathbb{Z} \to U(1)$ which takes $\mu \mapsto \omega \in S^1$. Then $\rho(M_K, \widetilde{\psi}) = \sigma_\omega(K)$ the Levine-Tristram signature. In the infinite dimensional case, $\rho(M_K, \widetilde{\psi}) = \int_{S^1} \sigma_\omega(K)\, d\omega$.

Step 3:

Define $\sigma_\psi : J(H) \times J(H) \to \mathbb{Z}$ or $\mathbb{R}$. Recall the 4-manifold $W = W(f, g) =$



$\cup$ some other globs

Define

$$\sigma_\psi(f,g) = \begin{cases} \sigma(W;\widetilde{\psi}) - n \cdot \sigma(W), & \text{in the finite dimensional case,} \\ \sigma^{(2)}(W,\widetilde{\psi}) - \sigma(W), & \text{in the infinite dimensional case} \end{cases}$$

$U(n) \subset \mathrm{Aut}(\mathbb{C}^n)$

$C_*(W;\widetilde{\psi}) := C_*(\widetilde{W}) \bigotimes_{\widetilde{\psi}} \mathbb{C}^n$

Let $\langle\ ,\ \rangle$ denote the $\mathbb{Z}[\pi_1 W]$-valued intersection form on $C_2(\widetilde{W})$.

Define $\langle c \otimes \vec{v}, d \otimes \vec{w} \rangle = \vec{v} \cdot \left(\widetilde{\psi}(\langle c, d \rangle) \bigstar \vec{w} \right)^*$

This is a hermitian form!

$\sigma(W;\widetilde{\psi})$ is the signature of this form.

Properties

Proposition 1 (APS, Ramachandran)

Given ψ *and* $f, g \in J(H)$, $\sigma_\psi(f,g) = \rho_\psi(f) + \rho_\psi(g) - \rho_\psi(f\,g)$, *i.e.* $\sigma_\psi = \delta\,\rho_\psi$ *with* $\mathbb{R}$*-coefficients.*

Theorem 2 (CHH)

Given ψ *and* $f, g \in J(H)$,

$$|\sigma_\psi(f,g)| \leq \begin{cases} 4n\,g(\Sigma), & \textit{in the finite dimensional case} \\ 4\,g(\Sigma), & \textit{in the infinite dimensional case} \end{cases}$$

Proposition 3 (CHH)

$\delta\,\sigma_\psi = 0$

Properties

A1 ρ_ψ are quasimorphisms (Prop 1 and Thm 2)

A2 When restricted to a certain subgroup of $J(H)$, ρ_ψ are homomorphisms

A3 When restriced to this subgroup times itself, the σ_ψ vanish (Prop 1)

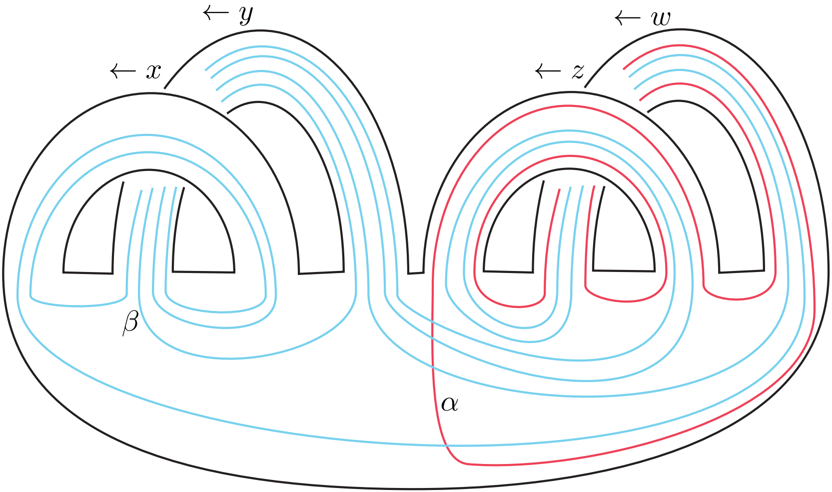
A4 σ_ψ are bounded 2-cocycles (Thm 2 and Prop 3)

A5 $[\sigma_\psi] \in \ker\left(H_b^2\left(J(H);\mathbb{Z}\right) \to H_b^2\left(J(H);\mathbb{R}\right)\right)$ (Prop 1)

Main Example:

Let $r \geq 2$, and fix two curves on $\Sigma_{2,1}$

$$\begin{aligned} \alpha &= [z, w] \\ \beta_r &= [z^{r-1} w^{-1}, x^{-(r+1)} y^{-1}][x^{-1}, y] \end{aligned}$$


$\leftarrow y$
$\leftarrow w$
$\leftarrow x$
$\leftarrow z$
β
α

α and β_r are separating curves, so D_α and $D_\beta \in \mathcal{K}$.
Let $f_{r,N} := (D_\alpha \circ D_{\beta_r})^{N+1}$
Let $\rho_\omega = \rho_\psi$ where $\psi : F/F_3 \twoheadrightarrow F/F_2 = \mathbb{Z}^4 \twoheadrightarrow \mathbb{Z} \to U(1)$ which maps each of x, y, z, w to a chosen ω.

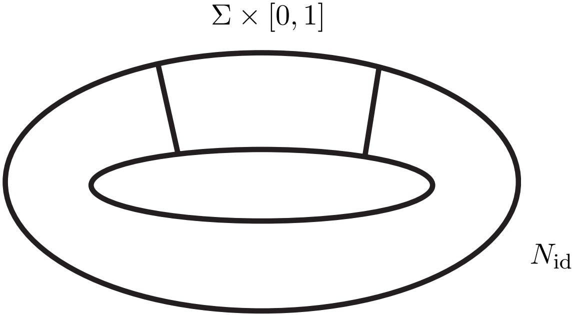
Lemma 4

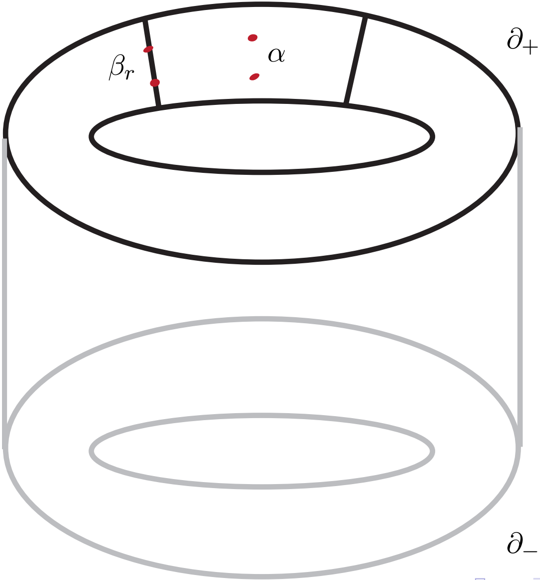
Let $r \geq 2,\ N \geq 0,\ G_r(t) = (t^r - 1)(t^{-r} - 1),\ \omega \neq 1 \in S^1 = U(1)$. *Then* $\rho_\omega(f_{r,N}) + 2(N+1) = \operatorname{sign}(C_{r,N}(\omega))$, *where*

$$C_{r,N}(\omega) = \begin{pmatrix} A & \overline{G_r(\omega)}B^\top \\ G_r(\omega)B & A \end{pmatrix}_{2N \times 2N}$$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}_{N \times N}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

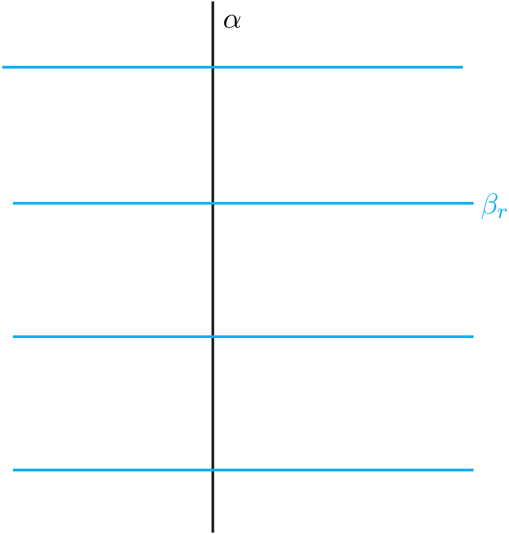
Proof: $\Sigma \times [0,1] \subset N_{\text{id}}$
Let $\mathcal{S} = \{\alpha \times \{1/2\}, \beta_r \times \{0\}\} \subset \Sigma \times [0,1]$
Let $X = N_{\text{id}} \times [0,1] \cup +1$-framed 2-handles along $\mathcal{S} \subset N_{\text{id}} \times \{1\}$

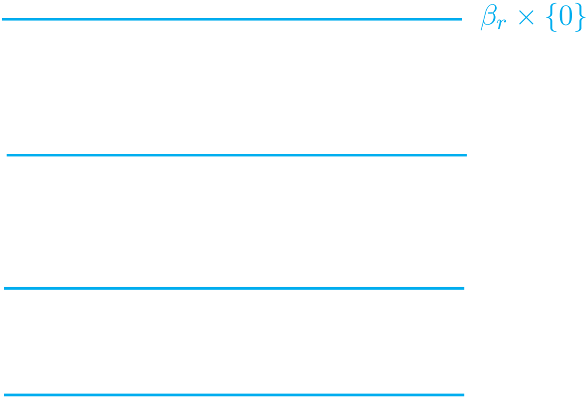

$\Sigma \times [0,1]$
N_{id}


β_r
α
$\partial_+ X = N_{D_\alpha \circ D_{\beta_r}}$
X
$\partial_- X = N_{\mathrm{id}}$

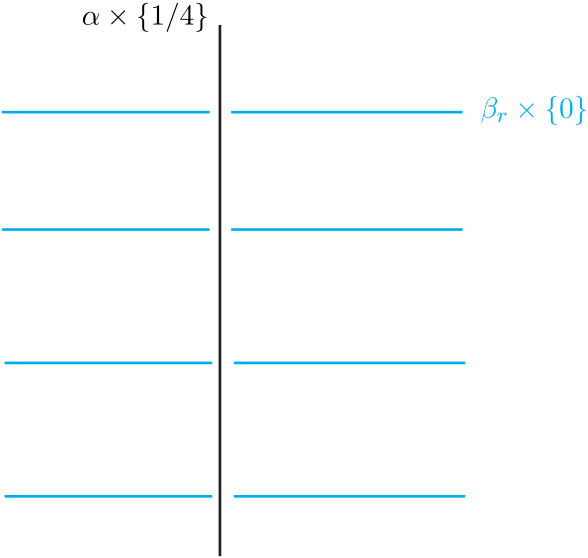
- $N_{\text{id}} = \#_4 S^1 \times S^2 = \partial \natural_4 S^1 \times B^3$
- Let $W = X \cup \natural_4 S^1 \times B^3$
- $\widetilde{\psi} : \pi_1(N_{D_\alpha \circ D_{\beta_r}}) \to U(1)$ extends across W.
- $\therefore \rho_\omega(D_\alpha \circ D_{\beta_r}) = \sigma(W; \widetilde{\psi}) - \sigma(W)$.
- instead of $D_\alpha \circ D_{\beta_r}$, we want powers of this homeomorphism
- Attaching set: $\mathcal{S} = \{\alpha \times \{\frac{2i+1}{2N+2}\}, \beta_r \times \{\frac{2i}{2N+2}\} : i = 0, \ldots, N\}$
- Let's do $(D_\alpha \circ D_{\beta_r})^2$.

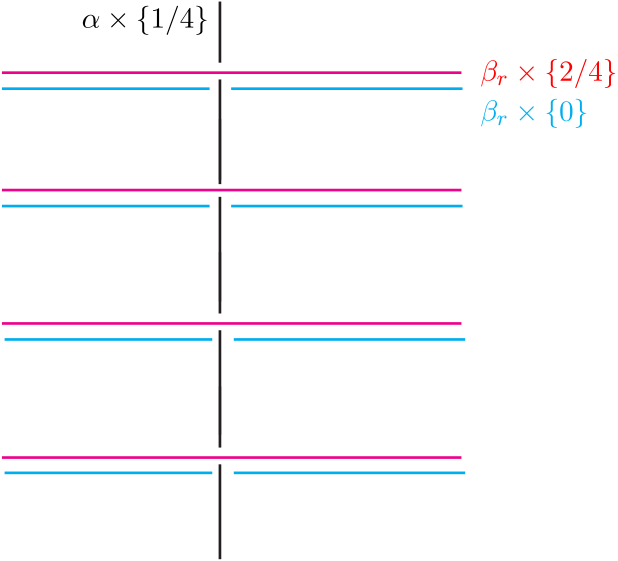
α and β_r interact only in a small rectangle:

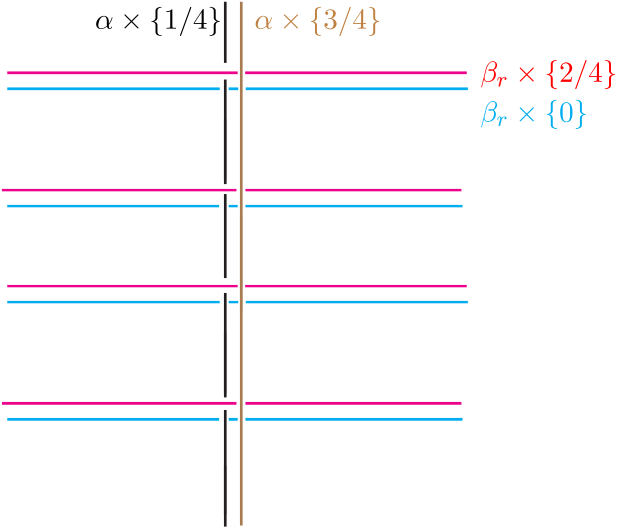




$\beta_r \times \{0\}$


$\alpha \times \{1/4\}$
$\beta_r \times \{0\}$


$\alpha \times \{1/4\}$
$\beta_r \times \{2/4\}$
$\beta_r \times \{0\}$

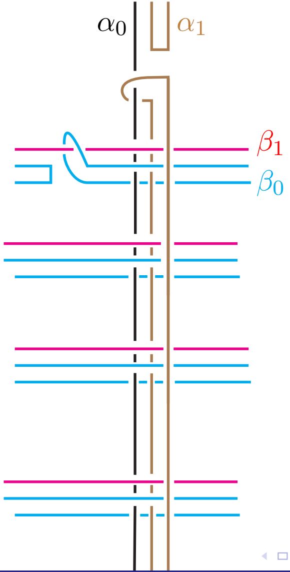

$\alpha \times \{1/4\}$
$\alpha \times \{3/4\}$
$\beta_r \times \{2/4\}$
$\beta_r \times \{0\}$

The $\alpha \times \{t\}$ and $\beta_r \times \{t\}$ curves are nullhomologous on $\Sigma \times \{t\}$.
In fact, they bound disjoint surfaces in W
After attaching $+1$-framed 2-handles, we see the intersection form of W is

$$\bigoplus_{2N+2} \langle +1 \rangle$$

so $\sigma(W) = 2N + 2$

Slide $\alpha \times \{3/4\}$ over $\alpha \times \{1/4\}$ (and same for β_r's).



α_0 and β_1 do not contribute to $\sigma(W;\widetilde{\psi})$.
α_1 and β_0 bound discs now.
These discs intersect geometrically 4-times
Taking into account these intersections with $\mathbb{Z}[\pi_1 W]$-weights, we get the expression

$$-1 + z^{r+1}w^{-1} - z^{r+1}w^{-1}z^{1-r}y^{-1} + z^{r+1}w^{-1}z^{1-r}y^{-1}wz^{r+1}$$

Using $\widetilde{\psi}$ to map to $U(1)$, this expression becomes $-G_r(\omega)$.
End of proof of lemma.

Lemma 5

$$sign\left(C_{r,2N}(\omega)\right) = \begin{cases} 4N, & \textit{if } \omega^r = 1 \\ 0, & \textit{if } \omega^r = \pm i \end{cases}$$

Proof.

In the first case, the matrix is $A \oplus A$. In the second, the matrix is

$$\begin{pmatrix} A & 2B^\top \\ 2B & A \end{pmatrix}$$

Have fun. ☐

Main results

Let $k \geq 1$, $\omega_k := \exp(2\pi i/4^k)$, and $\rho_k = \rho_{\omega_k}$.

Theorem A

The set of quasimorphisms $\{\rho_k\}$ is a linearly independent set in $\widehat{Q}(\mathcal{K})$.

Proof:
Suppose $\sum_{l=1}^{n} a_l\, \rho_{k_l} = 0$ in $\widehat{Q}(\mathcal{K})$, i.e. $\sum a_l\, \rho_{k_l}$ is a bounded function. $(k_1 < \cdots < k_n$, and $a_l \neq 0)$
Recall $f_{r,N} = (D_\alpha \circ D_{\beta_r})^{N+1}$. Note

$$\omega_k^{4^j} = \begin{cases} i, & \text{if } j = k-1 \\ 1, & \text{if } j \geq k \end{cases}$$

By Lemmas 4 and 5,

$$\rho_k(f_{4^j,2N}) = \begin{cases} -2(2N+1), & \text{if } j = k-1 \\ -2, & \text{if } j \geq k \end{cases}$$

Behavior of ρ_k on $f_{4^{k-1},2N}$:

grows linearly in N!

But this family of homeomorphisms $f_{4^{k-1},2N}$ *is not detected by* $\rho_{k'}$ if $k' < k$. This ends the proof.

Remark

Every nontrivial linear combination of ρ_k is unbounded on the subgroup generated by $D_\alpha D_{\beta_r}$ *for some* r.

Lemma 6

Let $D = D_\alpha$ *or* D_{β_r}. *On the subgroup generated by* D, *each* ρ_k *is a bounded function.*

Theorem B

The set of signature 2-cocycles $\{\delta\rho_k\}$ is a linearly independent set $H_b^2(\mathcal{K};\mathbb{R})$.

Proof.

Recall the exact sequence

$$0 \to H^1(G;\mathbb{R}) \to \widehat{Q}(G) \xrightarrow{\delta} H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R}).$$

Thus, $\sum a_l\,\delta\rho_{k_l} = 0$ iff $\sum a_l\,\rho_{k_l} = \phi + b$, where ϕ is a homomorphism and b is a bounded function. By Lemma 6, ϕ must be trivial on the subgroup generated by D_α and D_{β_r}. Thus this linear combination must be bounded, which we disproved in Theorem A. ☐