COMPACT OPERATORS ON $p$-ADIC BANACH SPACES

1. Introduction

In the construction of eigenvarieties, the space of overconvergent modular forms arises as an infinite-dimensional $p$-adic Banach space. The $U_p$ operator acts on this space as a compact operator, and cuts out a finite-dimensional locus on which it acts with a bounded slope. This requires an analogue of Fredholm–Riesz theory for $p$-adic Banach spaces. Today we will study the spectral theory of compact operators on non-archimedean Banach spaces in general. The construction of eigenvarieties will be carried out in the next few lectures.

1.1. Acknowledgements. Everything in these notes were taken — often verbatim — from the original articles [Ser62], [Col97], [Buz07]. I have also relied heavily on the English translation\footnote{Available at \url{http://vbbrt.org/writings/serre-3.pdf}} of [Ser62] by Jonathan Pottharst.

2. Fredholm–Riesz Theory à la Serre

2.1. Basis definitions. Throughout, $K$ will denote a field complete with respect to a non-trivial non-archimedean valuation $|\cdot|$. Let $\mathcal{O}$ be its valuation ring, $\mathfrak{m}$ its maximal ideal and $k$ its residue field.

Definition 1. A Banach space over $K$ is a complete normed vector space over $K$ whose norm $|\cdot|$ satisfies the ultrametric inequality

$$|x + y| \leq \sup(|x|, |y|)$$

for all $x, y \in E$. (Recall that the definition of a normed vector space requires that $|\lambda x| = |\lambda||x|$ for all $\lambda \in K$ and $x \in E$.)

Definition 2. We say a Banach space $E$ over $K$ is orthonormalizable if there exists a subset $\{e_i\}_{i \in I}$ of $E$ such that every element $x$ of $E$ can be written uniquely as $\sum_{i \in I} x_i e_i$ with $x_i \in K$, $\lim_{i \to \infty} x_i = 0$ and $|x| = \sup_{i \in I} |a_i|$. If this holds, we say $\{e_i\}$ is an orthonormal basis of $E$.

Note that this condition implies that $|e_i| = 1$ for all $i$. Here the statement $\lim_{i \to \infty} x_i = 0$ means that for all $\epsilon > 0$, there are only finitely many $i \in I$ with $|x_i| > \epsilon$. In particular, this is always true if $I$ is finite, and is the usual condition if $I = \mathbb{Z}_{\geq 0}$. In general, this implies only countably many $a_i$ can be non-zero.
Example 3. Let $I$ be a set, and $c(I)$ be the set of families $(x_i)_{i \in I}$, $x_i \in K$, such that \( \lim_{i \to \infty} x_i = 0 \). Then $c(I)$ is an orthonormalizable Banach space over $K$, with the supremum norm $\|(x_i)_{i \in I}\| = \sup_{i \in I} |x_i|$. An orthonormal basis given by the “standard basis”.

Example 4. More generally, given any Banach space $E$ over $K$, we define $c_E(I)$ to be the set of families of elements of $E$ indexed by $I$ that tend to 0, equipped with the supremum norm. With this notation, $c(I) = c_K(I)$. The space $c_E(I)$ is a Banach space over $K$, and is orthonormalizable if and only if $E$ is.

In general, showing that $E$ is orthonormalizable amounts to specifying an isometric isomorphism $E \cong c(I)$ for some set $I$. An algebraic approach comes from the following

Lemma 5. Let $E_0 = \{ x \in E : |x| \leq 1 \}$, and $\overline{E} = E_0/\mathfrak{m}E_0$. For a set $(e_i)_{i \in I}$ of elements of $E$ to be an orthonormal basis of $E$, it is necessary and sufficient that $e_i \in E_0$ and their images $\overline{e_i} \in \overline{E}$ form a basis (in the algebraic sense) of $\overline{E}$ as a $k$-vector space.

Proposition 6. If the valuation of $K$ is discrete, then every Banach space over $K$ is isomorphic as a topological vector space to a space $c(I)$ for some set $I$.

Note however that the above isomorphism is not isometric in general. For the purpose of the spectral theory of compact operators, only the topology matters and we don’t lose too much by restricting our attention to spaces of the form $c(I)$. We will make this comment more precise when we study the Fredholm determinant.

If $E$ and $F$ are two Banach spaces, we denote by $L(E,F)$ the set of continuous linear maps of $E$ into $F$. As in the classical case, a linear map $u : E \to F$ is continuous if and only if it is bounded, i.e.

$$|u| := \sup_{x \neq 0} \frac{|ux|}{|x|} < \infty,$$

and $L(E,F)$ is a Banach space under this norm. However, we cannot in general write $|u|$ as $\sup_{|x|=1} |ux|$ or $\sup_{|x| \leq 1} |ux|$, because it may not be possible to scale an element of $E$ by $K$ to attain any prescribed norm — for example, the two sets $|E|$ and $|K|$ may not be equal.

If $E = c(I)$ with canonical orthonormal basis $(e_i)_{i \in I}$, then to each $u \in L(E,F)$ we associate $(ue_i)_{i \in I} \in b_F(I)$, the space of bounded families of elements of $F$ equipped with the supremum norm. It is immediate that this establishes an isometric isomorphism

$$L(E,F) \cong b_F(I).$$

In particular, the dual $E^* := L(E,K)$ of $E$ is isomorphic to the Banach space $b(I)$ of bounded families of elements of $K$.

In case $F = c(J)$ is orthonormalizable as well, we can give a nice description of $L(E,F)$ in terms of matrix coefficients. Let $f_j$ be the canonical orthonormal basis of $c(J)$. For each $i \in I$ we write

$$ue_i = \sum_{j \in J} n_{ij} f_j$$

(note that the indexing here — following Serre — is opposite to standard treatments in linear algebra; $n_{ij}$ is the coefficient on row $j$ and column $i$). Then

- There exists $C > 0$ such that $|n_{ij}| < C$ for all $i,j$.
- For each $i$, $\lim_{j \to \infty} n_{ij} = 0$ (i.e. the matrix entries tend to 0 down each column).
Conversely, any collection of elements of $K$ satisfying these two conditions are the matrix coefficients of a continuous linear map of $E$ into $F$.

Let me point out quickly that many theorems about Banach spaces over $\mathbb{R}$ and $\mathbb{C}$ from classical functional analysis are valid for Banach spaces over $K$. Examples include the open mapping theorem and the closed graph theorem. The ultrametric inequality often makes things easier to prove. However, the Hahn–Banach theorem does not hold in general, so the existence of linear functionals is more subtle than the classical case. In fact, there exists an infinite-dimensional Banach spaces over $K$ whose dual space is zero!

2.2. Compact operators. Let $E$ and $F$ be two Banach spaces.

**Definition 7.** A map $u \in L(E, F)$ is compact (or completely continuous) if it lies in the closure of the subspace of $L(E, F)$ consisting of continuous linear maps of finite rank, i.e. if there exists a sequence of finite rank maps $u_i \in L(E, F)$ such that $u_i \to u$. The set of compact operators is denoted $C(E, F)$.

It is clear that pre- or post-composing a compact linear map with any continuous linear map gives another compact linear map. In particular, $C(E, E)$ is a closed two-sided ideal of the algebra $L(E, E)$.

If $F = c(J)$, for every $u \in L(E, F)$ we can write $ux = (w_j x)_{j \in J}$, where $w_j \in E^*$. This association defines an isometric isomorphism

$$C(E, F) \cong c_{E^*}(J).$$

Again, if $E = c(I)$ is orthonormalizable as well, we have a nice description of the space $C(E, F)$ — it corresponds precisely to the collection of matrix coefficients $(n_{ij})_{i \in I, j \in J}$ satisfying

- There exists $C > 0$ such that $|n_{ij}| < C$ for all $i, j$.
- $\lim_{j \to \infty} \sup_{i \in I} |n_{ij}| = 0$ (i.e. the matrix entries tend to 0 down each column uniformly).

Finally, we include the following proposition to explain the connection with the classical setting. In the terminology of functional analysis, Banach spaces over a locally compact complete non-archimedean field satisfies the approximation property.

**Proposition 8.** Suppose that $K$ is locally compact, and $E$ and $F$ are Banach spaces over $K$. Then $u \in L(E, F)$ is compact if and only if it sends every bounded subset of $E$ to a relatively compact subset of $F$.

We will not need to use this though.

2.3. Topological tensor products.

**Definition 9.** A topological tensor product of $E$ and $F$ is a Banach space $E \hat{\otimes} F$, equipped with a continuous bilinear map $E \times F \to E \hat{\otimes} F$ written $(x, y) \mapsto x \hat{\otimes} y$, satisfying the following universal property: every continuous bilinear map of $E \times F$ into a Banach space $X$ factors uniquely as

$$E \times F \longrightarrow E \hat{\otimes} F \longrightarrow X.$$
where \( E \otimes F \to X \) is a continuous linear map.

A topological tensor product can be constructed as follows. First we introduce a seminorm on \( E \otimes F \) (algebraic tensor product)

\[
|z| = \inf \left( \max_i (|x_i||y_i|) \right)
\]

where the infimum is taken over all representations of \( z \) as a finite sum \( \sum x_i \otimes y_i, x_i \in E, y_i \in F \). Then \( E \otimes F \) is defined by separating \( E \otimes F \) for this seminorm and completing the normed vector space obtained. The formula above defines norm on \( E \otimes F \), and the canonical map \( E \otimes F \to E \otimes F \) is an injection. As usual this is unique up to unique isomorphism.

It is easy to show that for any Banach space \( L \) and set \( J \), the topological tensor product \( L \otimes c(J) \) is isomorphic to the space \( c_L(J) \). As a consequence, we can relate this to the space of compact operators.

**Corollary 10.** Let \( E \) be a Banach space and \( F = c(J) \). Then the topological tensor product \( E^* \otimes \hat{F} \) is isomorphic to the space \( C(E,F) \) of compact operators from \( E \) into \( F \).

### 2.4. Fredholm determinant and resolvent

As an algebraic preliminary, let’s consider this setting: \( L \) will be a free module over a commutative ring \( R \), and \( f \) will be an endomorphism of \( L \) whose image is contained in \( M \), a direct summand of \( L \) that is free of finite rank. The polynomial \( \det(1 - tf) \in R[t] \) is well-defined, and is easily seen to be independent of the choice of \( M \); we write it as \( \det(1 - tf) \). We have the following expansion from linear algebra

\[
\det(1 - tf) = 1 + c_1 t + c_2 t^2 + \cdots
\]

where \( c_m = (-1)^m \text{Tr}(\wedge^m f) \) (in particular, \( c_1 = -\text{Tr}(f) \)). More explicitly, suppose \( \{e_i\}_{i \in I} \) is a basis for \( L \), and \( f \) has matrix \( (n_{ij}) \) with respect to this basis (remember we’re following Serre’s convention of indexing), then

\[
c_m = (-1)^m \sum_{S \subseteq I} \sum_{\sigma \in \text{Sym}(S)} \text{sgn}(\sigma) \prod_{i \in S} n_{i,\sigma(i)}
\]

(i.e. \( c_m \) is the sum of all \( k \times k \) minors corresponding to row and column indices that are equal).

Now let \( E = c(I) \) be an orthonormalizable Banach space over \( K \), and \( u \) be a compact operator on \( E \). Our goal is to define the **Fredholm determinant** \( \det(1 - tu) \) of \( u \), which will be a power series in \( t \). Suppose first that \( |u| \leq 1 \). Let \( E_0 = \{ x \in E : |x| \leq 1 \} \); we have \( u(E_0) \subseteq E_0 \). Let \( a \subseteq \mathcal{O} \) be any non-zero ideal (thus \( a \subseteq m \)). The endomorphism \( u \) defines an endomorphism \( u_a \) of \( E_a := E_0/aE_0 \). If \( \{e_i\}_{i \in I} \) is an orthonormal basis of \( E \), the images of \( e_i \) form a basis (in the algebraic sense) of \( E_a \) as an \( \mathcal{O}/a \)-module. As usual denote by \( (n_{ij}) \) the matrix of \( u \) with respect to \( (e_i)_{i \in I} \). Since \( u \) is compact, we have \( \lim_{j \to \infty} \sup_i |n_{ij}| = 0 \), which means \( n_{ij} \in a \) for all but finitely many \( j \). From this we see that the image of \( u_a \) is contained in a submodule of finite type of \( E_a \), so the polynomial \( \det(1 - tu_a) \in (\mathcal{O}/a)[t] \) is well-defined. As \( a \) varies over all non-zero ideals (i.e. a system of open neighborhoods of \( 0 \in \mathcal{O} \)), these polynomials form an inverse system and we define their inverse limit to be the power series \( \det(1 - t) \in \mathcal{O}[[t]] \), whose coefficients tend to 0. For a general compact operator \( u \), we simply choose a non-zero scalar \( c \in K \) such that \( |cu| \leq 1 \) and define \( \det(1 - tcu) \in \mathcal{O}[[t]] \). Scaling the coefficient of \( t^m \) by \( c^{-m} \), we get \( \det(1 - tu) \in K[[t]] \).
It follows from the construction that the Fredholm determinant has the same expansion as in the finite rank case
\[
\det(1 - tu) = \sum_{m=0}^{\infty} c_m t^m \text{ with } c_m = (-1)^m \sum_{S \subseteq I} \sum_{\sigma \in \text{Sym}(S)} \text{sgn}(\sigma) \prod_{i \in S} n_{i,\sigma(i)}
\]
(note that the sum defining \(c_m\) is indeed convergent, because \(\lim_{j \to \infty} \sup_{i \in I} |n_{ij}| = 0\) and hence the products \(\prod_{i \in S} n_{i,\sigma(i)}\) tend to 0), and that \(\det(1 - tu)\) coincides with the polynomial defined above if \(u\) is finite rank.

**Proposition 11.** Let \(u\) be a compact operator on \(E = c(I)\).

1. The series \(\det(1 - tu)\) is an entire function of \(t\), i.e. its radius of convergence is infinite.
2. Let \(u_n\) be a series of compact operators of \(E\) that tends to \(u\) (necessarily compact). Then \(\det(1 - tu_n)\) converges to \(\det(1 - tu)\) uniformly in the coefficients.

**Remark.** By (2), we see that the notion of Fredholm determinant depends only on the topology of \(E\) but not the choice of norm or orthonormal basis. This will allow us to generalize the theory to Banach spaces that are “almost” orthonormalizable — see below.

**Proof.**

1. Let \(\{r_j\}_{j \in I}\) be the family of positive real numbers \(r_j = \sup_{i \in I} |n_{ij}|\), arranged in decreasing order (which is possible since only countably many of them are non-zero). If \(S\) is a subset of \(I\) of size \(m\), then each product \(\prod_{i \in S} n_{i,\sigma(i)}\) contains \(m\) distinct indices \(j\), so it has norm at most \(r_1 \cdots r_m\), and so for any fixed \(R > 0\),
\[
|c_m| R^m \leq (r_1 \cdots r_m) R^m = (r_1 R) \cdots (r_m R).
\]
Since \(\lim_{j \to \infty} r_j = 0\), we conclude that \(|c_m| R^m \to 0\) as \(m \to \infty\).

2. This is very similar to the above, and we just have to refine the inequalities on \(\prod_{i \in S} n_{i,\sigma(i)}\) in a uniform manner. The proof will be omitted. \(\square\)

We mention a few other properties of the Fredholm determinant.

**Proposition 12.** Let \(E = c(I)\).

1. If \(u, v \in C(E, E)\), then
\[
\det((1 - tu)(1 - tv)) = \det(1 - tu) \cdot \det(1 - tv).
\]
(The left hand side makes sense because \((1 - tu)(1 - tv) = 1 - t(u + v - tuv)\) and \(u + v - tuv\) is compact.)

2. Let \(F = c(J)\). If \(u \in C(E, F)\) and \(v \in L(F, E)\), then \(vu \in C(F, F)\) and \(vu \in C(E, E)\) and
\[
\det(1 - tuv) = \det(1 - tvu).
\]

3. Let \(I = I' \cup I''\) be a partition of \(I\). Suppose \(u \in C(E, E)\) maps \(E' = c(I')\) into itself. Let \(u'\) be the restriction of \(u\) to \(E'\) and \(u''\) be the endomorphism of \(E'' = c(I'')\) defined by passage to quotient from \(u\). Then \(u'\) and \(u''\) are compact, with
\[
\det(1 - tu) = \det(1 - tu') \cdot \det(1 - tu'').
\]
These can be proved easily in the finite rank case. For example, if \( u \) is of finite rank, then (2) holds because

\[
\text{Tr}\left(\bigwedge^m (uv)\right) = \text{Tr}\left(\bigwedge^m u \circ \bigwedge^m v\right) = \text{Tr}\left(\bigwedge^m v \circ \bigwedge^m u\right) = \text{Tr}\left(\bigwedge^m (vu)\right).
\]

In general, we can reduce to the finite rank case by reducing modulo \( a \).

Next we define the Fredholm resolvent. As usual, we denote \( E = c(I) \) and \( u \in C(E, E) \) with matrix \((n_{ij})\). We put

\[
\det(1 - tu) = \sum_{m=0}^{\infty} c_m t^m \in K[[t]].
\]

The Fredholm resolvent is defined to be the formal power series

\[
P(t, u) = \frac{\det(1 - tu)}{1 - tu} = \sum_{m=0}^{\infty} v_m t^m.
\]

By the recurrence relation \( v_0 = 1 \) and \( v_m = c_m + uv_{m-1} \), we see that \( v_m \in K[u] \) (which can be thought of as an element of \( L(E, E) \)), so \( P(t, u) \in K[u][[t]] \).

**Proposition 13.** The Fredholm resolvent \( P(t, u) \) is an entire function of \( t \) with values in \( L(E, E) \).

Again, the basic idea is the same as the proof for \( \det(1 - tu) \). The calculations are more sophisticated but not too enlightening, so we will omit the proof.

2.5. Fredholm–Riesz theory. We're almost ready to prove the main theorem of Serre. Since \( \det(1 - tu) \in K[[t]] \) is an entire function of \( t \), we can substitute for \( t \) an arbitrary value \( a \in K \). It turns out this process detects whether the operator \( 1 - au \) is invertible.

**Lemma 14.** Let \( a \in K \). Then \( 1 - au \) is invertible in \( L(E, E) \) if and only if \( a \) is not a zero of \( \det(1 - tu) \).

**Proof.** If \( a \) is not a zero of \( \det(1 - tu) \), then we have

\[
(1 - au) \cdot P(a, u) = P(a, u) \cdot (1 - au) = \det(1 - au) \neq 0
\]

as elements of \( L(E, E) \), so \( 1 - au \) is invertible. Conversely, if \( 1 - au \) is invertible, we write \( (1 - au)^{-1} = 1 - v \). Thus \( v = -au + auv \) is a compact operator as well, so \( \det(1 - v) \in K \) is well-defined. By Proposition 12(1), we have

\[
\det(1 - au) \cdot \det(1 - v) = \det((1 - au)(1 - v)) = 1
\]

and hence \( \det(1 - au) \neq 0 \). \qed

**Theorem 15.** Let \( a \in K \) be a zero of order \( h \) of the Fredholm determinant \( \det(1 - tu) \in K[[t]] \). Then \( E \) can be decomposed uniquely into closed \( u \)-stable subspaces

\[
E = N(a) \oplus E(a)
\]

such that \( (1 - au)^h = 0 \) on \( N(a) \) and \( 1 - au \) is invertible on \( E(a) \). Moreover, \( \dim_K N(a) = h \).
Proof. The proof makes use of finite differences, which we now recall. If \( f = \sum_{m=0}^{\infty} a_m t^m \) is in \( K[[t]] \) and \( s \in \mathbb{Z}_{\geq 0} \), we define
\[
\Delta^s f = \sum_{m=0}^{\infty} \binom{m+s}{s} a_{m+s} t^m \in K[[t]]
\]
We can easily check that \( \Delta^s \) takes an entire function to an entire function. To say that \( a \) is a zero of order \( h \) of \( H(t) := \det(1 - tu) \) means that \( \Delta^s H(a) = 0 \) for \( s < h \) and \( \Delta^s H(a) = c \neq 0 \). Applying \( \Delta^s \) to the relation \((1 - tu) \cdot P(t, u) = H(t)\), we obtain
\[
(1 - tu) \Delta^s P(t, u) - u \Delta^{s-1} P(t, u) = \Delta^s H(t).
\]
We put \( v_s = \Delta^s P(a, u) \in L(E, E) \). The above identity evaluated at \( t = a \) then gives
\[
(1 - au) \cdot v_0 = 0
\]
\[
(1 - au) \cdot v_1 - u \cdot v_0 = 0
\]
\[
\ldots
\]
\[
(1 - au) \cdot v_{h-1} - u \cdot v_h = 0
\]
\[
(1 - au) \cdot v_h - u \cdot v_{h-1} = c \neq 0.
\]
This implies \((1 - au)^{s+1} v_s = 0 \) for \( s < h \).

Now we put \( e = c^{-1}(1 - au) \cdot v_h \) and \( f = -c^{-1} u \cdot v_{h-1} \). The last equation shows that \( e + f = 1 \), and we have \( fe^h = 0 \) since \((1 - au)^h v_{h-1} = 0 \) (commutativity is automatic since \( v_s \) is a power series in \( u \)).

Expanding \((e + f)^h = 1\), we obtain
\[
e^h + (he^{h-1} f + \cdots + he^{h-1} f + f^h) = 1
\]
and so we put \( p = e^h \) and \( q = he^{h-1} f + \cdots + he^{h-1} f + f^h \). They satisfy \( p + q = 1 \) and \( pq = 0 \), and hence \( p^2 = p \) and \( q^2 = q \). Thus \( p \) and \( q \) are projections.

Finally we put \( N(a) = \ker(p) = \im(q) \) and \( F(a) = \ker(q) = \im(p) \), which yields the decomposition
\[
E = N(a) \oplus F(a).
\]
Using the expressions for \( p \) and \( q \) above, it is easy to show \((1 - au)^h = 0 \) on \( N(a) \) and invertible on \( F(a) \), and the uniqueness of such a decomposition is immediate.

It remains to show \( \dim_K N(a) = h \). Let \( W \) be a finite-dimensional \( u \)-stable subspace of \( N(a) \). Then \( W \) is direct summand of \( E \), and \( E/W \) is of the form \( c(J) \). By Proposition 12.3, \( \det(1 - tu) \) is divisible by \( \det(1 - tu|_W) = (1 - ta^{-1})^{\dim_K W} \), which implies \( \dim_K W \leq h \). But \( W \) is an arbitrary finite-dimensional subspace of \( N(a) \), so we have shown \( \dim_K N(a) \leq h \). Consider the identity
\[
\det(1 - tu) = \det(1 - tu|_{N(a)}) \cdot \det(1 - tu|_{F(a)}) = (1 - ta^{-1})^{\dim_K N(a)} \det(1 - tu|_{F(a)}).
\]
Since \( a \) is a zero of \( \det(1 - tu) \) of order \( h \) but \( a \) is not a zero of \( \det(1 - tu|_{F(a)}) \) (as \( 1 - au \) is invertible on \( F(a) \)), we obtain \( \dim_K N(a) = h \) we desired. \( \square \)

Remark. Serre remarked that an analogous decomposition for \( E \) holds if we are given any polynomial factor of \( \det(1 - tu) \); in the theorem above, this polynomial factor is just \((1 - a^{-1} t)^h \). We will state the result more precisely when we talk about the generalizations by Coleman and Buzzard in the following.
We obtain the following easy corollary which is analogous to the classical Fredholm alternative for compact operators on Banach spaces.

**Corollary 16** (Fredholm alternative). *For each \( a \in K \), the image of \( 1 - au \) is a closed subspace of \( E \) of finite codimension equal to \( \dim_K \ker(1 - au) \).*

Serre then considered some applications of this theory, including the zeta function of hypersurfaces, which was the original motivation for Serre. Since they are irrelevant to the construction of eigenvarieties, we will omit them here.

3. **Fredholm–Riesz Theory à la Coleman and Buzzard**

In the 90’s, Coleman extended Serre’s theory to a family, which laid the foundation for the construction of the eigencurve due to himself and Mazur. More precisely, he considered orthonormalizable Banach modules over Banach algebras and defined the Fredholm determinant for compact operators. Given a factorization of the Fredholm determinant into relatively prime factors, he proved a corresponding direct sum decomposition of the underlying Banach module, which is completely analogous to Serre’s theory. However, Buzzard noted that some of the proofs were not complete, and filled in more detail in the special case when the base ring is commutative and Noetherian. At the same time he generalized the theory to direct summands of potentially orthonormalizable Banach modules. We will closely follow the treatment by Buzzard, and state the main results without addressing much of the technicality.

3.1. **Basic definitions.** As before, let \( K \) be a field complete with respect to a non-trivial non-archimedean valuation. A *\( K \)-Banach algebra* is a \( K \)-algebra equipped with a function \(| \cdot | : A \to \mathbb{R}_{\geq 0} \) satisfying

- \(|1| \leq 1\), and \(|a| = 0\) if and only if \( a = 0\),
- \(|a + b| \leq \max\{|a|, |b|\}\),
- \(|ab| \leq |a||b|\),
- \(|\lambda a| = |\lambda||a|\) for all \( \lambda \in K \), where \( |\lambda| \) is the valuation on \( K \),

such that \( A \) is complete with respect to this norm. Note that we don’t require the norm to be multiplicative; the inequality ensures that multiplication is continuous.

Following Buzzard, we will assume throughout that \( A \) is commutative and Noetherian. A *Banach \( A \)-module* is an \( A \)-module \( M \) equipped with a function \(| \cdot | : M \to \mathbb{R}_{\geq 0} \) satisfying

- \(|m| = 0\) if and only if \( a = 0\),
- \(|m + n| \leq \max\{|m|, |n|\}\),
- \(|am| \leq |a||m|\) for all \( a \in A \),

such that \( M \) is complete with respect to this norm. We have defined three different valuations \(| \cdot |\) on \( K \), \( A \) and \( M \) respectively, but there will not be any ambiguity.

If \( M \) and \( N \) are Banach \( A \)-modules, then \( M \oplus N \) has the natural structure of a Banach \( A \)-module, with norm given by \(|m \oplus n| = \max\{|m|, |n|\}\). For example, \( A^r \) has the natural structure of a Banach \( A \)-module.

We define orthonormalizable Banach \( A \)-modules exactly as before, and they are precisely the ones isometrically isomorphic to \( c_A(I) \). Our prior discussion of continuous and compact linear maps between orthonormalizable Banach \( A \)-modules, as well as their characterizations in terms of matrix coefficients, carry over without change. We remark that for compact linear
maps, the characterization depends on the Noetherianness of $A$. The subtle issue is that in general it is not clear whether any finite submodule of an orthonormalizable Banach module is contained in a finite free module.

For each compact operator $u$ on an orthonormalizable Banach $A$-module $M$, we can define its Fredholm determinant as before, which satisfies the exact same properties. Moreover, the fact that the Fredholm determinant depends only on the topology of $M$ but not its norm allows Buzzard to make a slight generalization of the theory.

### 3.2. Potentially orthonormalizable modules.

**Definition 17.** A Banach $A$-module is potentially orthonormalizable if there is a bounded collection $\{e_i\}_{i \in I}$ of elements of $M$ with the following properties:

- Every element $x$ of $E$ can be uniquely written as $\sum_{i \in I} a_i e_i$ with $a_i \in A$, $\lim_{i \to \infty} a_i = 0$
- There exist positive constants $c_1$ and $c_2$ such that for all $m = \sum_{i \in I} a_i e_i$ in $M$, we have $c_1 \sup_i |a_i| \leq |m| \leq c_2 \sup_i |a_i|$.

If this holds, we say $\{e_i\}$ is a potentially orthonormal basis for $M$.

More conceptually, this means there exists a norm on $M$ equivalent to the given norm, for which $M$ becomes an orthonormalizable $A$-module. This is arguably a more natural notion than being orthonormalizable, because many notions, such as the Fredholm determinant, depends on the topology of $M$ but not the precise norm. Note that to say a module is orthonormalizable is equivalent to saying that it is isometrically isomorphic to some $c_A(I)$, and to say that it is potentially orthonormalizable means that it is isomorphic to some $c_A(I)$ as Banach modules. There are Banach $A$-modules that are potentially orthonormalizable but not orthonormalizable:

**Example 18.** Let $A = K = \mathbb{Q}_p$ and $M = \mathbb{Q}_p/(\sqrt{p})$ with its usual norm. Then $|M| \neq |A|$, which shows $M$ is not orthonormalizable, but it is potentially orthonormalizable.

Since the notion of Fredholm determinant depends only on the topology, it is well-defined for compact operators on potentially orthonormalizable Banach $A$-modules.

The Fredholm determinant behaves well under base change. Again, we can define the topological tensor product as before. If $h : A \to B$ is a continuous homomorphism of commutative Noetherian $K$-Banach algebras, and $M$ is a potentially orthonormalizable Banach $A$-module with potentially orthonormal basis $\{e_i\}_{i \in I}$, then $M \hat{\otimes}_A B$ is a potentially orthonormalizable Banach $B$-module with potentially orthonormal basis $\{e_i \otimes 1\}_{i \in I}$. Now suppose we have a compact linear map $\phi : M \to N$ between potentially orthonormalizable Banach $A$-modules, with matrix $(n_{ij})$ with respect to potentially orthonormal bases $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$, then $\phi \otimes 1 : M \hat{\otimes}_A B \to N \hat{\otimes}_A B$ is also compact, with matrix $(h(n_{ij}))$ with respect to the bases $\{e_i \otimes 1\}_{i \in I}$ and $\{f_j \otimes 1\}_{j \in J}$. With this, it follows that if $\det(1 - t\phi) = \sum c_m t^m$, then $\det(1 - t(\phi \otimes 1)) = \sum h(c_m) t^m$.

### 3.3. Projective modules.

Buzzard further generalizes the theory of compact linear maps to the natural analogue of projective modules.

**Definition 19.** A Banach $A$-module $P$ is projective if there is a Banach $A$-module $Q$ such that $P \oplus Q$, equipped with its usual norm, is potentially orthonormalizable.
Equivalently, projective Banach $A$-modules satisfy the usual universal property: for every surjection $f : M \to N$ of Banach $A$-modules and every continuous map $\alpha : P \to N$, $\alpha$ lifts to a continuous map $\beta : P \to M$ such that $f \beta = \alpha$.

Remark. Buzzard considers it “slightly disingenuous” to call such modules projective, because there are epimorphisms in the category of Banach $A$-modules whose underlying module map is not surjective. However, in these notes we will retain the more familiar terminology for simplicity.

Let us verify that the Fredholm determinant is well-defined for any compact operators $\phi : P \to P$ on a projective Banach $A$-module $P$. Choose $Q$ such that $P \oplus Q$ is potentially orthonormalizable, and define $\det(1-t\phi) = \det(1-t(\phi \oplus 0))$; note that $\phi \oplus 0 : P \oplus Q \to P \oplus Q$ is clearly compact. This definition may a priori depend on the choice of $Q$. If $R$ is another Banach $A$-module such that $P \oplus R$ is potentially orthonormalizable, then so is $P \oplus Q \oplus P \oplus R$, and the maps $\phi \oplus 0 \oplus 0 \oplus 0$ and $0 \oplus 0 \oplus \phi \oplus 0$ are conjugate via an isometric $A$-module isomorphism, and hence have the same Fredholm determinant. But by quotienting out a subspace on which the operator acts as 0, we see that these two maps have the same Fredholm determinant as $\phi \oplus 0 : P \oplus Q \to P \oplus Q$ and $\phi \oplus 0 : P \oplus R \to P \oplus R$ respectively, so the Fredholm determinant $\det(1-t\phi)$ is independent of the choice of $Q$.

This trick also allows us to prove the same properties of the Fredholm determinant over projective Banach modules, but we will not repeat them here.

3.4. Fredholm–Riesz theory. In order to generalize Serre’s theory, Coleman made use of the theory of resultants between polynomials and power series. We will leave that in a blackbox. Let $A\{\{t\}\}$ be the subring of $A[[t]]$ consisting of power series $\sum_{m=0}^{\infty} a_m t^m$ which are entire, i.e. whose radii of convergence are infinite.

Let $M$ be a projective Banach $A$-module, and $\phi : M \to M$ be a compact operator with Fredholm determinant $\det(1-t\phi) \in A\{\{t\}\}$. We define the Fredholm resolvent of $\phi$ to be

$$\frac{\det(1-t\phi)}{1-t\phi} \in A[\phi][[t]].$$

Exactly as before, this is an entire function of $t$ with values in $L(M,M)$. Coleman proved the following:

Lemma 20. If $Q(t) \in A[t]$ is a monic polynomial of degree $n$, then $Q$ and $\det(1-t\phi)$ generate the unit ideal in $A\{\{t\}\}$ if and only if $Q^*(\phi)$ is an invertible operator on $M$, where $Q^*(t) := t^n Q(t^{-1})$.

This specializes to Serre’s lemma upon taking $Q(t) = t - a$.

Before we state the analogue of Fredholm–Riesz theory for compact operators on projective Banach $A$-modules, we need to make some remarks on zeros of power series. If $f = \sum_{m=0}^{\infty} a_m t^m$ is in $A[[t]]$ and $s \in \mathbb{Z}_{\geq 0}$, recall that we have defined

$$\Delta^s f = \sum_{m=0}^{\infty} \binom{m+s}{s} a_{m+s} t^m \in A[[t]].$$

This satisfies:
• If \( f, g \in A[[t]] \), then
\[
\Delta^s(fg) = \sum_{i=0}^s \Delta^i(f)\Delta^{s-i}(g).
\]

• \( \Delta^s \) stabilizes \( A\{\{t\}\} \).

We say that \( a \in A \) is a zero of order \( h \) of \( H \in A\{\{t\}\} \) if \( (\Delta^s H)(a) = 0 \) for \( s < h \) and \( (\Delta^s H)(a) \) is a unit in \( A \). The second condition is hardwired into the definition because we need to divide by \( (\Delta^s H)(a) \) in the proof. Note that with this definition, some zeros do not have an order! If \( h > 0 \) and \( H = 1 + a_1t + \cdots \), then \( a \) is a unit. By induction on \( h \), we get that \( H(t) = (1 - a^{-1}t)^h G(t) \), where \( G(t) \in A\{\{t\}\} \) and \( G(a) \) is a unit in \( A \).

**Theorem 21.** Let \( a \in A \) be a zero of \( \det(1 - t\phi) \) of order \( h \). Then there is a unique decomposition \( M = N \oplus F \) into closed \( \phi \)-stable submodules such that \( (1 - a\phi)^h \) is invertible on \( N \) and \( 1 - a\phi \) is invertible on \( F \). Moreover, \( N \) is projective of rank \( h \), and if \( h > 0 \) then \( a \) is a unit and the Fredholm determinant of \( \phi \) on \( N \) is \( (1 - a^{-1}t)^h \).

Finally, we have the following generalization of Serre’s remark.

**Theorem 22.** Suppose \( \det(1 - t\phi) = Q(t)S(t) \) where \( S = 1 + \cdots \in A\{\{t\}\} \) and \( Q = 1 + \cdots \in A[[t]] \) is a polynomial of degree \( n \) whose leading coefficient is a unit, and which is relatively prime to \( S \). Then there is a unique direct sum decomposition \( M = N \oplus F \) into closed \( \phi \)-stable submodules such that \( Q^*(\phi) \) is zero on \( N \) and invertible on \( F \). Furthermore, \( N \) is projective of rank \( n \) and the Fredholm determinant of \( \phi \) on \( N \) is \( Q(T) \).

**Remark.** \( N \) and \( F \) are the kernel and image of a projector which lies in the closure of \( A[\phi] \) in \( \text{L}(M,M) \).

**References**

