PROGRAM FOR THE SEMINAR: GEOMETRY OF FUNDAMENTAL LEMMAS

1. Introduction

The Fundamental Lemma of Langlands-Shelstad (LSFL) is a pivotal ingredient in the Langlands program. It has appeared indispensable to many major modern-day achievements in number theory, including Fermat's last theorem, the local Langlands correspondence, the zeta function of Shimura varieties, the construction of Galois representations, the Birch and Swinnerton-Dyer formula, just to name a few. It also has variants in the setting of relative trace formulas, with applications to the Gan-Gross-Prasad conjectures, etc. After decades of work of many people, the LSFL was proved by Ngô Bao Châu around 2010. The proof was geometric in nature, a beautiful blend of algebraic geometry and topology, using Hitchin fibrations, Weil conjectures, perverse sheaves, etc.

The aim of our seminar is to study the geometric nature of this proof. To make it more accessible (even for non number-theorists), we shall first study Yun's proof [Yun11] of the Jacquet-Rallis Fundamental Lemma (JRFL). This should minimize the prerequisites from automorphic forms and trace formulas in order to understand Ngô's key geometric ideas. All are welcome to participate.

2. References

In the first stage of this seminar we will study the paper [Yun11]. For Ngô's proof of the original LSFL, there are many great expository articles apart from the original references [Ngô10] and [Ngô06]. Ngô's ICM report [Châ10] and Utah lectures (available on his website) are highly recommended. There is also Nadler's article [Nad12], and Hales's Bourbaki report [Hal12], which discuss motivations. The pair of chapters [DDT11] [Châ11] in the Paris book project is also a great introduction to the subject.

3. Tentative list of the first few talks

1. Global motivation for the fundamental lemmas. Suvery of results. List of major applications. Discuss both endoscopy case and GGP case (two talks if better);

2. [Yun11] 2.1-2.5.2: Define orbital integrals; Interpret them as lattice counting; State the Jacquet-Rallis fundamental lemma (JRFL).

3. [Yun11] 2.5.3-2.7, (cf. also [JR11]): Prove simple cases of JRFL; Dedcue the group version from the Lie algebra version; (Discuss Jacquet-Rallis's original computation for n=3;) Give geometric reformulation of JRFL using local moduli spaces.

4: [Yun11] 3.1-3.3: Introduce global moduli spaces and spetral curves;

5. [Yun11] 3.4-3.5: Prove the product formula and the smallness of the invariant maps.

6. [Yun11] 4: Prove the global matching theorem.

7. [Yun11] 5: Prove JRFL using the global matching theorem.

Some of the talks may be easier/more difficult and we can adjust the schedule accordingly.

4. Future plans

After [Yun11] we could continue to read the more difficult papers of Ngô [Ngô10] [Ngô06], and maybe the more recent developments in these geometric methods, e.g. the "non-anisotropic case" [CL10] [CL12] or the geometrization [FN11]. Alternatively, after studying [Yun11] we could try to understand how to prove the global Gan-Gross-Prasad using JRFL following Wei Zhang's papers [Zha14a] [Zha14b]. We shall decide the route according to people's interest.

References

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