Polynomial invariants of knots

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Abstract

The main problem of knot theory is to classify knots and links. By the year 1900, Tait has enumerated prime knots up to ten crossings. However, he assumed three conjectures, at present called his name. They have not been proved until the papers of Jones in 1980's. In this paper we will introduce two most powerful methods of constructing polynomial invariants – from a knot group by the notion of derivative and from diagrams by skein relations.

1 Introduction

By a knot we mean a smooth or PL embedding $K: S^1 \to \mathbb{R}^3$ of an oriented circle into a 3-dimensional space, also oriented. Generally, an embedding (smooth or PL) of a disjoint sum of a finite number of circles (all oriented) is called a link. The circles are called components of a link. By this definition a knot is just a link with one component.

All knots create a space

$$X = \{ K: S^1 \to \mathbb{R}^3 | K - \text{is a knot} \}$$

Two knots are regarded as equivalent if they lie in the same path component of $X$ (considered with the compact-open topology). That means there is a path in $X$ from one knot to the other. Later we will not differ equivalent knots and by a knot we will often mean its class of equivalence.

The unknot is a standard embedding of a circle

$$U: S^1 \ni t \mapsto (\sin(t), \cos(t), 0) \in \mathbb{R}^3$$

and any knot equivalent to it is called trivial. Equivalence of links is defined in a similar way.

For better understanding we can imagine a knot as a string knotted in a space with ends glued together. Then paths in a knot space are just simple deformations of this string – no cutting or gluing is allowed.

Let $K$ be any knot. By $K^m$ we will mean a mirror knot, which is a knot symmetric to $K$ relating to some plain $A \subset \mathbb{R}^3$. The definition is correct as for different plains we obtain equivalent knots. Taking $A = \mathbb{R}^2 \times 0$ we have

$$K^m(t) = (x(t), y(t), -z(t)) \quad \text{for} \quad K(t) = (x(t), y(t), z(t))$$
A reverse knot $K'$ is obtained by changing orientation of the circle: $K'(t) = K(-t)$.

Of course $U^m = U^n = U$. However, there exist knots for which neither of these two operations preserve the knot class.

## 2 Alexander polynomials

Having any finitely-presented group $G$ we can compute a polynomial, called the Alexander polynomial, which is independent from the presentation. Applying this to the knot group, we obtain a set of knot invariants, which are quite easy to compute. The basic step is to create some matrix over the group ring $\mathbb{Z}G_{ab}$ where $G_{ab}$ is the abelianization of $G$. The Alexander polynomial is just the greater common divisor of some minors.

At first we will introduce the notions of a group ring and a derivative.

**Definition 2.1.** Let $G$ be any group. By a group ring $\mathbb{Z}G$ of $G$ we mean a ring of all formal sums

$$\sum_{g \in G} n_g g$$

with $n_g = 0$ for almost every $g \in G$. The operations of sum and multiplication are defined naturally by

$$\sum_{g \in G} n_g g + \sum_{g \in G} m_g g = \sum_{g \in G} (n_g + m_g) g$$

$$\left( \sum_{g \in G} n_g g \right) \cdot \left( \sum_{g \in G} m_g g \right) = \sum_{g \in G} \left( \sum_{h \cdot g} (n_h m_h) \right) g$$

Notice, that sum and multiplication are defined in a similar way like for polynomials with several variables, but here the ‘variables’ are not free - there are some relations between them derived from the structure of the underlaying group. However, for a free group of rank $n$ we will obtain exactly a ring of Laurent polynomials with $n$ variables $\mathbb{Z}[t_1^\pm \ldots, t_n^\pm]$. A group ring does not need to be commutative - this is only the case of an abelian group.

**Definition 2.2.** A map $D: \mathbb{Z}G \to \mathbb{Z}G$ is called a derivative if for any $x, y \in \mathbb{Z}G$: 2
(d1) \( D(x + y) = D(x) + D(y) \)
(d2) \( D(xy) = D(x)y + xD(y) \)

where

\[
\varepsilon : ZG \ni \sum_{g \in G} n_g g \rightarrow \left( \sum_{g \in G} n_g \right) 1_G \in ZG
\]

is called an augmentation map.

**Example 2.3.** For a free group \( F \) generated by \( x_1, \ldots, x_n \) we can define partial derivatives \( \frac{\partial}{\partial x_j} : ZF \rightarrow ZF \) by the formula

\[
\frac{\partial x_i}{\partial x_j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}
\]

From the universal property of a free group it can be extended to the whole group \( F \) and then to the whole group ring \( ZF \) by (d1).

Assume \( G = (X : R) \) is finitely-presented with generators \( X = \{x_1, \ldots, x_n\} \) and relations \( R = \{r_1, \ldots, r_m\} \). Denoting by \( F \) a free group generated by \( x_1, \ldots, x_n \) and by \( H \) its normal subgroup generated by the relations \( r_1, \ldots, r_m \) we have

\[ G \cong F/H. \]

Let \( \gamma : F \rightarrow G \) be the quotient homomorphism and \( A : G \rightarrow G_{ab} \) the abelianization of \( G \). The derivative matrix \( J \) of the presentation \( (X : R) \) is defined as

\[
J = \begin{bmatrix}
\frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n}
\end{bmatrix}
\]

Applying the defined homomorphisms we obtain a matrix \( A = A\gamma(J) \) over a commutative ring \( ZG_{ab} \) called the Alexander matrix of the presentation \( G = (X : R) \).

**Example 2.4.** A symmetric group \( S_3 \) has a presentation

\[
(*) \quad S_3 = \langle a, b : a^2 = b^3 = (ab)^2 = 1 \rangle
\]

(i.e. \( a = (1 2 3), b = (1 2) \)). Treating relations as elements of a free group \( F = F(a, b) \) we calculate the derivative matrix:

\[
J = \begin{bmatrix}
\frac{\partial a^3}{\partial a} & \frac{\partial a^3}{\partial b} & \frac{\partial a^3}{\partial (ab)^2} \\
\frac{\partial b^3}{\partial a} & \frac{\partial b^3}{\partial b} & \frac{\partial b^3}{\partial (ab)^2} \\
\frac{\partial (ab)^2}{\partial a} & \frac{\partial (ab)^2}{\partial b} & \frac{\partial (ab)^2}{\partial (ab)^2}
\end{bmatrix} = \begin{bmatrix}
1 + a + a^2 & 0 \\
0 & 1 + b \\
1 + ab & a + aba
\end{bmatrix}
\]

Denote \( A(a) = \bar{a}, A(b) = \bar{b} \). In \( S_3 \) we have \( abab = b^2 = 1 \) what under abelianization gives \( \bar{a}^2 = 1 \). As \( \bar{a}^3 = 1 \) it gives that \( \bar{a} = 1 \). Following this, we get

\[ A(S_3) \cong C_2 = (t : t^2 = 1) \] with \( t = \bar{b} \)

As a result the Alexander matrix of the presentation \( (*) \) is

\[
A = \begin{bmatrix}
3 & 0 \\
0 & 1 + t \\
1 + t & 1 + t
\end{bmatrix}
\]
The abelianization of a knot group is infinite cyclic \( C_\infty = \langle t : \rangle \) for any knot and the Alexander matrix for any presentation of that group is a matrix over the ring of Laurent polynomials \( \mathbb{Z}[t^{\pm 1}] \) which is a unique factorization domain. Especially, any finite set of these polynomials has a greatest common divisor (g.c.d.)

**Definition 2.5.** Let \( K \) be any knot and \( A \) the Alexander matrix of some presentation of its group with \( n \) columns and \( m \) rows. The Alexander polynomial \( \Delta_1(K) \) of a knot \( K \) is a g.c.d. of all \( (n - 1) \times (n - 1) \) minors of the matrix \( A \).

The Alexander polynomial is well-defined as for knot groups we always have \( m \geq n - 1 \). Furthermore, this notion is somewhat independent on the presentation of the knot group:

**Theorem 2.6.** Alexander polynomials computed for different presentations differ by \( \pm t^{\pm n} \), which is a unit:

\[ \Delta_1(K) = \pm t^{\pm n} \Delta'_1(K) \]

Further we will write \( a \equiv b \) for elements differing by a unit.

Sometimes this polynomial is called the first Alexander polynomial – in a similar way other polynomials can be defined. Moreover, we can define these polynomials also for links. However, they may have more than one variable.

**Example 2.7.** The knot group of the trefoil (fig. 1) has a presentation

\[ G = \langle a, b : a^2b^{-3} = 1 \rangle \]

Under abelianization we obtain

\[ \bar{G}_{ab} = \langle t : \rangle \text{ with } \bar{a} = t^3, \bar{b} = t^2. \]

Computing the Alexander matrix we have

\[ A = \begin{bmatrix} 1 + t^3 & -1 - t^2 - t^4 \end{bmatrix} \]

and \( GCD(1 + t^3, 1 + t^2 + t^4) = 1 - t + t^2 \). So \( \Delta_1(3_1) = 1 - t + t^2 \).

As the Alexander polynomial of an unknot is trivial, trefoil is indeed knotted.

### 3 Diagrams. Reidemeister moves

The original approach of J. W. Alexander (see [1]) was different and requires a notion of a diagram of a knot. Intuitively, the diagram is a picture of a knot on a plane with some information about crossings – which string goes above and which below.

**Definition 3.1.** For a knot \( K \) a projection \( \pi : \mathbb{R}^3 \to \mathbb{R}^2 \) is called regular if

- each point \( x \in \mathbb{R}^3 \) is an image of at most two points of the knot
• the set of double points (a double point is an image of exactly two points) is finite
• at each double point the image of the knot crosses itself transversally

\[ \begin{array}{c}
\text{Figure 2: Improper situations}
\end{array} \]

It can be proved that each knot possesses a regular projection. By cutting at each crossing one of the part of a knot we obtain a diagram. A diagram contains all needed information to restore the knot (with accuracy to equivalence). Moreover, the equivalence of knots can be easily stated in the category of diagrams:

**Theorem 3.2.** Let \( K_1, K_2 \) be knots with diagrams \( D_1, D_2 \). Then they are equivalent if and only if diagram \( D_2 \) can be obtained from \( D_1 \) by local deformations and moves \( R_1 - R_3 \) (**fig. 3**).

\[ \begin{array}{c}
\text{Figure 3: Reidemeister moves}
\end{array} \]

This theorem comes from K. Reidemeister (1927) and the moves \( R_i \) are called his name. In general, it is simpler to look for invariants of diagrams instead knots, as checking the independence on the Reidemeister moves is often quite easy. Also the original approach of Alexander to compute his polynomials uses diagrams. He enumerated the arcs and for any crossing wrote down a relation

\[ (1 - t)x - z - ty \]

according to the figure 4. The matrix of the system of linear equalities

\[ \begin{array}{c}
\text{Figure 4: A knot crossing}
\end{array} \]

obtained in this way is just an Alexander matrix of a special presentation
of a knot group.\footnote{ Wirtinger presentation; see \cite{[4, 7].}

**Example 3.3.** Having a diagram of a figure-eight knot with four crossings (fig. 5) we obtain a system of equalities

\[
\begin{align*}
(1 - t)x_1 &= x_3 - tx_4 \\
(1 - t)x_2 &= x_1 - tx_3 \\
(1 - t)x_3 &= x_1 - tx_2 \\
(1 - t)x_4 &= x_3 - tx_2
\end{align*}
\]

Then compute $3 \times 3$ minors of the matrix $A$ and their g.c.d., we have

\[
\Delta_1(4_1) = t^2 - 3t + 1.
\]

Figure 5: The diagram of the figure-eight knot

This approach gives us a better look on some aspects of the Alexander polynomials. For example, some problems become easily visible.

**Proposition 3.4.** Let $K$ be any knot. Then

\[
\Delta_1(K)(t) = \Delta_1(K^m)(t^{-1}) = \Delta_1(K^r)(t^{-1})
\]

**Proof.** To show the first equality notice that symmetric diagrams produce similar relations for any crossing - only the role of $t$ and $t^{-1}$ are exchanged:

\[
(1 - t)x = z - ty
\]

Reversing the orientation of a knot results in some changes:

\[
(1 - t)x = z - ty
\]

This proves the second equality. \qed
4 Skein relation

J. W. Alexander discovered an interesting relation, called the *skein relation*, satisfied by his polynomial. Having a diagram of an oriented knot we can create two new diagrams by changing or destroying one crossing. Denoting the diagrams as $\mathcal{X}$, $\mathcal{Y}$ or $\mathcal{Z}$ (outside this point they are identical) the relation is

$$\Delta_{\mathcal{Z}}(t) - \Delta_{\mathcal{X}}(t) = (t^{1/2} - t^{-1/2})\Delta_{\mathcal{X}}(t)$$

In early 1960’s J. Conway noticed, that this relation is sufficient to define a unique invariant (see [2]). He introduced a normalized Alexander polynomial, called now the *Conway polynomial*, by three conditions:

1. $\nabla_{K}(z)$ is a knot invariant
2. for an unknot $\nabla_{U}(z) = 1$
3. for any diagram $D$ the Conway-Alexander relation is satisfied:

$$\nabla_{\mathcal{Z}}(z) - \nabla_{\mathcal{Y}}(z) = z\nabla_{\mathcal{X}}(z)$$

Of course, for any knot $K$ we have $\nabla_{K}(t^{1/2} - t^{-1/2}) = \Delta_{K}(t)$.

**Example 4.1.** To compute the Conway polynomial of two disjoint circles, notice the following three diagrams:

The first two knots are trivial, so by the skein-relation:

$$z\nabla_{\mathcal{Z}}(z) = \nabla_{\mathcal{X}}(z) - \nabla_{\mathcal{Y}}(z) = 1 - 1 = 0$$

and as a result, $\nabla_{2U}(z) = 0$.

In the same way we can show, that the Conway polynomial of $n$ circles ($n > 1$) is 0.

**Example 4.2.** To compute the Conway polynomial of the trefoil, we will create a ‘computing tree’ (fig. 6). For each branched point we have a skein-relation:

$$\nabla_{\mathcal{Y}}(z) - \nabla_{\mathcal{X}}(z) = z\nabla_{\mathcal{X}}(z)$$

As we know $\nabla_{U}(z) = 1$ and $\nabla_{2U}(z) = 0$, so the Conway polynomial of the trefoil is

$$\nabla_{\mathcal{Y}}(z) = \nabla_{U}(z) + z(\nabla_{2U}(z) + z\nabla_{U}(z)) = 1 + z^2.$$
The above example shows the general algorithm of computing Conway polynomials. For a given diagram we create two others by changing or destroying one crossing. At each step the number of crossings does not increase. As any diagram becomes trivial by changing at most a half of its crossings, being aware of not changing the same crossing twice we will always obtain the unknot after some finite sequence of changes. According to this, for any knot we can create a 'computing tree' like in the example above with trivial links at bottom.

In 1984 another polynomial invariant satisfying a similar relation has been discovered by V. F. R. Jones. It was just an offshot of his work in a quite different area – the theory of operator algebra. For his achievements in understanding interactions between knot theory, geometry and quantum physics, Jones was awarded the Fields Medal in 1990.

The Jones polynomial is defined as follow:

1. $V_K(t)$ is a knot invariant
2. for an unknot $V_U(t) = 1$
3. for any diagram $D$ the following relation is satisfied:

$$t^{-1}V_X(t) - tV_X(t) = (t^{1/2} - t^{-1/2})V_Z(t)$$

This invariant is much more powerful than the Conway one as it can distinguish left-handed trefoil from the right-handed one. One can check in a similar way as in the case of Conway polynomial, that

$$V_{\Delta}^L(t) = -t^{-4} + t^{-3} + t^{-1} \quad \text{and} \quad V_{\Delta}^R(t) = -t^4 + t^3 + t.$$  

Moreover, the notion of Jones polynomials gives a green light to prove...
the Tait's conjectures, stated almost one hundred years before.\footnote{The first and the second conjectures of Tait were proved in 1987 in three different ways by Kauffman, Murasugi and Thistlethwaite. The approach of the Kauffman is nicely described in \cite{4}.}

After papers of Jones another polynomial invariant were being looked for, which could be a generalization of both Jones and Conway ones. It was simultaneously discovered in 1985 by P. J. Freyd, D. N. Yetter, J. Hoste, W. B. R. Lickorish, K. C. Millett and A. Ocneanu. Their work was published as a co-authored paper (\cite{5}) and the polynomial was called HOMFLY.

The HOMFLY polynomial is a unique two-variable Laurent polynomial $P(l, m)$ satisfying the relation:

$$lP_{\lambda}(l, m) - t^{-1}P_{\lambda}(l, m) + mP_{\lambda}(l, m) = 0$$

It contains all information carried by Jones and Conway ones. Indeed we have

$$\nabla_K(z) = P_K(1, -z)$$

$$V_K(t) = P_K(t^{-1}, t^{-1/2} - t^{1/2})$$

The HOMFLY polynomial can distinguish all prime knots up to 13 crossings. However, examples of different knots with same HOMFLY polynomial are known. On the other hand, the question about detecting unknottedness (does $P_K(l, m) = 1$ implies triviality of $K$) is still open.

References


