1. Introduction

Let me start with some motivations and generalities about $p$-adic $L$-functions.

Classical $L$-functions:
- These are defined by Euler products in some half-plane.
- Sometimes with hard work, we can establish their expected analytic properties (e.g. converse theorems).
- Zeros and special values hold deep arithmetic meaning (e.g. class numbers).

Often $p$-adic $L$-functions are the right bridge between these and the arithmetic objects.

$p$-adic $L$-functions:
- These admit no naïve definition.
- In general, given $M$ a “motive” or “algebraic automorphic representation”, a $p$-adic $L$-function associated with $M$ tends to have two properties:
  - it interpolates the special values $L(n, M \otimes \chi) \ p$-adically, for $n$ in some interval and $\chi$ a $p$-power conductor Dirichlet character;
  - it has reasonable growth properties.

One of the funny things is that even for individual $M$ (e.g. elliptic curves), there’s typically more than one $p$-adic $L$-function associated to it. Sometimes these properties determine it uniquely; sometimes they don’t. Sometimes the first property fails.

The point is this talk is to generalize modular forms over $\mathbb{Q}$ to automorphic representations of $GL(2)$ over arbitrary number fields.

2. Classical Modular Forms

Example 1. Let $f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z} \in S_k(\Gamma_1(N))$ be a newform and Hecke eigenform. The classical $L$-function is defined by

$$L(s, f) = \sum_{n \geq 1} a_n n^{-s} = \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s} \epsilon_f(p)} \prod_{p|N} \cdots.$$
Hecke proved that $L(s, f)$ is holomorphic in the $s$-plane. The key idea is to consider

$$\Lambda(s, f) = (2\pi)^{-s} \Gamma(s) L(s, f) = \int_0^\infty f(iy) y^{s-1} dy$$

which is easy to prove once you're told of this formula.

It turns out the special values of this function between 0 and $k$ have lots of symmetry to them. More precisely, Eichler and Shimura (in the 50’s and 60’s) showed that there exist nonzero complex numbers $\Omega_f^\pm \in \mathbb{C}$ such that for any $1 \leq j \leq k - 1$ and any Dirichlet character $\chi$, the ratio

$$L^{\text{alg}}(j, f \otimes \chi) = \frac{\Lambda(j, f \otimes \chi)}{\Omega_\pm \cdot \tau(\chi)}$$

lies in the field $\mathbb{Q}(\cdot \cdot \cdot, a_n, \cdot \cdot \cdot, \text{values of } \chi)$ (where $\chi(-1) = \pm (-1)^j$).

Now we have some algebraic numbers that we want to interpolate $p$-adically. The first people to realize this were Mazur and Swinnerton-Dyer. To get a $p$-adic $L$-function, we need one more piece of data, which explains the multiplicity of $p$-adic $L$-functions I mentioned before. We need to choose a refinement of $f$: a root $\alpha$ of $X^2 - a_p X + p^{k-1} \epsilon_f(p)$. It is conjectured that this always has two distinct roots. This is known in weight 2, and in higher weights it would follow from the Tate conjecture.

Let $G$ be the $\mathbb{Q}_p$-rigid analytic space such that $G(\overline{\mathbb{Q}_p}) \cong \text{Hom}_{cts}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}_p}^\times)$. This should be thought of as $p - 1$ copies of the open unit ball, since $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)$, via $\chi \mapsto (\chi_{\mu_{p-1}}, \chi(1+p) - 1)$. (For those of you who don’t know rigid analytic spaces, this is just some disconnected space, but Tate realized there is a way to make it act like a connected space where we can talk about sheaves.)

**Theorem 2** ($v_p(\alpha) = 0$ by Mazur–Swinnerton-Dyer, $v_p(\alpha) < k - 1$ by Amice–Vélu and Višik). There exists a unique $L_p(f, \alpha) \in \mathcal{O}(G) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\cdot \cdot \cdot, a_n, \cdot \cdot \cdot)$ such that for any character $\psi$ of the form $\psi(x) = x^j \cdot \chi(x)$ with $1 \leq j \leq k - 1$ and $\chi$ of finite order, we have

$$L_p(f, \alpha)(\psi) = (\cdot \cdot \cdot) \cdot L^{\text{alg}}(j, f \otimes \chi^{-1})$$

and $L_p(f, \alpha)$ doesn’t grow too quickly.

Why would a function like this be interesting? In two words, the answer is Iwasawa theory. In its roughest form, Iwasawa theory relates (conjecturally) these functions and arithmetically defined groups. For example, if we adjoin all $p$-power roots of unity to $\mathbb{Q}$, these groups relate to Selmer groups and Selmer modules. There’s a very precise conjecture that says these objects are related in some way. This has led to progress on the Birch–Swinnerton-Dyer conjecture. In many cases where BSD is known, it goes through this theorem in the form of what’s called the Main Conjecture. For example, Kato proved $\text{ord}_{\psi=x} L_p(E, \alpha) \leq \text{rank } E(\mathbb{Q})$ if $v_p(\alpha) = 0$.

For many years, $p$-adic $L$-functions were defined by constructing $p$-adic measures on $\mathbb{Z}_p$. In the mid 90’s, Glenn Stevens realized there is a very clean way to do this which is analogous to considering the integral (1) above.
How to prove these results? Let me first draw a picture before explaining it. For \( k \geq 2 \),

\[
S_k(\Gamma_1(N)) \xrightarrow{\omega} H^1(\Gamma_1(N), \mathcal{L}_{k-2})
\]

\[
\downarrow i
\]

\[
H^1(\Gamma_1(N), \mathcal{D}_{k-2})
\]

Let \( \mathcal{L}_\kappa(R) = R[X]_{\deg \leq \kappa} \) with the left action

\[
(\gamma \cdot p)(X) = (a + cX)^\kappa p \left( \frac{b + dX}{a + cX} \right)
\]

for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \).

Define a map \( \omega : S_k(\Gamma_1(N)) \rightarrow H^1(\Gamma_1(N), \mathcal{L}_{k-2}(\mathbb{C})) \) by

\[
\omega(f)(\gamma) = \int_{z_0}^{z_1} f(z)(z + X)^{k-2}dz \in H^1(\Gamma_1(N), \mathcal{L}_{k-2}(\mathbb{C})).
\]

It turns out that \( H^1(\Gamma_1(N), \mathcal{L}_{k-2}(\mathbb{C})) \cong S_k(\Gamma_1(N)) \oplus M_k(\Gamma_1(N)) \).

One thing to notice is that if we take \( z_0 \) to be 0 (which is not quite legal!) and \( \gamma \) to take 0 to the cusp at \( \infty \), this integral resembles the one defining \( \Lambda(s, f) \).

The \( \Omega^\kappa \) are defined such that they take a rational basis of \( H^1 \) to the canonical basis of \( S_k(\Gamma_1(N)) \oplus M_k(\Gamma_1(N)) \) under the isomorphism above.

Let \( R \) be any \( \mathbb{Q}_p \)-Banach algebra. Let \( \mathcal{A}_\kappa(R) = \{ f : \mathbb{Z}_p \rightarrow R \text{ locally constant} \} \) with the right action of \( \Gamma_0(p) \)

\[
(f \cdot \gamma)(x) = (cx + d)^\kappa f \left( \frac{ax + b}{cx + d} \right)
\]

for \( \gamma \in \Gamma_0(p) \) (note that this is well-defined since \( cx + d \) is a \( p \)-adic unit). Let \( \mathcal{D}_\kappa \) be the topological \( R \)-dual of \( \mathcal{A}_\kappa \) with its natural Fréchet topology and the dual left action of \( \Gamma_0(p) \). Its elements are distributions.

Stevens showed that the map \( \mathcal{D}_\kappa(\mathbb{Q}_p) \rightarrow \mathcal{L}_\kappa(\mathbb{Q}_p) \) given by

\[
\mu \mapsto \int (x + X)^\kappa \mu(x) := \sum_{i=0}^{\kappa} \binom{\kappa}{i} \mu(x^{\kappa-i})
\]

is \( \Gamma_0(p) \)-equivariant and surjective. We get a map

\[
i : H^1(\Gamma_1(Np), \mathcal{D}_{k-2}(\mathbb{Q}_p)) \rightarrow H^1(\Gamma_1(Np), \mathcal{L}_{k-2}(\mathbb{Q}_p))
\]

which is equivariant for the Hecke actions of \( T_l \) (for \( l \nmid Np \)) and \( U_p \).

**Theorem 3** (Stevens). The map \( i \) is an isomorphism on the subspace where the eigenvalues of \( \mathbb{Q}_p \) have valuation \( < k - 1 \).

Now do the following. Given \( f \) and \( \alpha \) as before, there exist elements

\[
\psi_{f,\alpha}^+ \in H^1(\Gamma_1(Np), \mathcal{L}_{k-2}(\mathbb{Q}_p(f, \alpha)))
\]

and
such that $T_i \cdot v = a_i \cdot v$, $U_p \cdot v = \alpha \cdot v$ and $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \cdot v = \pm v$. The elements $v_{f,\alpha}^\pm$ are unique up to scaling. Using $i$, we lift $v_{f,\alpha}^\pm$ canonically to

$$\delta_{f,\alpha}^\pm \in H^1(\Gamma_1(Np), D_{k-2}(Q_p(f, \alpha))) \sim \text{Hom}(\text{Div}^0 \mathbb{P}^1(Q), D_{k-2})_{\Gamma_1(Np)}$$

(the latter with action $l(\gamma D) = \gamma \cdot l(D)$). Now $\delta_{f,\alpha}^\pm((\infty - 0)) \in D(Q_p(f, \alpha))$ turns out to be $L_p(f, \alpha)$ in the sense that we can evaluate this function on all characters and it satisfies the interpolation properties.

3. GL(2) over Number Fields

For the remainder of this talk, we will see how this picture generalizes to GL(2) over arbitrary number fields. I will only make those assumptions that are morally necessary.

Fix $\iota: \mathbb{C} \sim \overline{\mathbb{Q}}_p$. Fix $F/\mathbb{Q}$ of degree $d = r_1 + 2r_2$. Let $\Sigma = \text{Hom}(F, \mathbb{C}) \sim \text{Hom}(F, \overline{\mathbb{Q}}_p)$. $\sigma$ will denote an element of $\Sigma$. If $v$ is a place over $p$, $\sigma | v$ means $\iota \circ \sigma$ induces $v$.

**Definition 4.** A cohomological weight is a pair $((\kappa_\sigma)_{\sigma \in \Sigma}, w) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$ such that

1. $2 \mid (\kappa_\sigma - w)$ for all $\sigma \in \Sigma$;
2. $\kappa_\sigma \geq 0$.

Let $L_{\kappa, w}(C) = \mathbb{C}[\cdots, X_\sigma, \cdots]_{\sigma \in \Sigma, \deg X_\sigma \leq \kappa_\sigma}$. Define

$$(\gamma \cdot p)(X) = (a + cX)^{\kappa} p \left( \frac{b + DX}{a + cX} \right) (\det \gamma)^{\frac{w - \kappa}{2}}$$

for $\gamma \in \text{GL}_2(F)$, where we use shorthand notations like

$$(a + cX)^{\kappa} := \prod_{\sigma} (\sigma(a) + \sigma(c)X_\sigma)^{\kappa_\sigma}.$$
The analogue of Stevens’ theorem is less satisfactory in this setting, so we will just define
the key property we want. Denote $Y = Y_1(np)$ for simplicity.

**Definition 5.** $(\pi, \alpha)$ is noncritical if the induced map $H^1(Y, D_{\kappa,w}) \to H^*(Y, L_{\kappa,w})$ is an
isomorphism after localizing at $m_{\pi,\alpha}$.

For each sign $\epsilon$, let $\phi_{\pi,\alpha}^\epsilon$ be a generator. Then there exists a canonical lift $\delta_{\pi,\alpha}^\epsilon \in H^q(Y, D_{\kappa,w})$.

It is much harder to define modular symbols in this setting, let alone work with them in
a meaningful way.

Let $\Gamma_F$ be the maximal quotient of $A_F^\times$ such that any $p$-adically continuous Hecke character
unramified outside of $p$ and $\infty$ factors through $\Gamma_F$. In other words,

$$\Gamma_F = A_F^\times/(F^\times \cdot \mathcal{O}_F^{(p)} \cdot F_{\infty,+})^\times.$$

This is still quite abstract. The way to understand this is that it sits in the exact sequence

$$1 \to (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times / \mathcal{O}_F^{\times +} \to \Gamma_F \to \text{Cl}_F^+ \to 1.$$

Then $D(\Gamma_F)$ is the space of locally analytic distributions on $\Gamma_F$, i.e. the dual of locally
analytic $f : \Gamma_F \to O_p$.

**Theorem 6.** There exists a canonical linear map

$$\text{Per}_{\kappa,w} : H^q(Y, D_{\kappa,w}(R)) \to D(\Gamma_F) \otimes_{\mathbb{Q}_p} R$$

which “reverses modular symbols” when $F = \mathbb{Q}$.

The idea is to define some Shintani cone $C_\infty \subset Y$ of dimension $q$ and $\beta : \mathcal{A}(\Gamma_F) \to
H^0(C_\infty, t^* A_{\kappa,w})$, and consider

$$H^q_c(Y, D_{\kappa,w}) \xrightarrow{\text{req}} H^q_c(C_\infty, t^* D_{\kappa,w}) \to H_c(C_\infty, t^* D_{\kappa,w})^{\text{cotangent}} \to D(\Gamma_F).$$

The final result is that we can use these to define $p$-adic $L$-functions.

**Definition 7.**

$$L_p(\pi, \alpha) = \bigoplus_{\epsilon} \text{Per}_{\kappa,w}(\delta_{\pi,\alpha}^\epsilon) \in D(\Gamma_F).$$

**Theorem 8.** $L_p(\pi, \alpha)$ enjoys the following properties:

1. **interpolation:** $L_p(\pi, \alpha)(Nj \cdot \chi) = (\cdots)L^{\text{alg}}(j, \pi \otimes \chi^{-1})$

   for all $\frac{w-\kappa^-}{2} + 1 \leq j \leq \frac{w+\kappa^-}{2} + 1$, where $\kappa^- = \inf_{\sigma} \kappa_{\sigma}$

2. **growth**

3. they vary well in $p$-adic families

4. If $F$ is totally real and Leopoldt, then (1), (2), (3) determine $L_p(\pi, \alpha)$ uniquely.