Abstract. $p$-adic Hodge theory, broadly speaking, is the study of representations of the absolute Galois group of $\mathbb{Q}_p$ acting on $\mathbb{Q}_p$-vector spaces. Classical Hodge theory concerns the relationship between the singular and de Rham cohomologies of a compact Kähler manifold. $p$-adic Hodge theory began as the search for a similar theory relating the étale and de Rham cohomologies of varieties over $p$-adic fields. Over time, however, it’s grown more broadly into a subject with its own rich inner life and with many applications in number theory.

During these two talks, I hope to explain some of the motivation and philosophy of $p$-adic Hodge theory, as well as some of its successes. Some applications which I hope to touch on include: properties of Galois representations associated with modular forms, good reduction of abelian varieties, modularity of elliptic curves, the Fontaine–Mazur conjecture, special values of $L$-functions, and some results on Hodge numbers of varieties over number fields.

Let us start with some notations which will be in effect the whole time. Let $p$ be a prime, and let $K$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}_p$ in $K$, with maximal ideal $\mathfrak{m}_K = (\varpi_K)$ and residue field $k = \mathcal{O}_K/\mathfrak{m}_K$. Let $\overline{K}$ be an algebraic closure of $K$. Our main object of study is $G_K = \text{Gal}(\overline{K}/K)$.

Definition 1. A $p$-adic representation of $G_K$ is a $\mathbb{Q}_p$-vector space $V$ equipped with a (continuous) action of $G_K$.

There are cases where $V$ is not finite-dimensional, and continuity requires a bit more care.

Let us recall the structure of $G_K$. There is an exact sequence

$$1 \rightarrow I_K \rightarrow G_K \rightarrow G_k \cong \hat{\mathbb{Z}} \rightarrow 1$$

where $I_K$ is the inertia subgroup. It sits in another exact sequence

$$1 \rightarrow P_K \rightarrow I_K \xrightarrow{\iota_p} \prod_{\ell \neq p} \mathbb{Z}_\ell \rightarrow 1$$

where $P_K$ is the “wild” inertia subgroup. The main thing for now is that $P_K$ is a pro-$p$ group. We can think of $G_K$ as having a filtration with three pieces: $\hat{\mathbb{Z}}$, almost $\hat{\mathbb{Z}}$ with $p$-part stripped away, and $P_K$. $P_K$ is what makes the representations of $G_K$ subtle in $p$-adic situations.

As a consequence, any continuous $\rho : G_K \rightarrow \text{GL}_n(\mathbb{Q}_\ell)$ with $\ell \neq p$ satisfies that $\rho(P_K)$ is finite. $p$-adic representations are much wilder than $\ell$-adic representations.

Let me give some examples.

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Example 2. Let $\chi : G_K \to L^\times$ be any continuous character, where $L/\Q_p$ is a finite extension. Take $V = L$ with action given by $g \cdot v = \chi(g)v$. We denote this as “$L(\chi)$”.

Example 3. $\Z_p(1) := \{(1 = x_0, x_1, x_2, \ldots) \in K, x_{i+1}^p = x_i \text{ for all } i \geq 0\}$. This is non-canonically isomorphic to $\Z_p$, and has an action by $G_K$. There exists a unique $\chi_{\text{cyc}} : G_K \to \Z^\times_p$ such that $g \cdot x = (x_0^{\chi_{\text{cyc}}(g)}, x_1^{\chi_{\text{cyc}}(g)}, \ldots)$. In other words, $g(\zeta) = \zeta^{\chi_{\text{cyc}}(g)}$ for any $p^\infty$-th root of unity $\zeta \in \overline{K}$.

Define $\Q_p(1) = \Z_p(1) \otimes_{\Z_p} \Q_p$, and $M(n) := M \otimes_{\Q_p} \Q_p(1)^{\otimes n}$ where $M$ is any $\Q_p[G_K]$-module. Note $\Q_p(1) \simeq \Q_p(\chi_{\text{cyc}})$, but non-canonically so.

Example 4. Let $A$ be an abelian variety over $K$. The $\ell$-adic Tate module is $T_\ell A := \lim_{\leftarrow n} A(K)[\ell^n]$ which is a finite free $\Z_\ell$-module of rank $2 \dim A$. The rational Tate module is $V_\ell A := T_\ell A \otimes_{\Z_\ell} \Q_\ell$.

We will start posing some questions which we will eventually answer using $p$-adic Hodge theory. First we have a theorem of Serre and Tate, sometimes called the Néron–Ogg–Shafarevich criterion.

Theorem 5 (Serre–Tate). A has good reduction if and only if $I_K$ acts trivially on $V_\ell A$ for some (any) $\ell \neq p$.

A natural question is:

Question 1. Is there a similar good reduction criterion involving $V_p A$?

Example 6. Let $\Delta = q \prod_{n=1}^\infty (1 - q^n)^{24} = \sum_{n=1}^\infty \tau(n)q^n$.

Theorem 7 (Deligne). There exists a unique representation $\rho_{\Delta,p} : G_{\Q} \to \GL_2(\Q_p)$ such that $\text{tr } \rho_{\Delta,p}(\text{Frob}_\ell) = \tau(\ell)$ for any prime $\ell \neq p$.

Note $\text{Frob}_\ell \in G_{\Q} \hookrightarrow G_{\Q}$.

Question 2. Can we recover $\tau(p)$ from $\rho_{\Delta,p}|_{G_{\Q_p}}$ somehow?

Let me give one more example which leads to more general things.

Example 8. Let $X$ be any smooth proper variety over $K$. Then $H^n_{\text{et}}(X_{\overline{K}}, \Q_p) := \left( \lim_{\leftarrow j} H^1_{\text{et}}(X_{\overline{K}}, \Z/p^j\Z) \right) \otimes_{\Z_p} \Q_p$ is a $p$-adic representation.

If $X = A$ and $n = 1$, then $H^1_{\text{et}}(A, \Q_p) \cong \text{Hom}_{\Q_p}(V_p A, \Q_p)$.

What sort of structure might we expect from such an object? Let us recall the complex analogue. Let $X$ be a compact Riemannian manifold. Then the singular cohomology is related to differential objects via the de Rham isomorphism $H^n(X, \Z) \otimes \C \cong H^n_{\text{dR}}(X, \C)$.
If $X$ has more structure, say $X$ is Kähler, then

$$H^n_{dR}(X, \mathbb{C}) = \bigoplus_{i} H^{n-i}(X, \Omega^i_X)$$

where $\Omega^i_X$ is the sheaf of holomorphic $j$-forms.

**Question 3.** Is there some relationship between the étale cohomology groups $H^*_\text{ét}(X_{\overline{K}}, \mathbb{Q}_p)$ and $H^*_\text{dR}(X)$ or $H^*(X, \Omega^i_X)$?

The latter groups are very coherent and should really be thought of as the analogous objects of holomorphically defined cohomology.

**Conjecture 9** (Tate (1967), proved by Faltings (1989)). For any $X$ as above, there is a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{Q}_p \cong \bigoplus_{i=0}^n H^{n-i}(X, \Omega^i_X) \otimes_K \mathbb{C}_p(-i)$$

where $\mathbb{C}_p$ is the $p$-adic completion of $\overline{K}$.

Tate proved this when $X = A$ is an abelian variety and $n = 1$.

There is a natural action of $G_K$ on both sides: on the left it acts diagonally, and on the right it is trivial on the Hodge cohomology groups. We may ask if this isomorphism is $G_K$-equivariant. There is a remarkable corollary. Before stating it we need another theorem.

**Theorem 10** (Tate (1967)).

$$H^0(G_K, \mathbb{C}_p(j)) = \begin{cases} K & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Then the conjecture implies that

**Corollary 11.**

$$\left( H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{Q}_p \mathbb{C}_p(j) \right)^{G_K} \cong \left( \bigoplus_{i=0}^n H^{n-i}(X, \Omega^i_X) \otimes_K \mathbb{C}_p(-i+j) \right)^{G_K} = H^{n-j}(X, \Omega^i_X).$$

Thus we have recovered the Hodge cohomology groups in some Galois-theoretic way.

The general philosophy and goals of $p$-adic Hodge theory are:

- Define and study interesting subcategories of the category of all $p$-adic representations of $G_K$.
- Relate them to representations occurring “in nature”.
- Use these ideas to solve actual problems.

One overarching philosophy of how to do these is due to Fontaine: define “interesting” period rings $B$, which are topological $\mathbb{Q}_p$-algebras with a $G_K$-action and some extra structures. For any $p$-adic representation $V$, one can form

$$\mathbb{D}_B(V) = (V \otimes \mathbb{Q}_p B)^{G_K}.$$

This is a module over $B^{G_K}$, which tends to be a field. This module inherits whatever extra structure $B$ has.

Let me try to recast the Hodge–Tate conjecture.
Example 12.

\[ B_{HT} := \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i). \]

By the theorem of Tate and the relation \( \mathbb{C}_p(i) \otimes_{\mathbb{C}_p} \mathbb{C}_p(j) = \mathbb{C}_p(i + j) \),

\[ B_{HT}^{G_K} = \bigoplus_{i} \mathbb{C}_p(i)^{G_K} = K. \]

We can restate the conjecture as follows. If \( V = H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \), then

\[ \mathcal{D}_{B_{HT}}(V) \cong \bigoplus_{j} H^{n-j}(X, \Omega^j_X). \]

The interesting period rings are \( B_\bullet \), where \( \bullet \in \{ \text{HT, dR, st, crys} \} \). These stand for “Hodge–Tate, de Rham, semistable and crystalline” respectively. For each of these adjectives, one defines

\[ \mathcal{D}_\bullet(V) := \mathcal{D}_{B_\bullet}(V). \]

\( \mathcal{D} \) is for Dieudonné.

For any one of these, one has a natural map

\[ \alpha_\bullet : \mathcal{D}_\bullet(V) \otimes_{B_\bullet^{G_K}} B_\bullet \rightarrow V \otimes_{\mathbb{Q}_p} B_\bullet \]

which is always injective. The proof is this is not trivial and requires a careful analysis of each of these rings.

**Definition 13.** \( V \) is “blah” if \( \alpha_{\text{blah}} \) is an isomorphism.

Let me spell out what it means to be a Hodge–Tate representation, which is the simplest category of all.

**Example 14.** \( V \) is Hodge–Tate if and only if

\[ V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i} \mathbb{C}_p(i)^{\oplus n_i} \]

with \( \sum_i n_i = \dim_{\mathbb{Q}_p} V \).

For any variety \( X \), \( H^n(X_{\overline{K}}, \mathbb{Q}_p) \) is Hodge–Tate. Indeed we have a more precise decomposition in terms of Hodge cohomology.

An answer to Question \( \square \) is given by the

**Theorem 15** (Coleman–Iovita). \( A \) has good reduction if and only if \( V_p A \) is crystalline.