Spindle

$(X, \cdot)$ is a spindle if the operation $(x, y) \mapsto xy$ is

1. idempotent: $xx = x$
2. distributive: $(xy)z = (xz)(yz)$

A spindle coloring $x y x z x z y x y x z$ satisfies

$x y x z x z y x y x z = x y x z x z y x y x z$

and $x y x z x z y x y x z = x y x z x z y x y x z$

Distributive homology

Chains are given by sequences $(x_n, \ldots, x_0)$

$C_n(X) := \mathbb{Z}$

Differential is given by faces $d_n := \sum_{i=0}^{n} (-1)^i d_i$

where $d_i$ is defined as follows:

$d^{i-1}d^i = \sum_{j=i}^{n} \sum_{k=j}^{n} (-1)^{j-k} d_j d_k$

Degenerate vs Normalized

Sequences with repetitions are degenerate.

$C^D_n(X) := \{ x_n, \ldots, x_0 \}$

It is a subcomplex of $C(X)$, since

Normalised complex has no repetitions:

$\tilde{C}^N_n(X) = C(X)/C^D(X)$.

Theorem $C(X) \cong C^N(X) \oplus C^D(X)$.

Filtration

$C^D(X)$ is filtered $F^0 \subset F^1 \subset F^2 \subset \ldots$ with

$F^p_n := \mathbb{Z} \begin{pmatrix} \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \end{pmatrix} : i \leq p$

Quotients fix the first repetition at position $p$:

$F^p_n / F^{p-1}_n = \tilde{C}_{n-p-2} \oplus C^N_p$

Differential acts on the first term only:

$\Sigma \pm \begin{pmatrix} \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \end{pmatrix}$

Homology of the graded associate chain complex $\text{gr} C^D = \bigoplus_p F^p / F^{p-1}$ are easy:

$H_n(\text{gr} C^D) = \bigoplus_{p+q=n} \tilde{H}_{q-2} \oplus C^N_p$

First main result – one-term case

Theorem $C^D_n(X)$ is isomorphic to $gC^D_n(X)$. Hence, $H^D_n = \bigoplus \tilde{H}_{q-2} \oplus C^N_p$.

Second main result – two-term case

Theorem If $\alpha + \beta = 0$, then $f : \text{Tot}(B) \to C$ is a filtered chain map with respect to the two-term differential. It induces an isomorphism

$H^D_n(X) \cong H_{n-2}(\tilde{C} \oplus C^N) = H^N_{n-2} \oplus H_{n-2}(C \oplus C^N)$.

Let’s see, what’s happening in $B$. Green arrows represent $f$.

Bicomplex from filtration

Again $F^0 \subset F^1 \subset F^2 \subset \ldots$ filters $C^D(X)$ with

$F^p / F^{p-1} = \tilde{C}_{n-p-2} \oplus C^N_p$

The map $f : \text{gr} C^D \to C^D$ given by blue strands is no longer a chain map. However, it fits nicely, when we replace $\text{gr} C^D$ with a bicomplex $B = [C^2] \oplus C^N$.

$\tilde{C}^N_{q-2} \oplus C^N_{p-1} \overset{id \oplus d}{\longrightarrow} \tilde{C}^N_{q-2} \oplus C^N_{p}$

Direct sums along diagonals form $\text{Tot}(B)$. Everything is cancelled except the middle column. But this is precisely $id \oplus d$.

$\begin{pmatrix} \Sigma \pm \begin{pmatrix} \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \end{pmatrix} \end{pmatrix}$

In the actual computation, one more relation is necessary:

This holds, because the red strand acts trivially.

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