The Yoneda embedding & representable functors

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Motivation

Fact: If $C$ is any category & $D$ is complete/incomplete, then $[C, D]$ is complete/incomplete.

For $[C, \mathbf{Set}]$ is complete & incomplete for any category $C$. It is also Cartesian closed.

The set $[C, \mathbf{Set}]$ has special elements, namely functors of the form $Y^X := \text{Mor}_C(X, -)$.

Similarly, we have functors $Y_x := \text{Mor}_C(-, x)$ in $[C^o, \mathbf{Set}]$.

Prop: There are functors $Y^* : C^o \to [C, \mathbf{Set}]$ and $Y : C \to [C^o, \mathbf{Set}]$ defined for morphisms in $C$ as follows:

If $g : A \to B$ then $Y^*: Y^B \Rightarrow Y^A$ is a natural transformation s.t. $(Y^g)_x = y^g^x$

$Y_g : Y^A \Rightarrow Y^B$

$(Y_g)_x = g_x$

Problem: Are $Y^*$ and $Y^g$ embeddings? In other words, are they faithful and full?

Yoneda embedding:

Recall that $\mathbf{Cat}$ is Cartesian closed with the evaluation map $e : C \times [C, D] \to D$
given by:

$(A, F) \mapsto FA$

$e(f, u) = \mu_B^f \circ F$ for $f : A \to B$, $u : F \Rightarrow G$

Exercise: Check that $e : C \times [C, D] \to D$ is a functor.

The Yoneda lemma is a first step to show that $Y^*$ and $Y^g$ are embeddings. It states that any functor $F : C \to \mathbf{Set}$ can be computed by looking at natural transformations between $Y^A$ and $F$.

Thm. (Yoneda Lemma)

Let $C$ be any category & $F : C \to \mathbf{Set}$ be any functor. Then for any $A \in \text{Ob} C$

the following map is a bijection natural in both $A$ and $F$:

$\phi : \text{Nat}(Y^A, F) \xrightarrow{\simeq} FA \ \alpha \mapsto \alpha_A (1_A)$.

If $F$ is contravariant, take $Y_A$ instead of $Y^A$. 
**Proposition:** Let \( \psi: A \to \text{Nat}(Y^A, F) \) be defined as \( \alpha \mapsto (\alpha: Y^A \to F) \) s.th. \( \alpha_Y \circ F = FF \).

Then \( (\gamma \circ \alpha)(a) = \psi(\alpha) = \alpha_Y(a) = F \alpha(a) = FF \alpha(a) = a \).

\[
(\gamma \circ \alpha)(\alpha) = \psi(\alpha) = \alpha_Y(\alpha) = \alpha_Y(\alpha)
\]

But \( \alpha: Y^A \to F \) so for any \( f: A \to B \).

\[
\begin{align*}
\text{Mor}(A, A) & \xrightarrow{\alpha} \text{FA} \\
\downarrow F & \downarrow F \\
\text{Mor}(A, B) & \xrightarrow{\alpha \circ B} \text{FB}
\end{align*}
\]

so \( \alpha_Y(\alpha) \circ B(f) = F \alpha_Y(\alpha)(f) = \alpha_Y(f) \), hence \( \alpha_Y(\alpha) = \alpha \).

**Naturality:** Let \( g: A \to B \), \( \alpha: F \to G \). We have to check the square commutes:

\[
\begin{array}{ccc}
\text{Nat}(Y^A, F) & \xrightarrow{\psi} & \text{FA} \\
\downarrow (\gamma \circ \alpha) & & \downarrow \mu_B \circ FG \\
\text{Nat}(Y^B, G) & \xrightarrow{\psi} & \text{GB}
\end{array}
\]

\[
\begin{align*}
\gamma \circ (\mu_B \circ FG \circ \psi)(\alpha) &= (\mu_B \circ FG \circ \psi)(\alpha)(1_B) = (\mu_B \circ \alpha)(\gamma)(1_B) = \alpha(\gamma)(1_B)
\end{align*}
\]

\[
\begin{align*}
\gamma \circ \psi(\alpha) &= \psi(\alpha \circ \gamma) = (\mu_B \circ \alpha \circ \gamma)(1_B) = \alpha(\gamma)(1_B)
\end{align*}
\]

\[
\square
\]

**Thm (Yoneda)** The functors \( Y^c: C^{op} \to [D, \text{Set}] \) and \( Y^d: C \to [D^{op}, \text{Set}] \) are full and faithful.

**Pf** By the Yoneda Lemma:

\[
\begin{align*}
\text{Mor}_c(A, B) &= Y^A(B) \cong \text{Nat}(Y^B, Y^A) \\
\text{Mor}_c(A, B) &= Y_A(A) \cong \text{Nat}(Y_A, Y_B)
\end{align*}
\]

Recall that \( F \) is full if \( \text{Mor}(A, B) \xrightarrow{F} \text{Mor}(FA, FB) \) is surjective, i.e. any morphism \( g: FA \to FB \) is an image of some \( f: A \to B \).

\( F \) is faithful, if the map is injective, i.e. \( FF = FG \Rightarrow FG \).

Therefore, \( F \) is full \& faithful, \( A \cong B \) if \( FA \cong FB \). (Prove this!)

\[\text{Cor: } A \cong B \text{ in } C \text{ iff } Y_A \cong Y_B \text{ iff } Y^A \cong Y^B\]

\[\text{If } \alpha \xrightarrow{\psi} \beta \text{ are functors s.th. } \text{Mor}_c(X, FA) \cong \text{Mor}_c(X, GA) \text{ for any } X \in D, A \in C, \text{ then } F \cong G.\]

**Pf** \( \text{Mor}_c(X, FA) \cong \text{Mor}_c(X, GA) \Rightarrow Y_A \cong Y_B \Rightarrow FA \cong GA \text{ (naturally in } A) \Rightarrow F \cong G.\)

\[\text{If } F \cong G \text{ \& } F \cong G', \text{ then } G \cong G'.\]

\[\text{If } F \cong G \text{ \& } F \cong G', \text{ then } F \cong G'.\]
Every category is isomorphic to a subcategory of $\textbf{Set}$.

(1) $\text{Ob} \rightarrow X : \{ f : A \rightarrow X \mid A \in \text{Ob} \}$

$\text{Mor} \rightarrow \overline{\text{X}} : \overline{f} = g \circ f$

Then $\overline{X} = \overline{Y} = X = Y$, since $X, Y = \text{colim}(L_x), \text{colim}(L_y)$ and $L_x, L_y \in \overline{X} = \overline{Y}$.

$\overline{g} = h \Rightarrow g = \overline{g}(L_x) = \overline{h}(L_x) = h$.

Remark: The functors $Y'$ and $Y$, are called Yoneda embeddings. They preserve limits but not colimits. Moreover, any functor $C \rightarrow \textbf{Set}$ is a colimit of some functors $Y^A$, i.e. the image of $Y'$ is "colimit dense" and similarly for $Y$.

Representable functors

Def: A functor $F : C \rightarrow \textbf{Set}$ is representable if for some $A \in \text{Ob} \bar{C}$, $F \cong YA$.

Cor: If $F : C \rightarrow \textbf{Set}$ is representable, then it preserves limits.

If $F : C \rightarrow \textbf{Set}$ is representable, then it maps colimits to limits.

Let $F \cong YA$ and $f : A \rightarrow X$ be any morphism. Then we have a commutative square

\[
\begin{array}{ccc}
\text{Mor}(A, A) & \xrightarrow{YA} & FA \\
\downarrow f \downarrow & & \downarrow f \downarrow \\
\text{Mor}(A, X) & \xrightarrow{YX} & FX \\
\end{array}
\]

Since $\phi$ is a bijection, there is such $f : A \rightarrow X$ for any $x \in FX$ and we have the following universal property of the pair $(A, u)$.

(∀x ∈ FX)(∃! f : A → X : Ff(u) = x).

Def: A pair $(A, u)$ having the property (r) is called a universal pair for $F$. The element $u \in FA$ is called a universal element of $F$.

Thm: A pair $(A, u)$ is universal for $F$ if there exists an $\alpha : YA \cong F$ s.t. $u = \alpha(A)$. \(\Rightarrow\) done above.

$\Rightarrow$ Let $\psi_X : \text{Mor}(A, X) \rightarrow FX$ be defined as $\psi_X(f) = Ff(u)$. Since $(A, u)$ is universal, $\psi$ is bijective (follows from (r)). Naturally follows from:

$(F \circ \psi_X)(f) = F(\psi_X(f)) = F(f(u)) = (\psi_Y \circ f)(u)$ where $\alpha : X \rightarrow Y$.
Examples

1. A constant functor \( \Delta_A : X \to A \) is representable iff \( A = (a) \) and \( C \) has an initial object \( \mathbf{1} \).

   \[ \text{PF} \leftarrow \Delta_A \cong \text{Mor}(\mathbf{1}, -) \]

   By \( (\alpha) \):
   \[
   \begin{array}{c}
   \xymatrix{
   \ast \ar[r]^-{\alpha} & A \\
   X \ar[u]_A \ar[r]_-{\alpha} & A \ar[u]_A
   }
   \end{array}
   \]

   so \( \forall x : A \cdot x = a \) and \( A = (a) \).

   Since \( \Delta_A \cong \text{Mor}(X, -) \) and \( \forall B \cdot \Delta_A(B) \) is a singleton, \( X \) is initial.

2. Let \( F = \text{Mor}(A, X) \times \text{Mor}(B, X) \) for fixed \( A, B \in C \). Then \( F \) is represented by the categor product \( A \times B \) and the universal element \( (i_A, i_B) \).

3. Let \( F = \text{Mor}(X, A) \times \text{Mor}(X, B) \) for fixed \( A, B \in C \). Then \( F \) is represented by the product \( A \times B \) and the universal element is \( (\pi_A, \pi_B) \).

Remark: In a similar way any pro limit/limit can be defined.

4. Let \( F : R-\text{Mod} \to \text{Set} \) be defined as

   \[ F(P) = \text{Bilin}_{R}(M, N; P) - \text{bilinear map from } M \times N \text{ to } P \text{ for } M, N \text{ fixed modules.} \]

   Then \( F \) is represented by the tensor product \( M \otimes N \) and a universal element

   \[ \omega : M \otimes N \to M \otimes N \]

5. \( F : \text{Vec}_R \to \text{Vec}_R \)

   \[ F V = V^*, \text{dual vector space} \]

   Then \( F \cong \text{Hom}_R(-, \mathbb{R}) \).

6. \( F : \text{Vec}_R \to \text{Set} \)

   \[ F V = \{ \text{Functions } X \to V \} \text{ for a fixed set } X \]

   Then \( F \) is represented by the vector space \( M_R X \) spanned by \( X \) with a universal map \( X \to M_R X \) being the standard inclusion.

7. \( F : \text{Set} \to \text{Set} \)

   \[ F X = \{ f : R \to X \mid f(x + t) = f(x) \} \]

   Then \( F \) is representable by the set \( \text{Mor}_\text{Set}(X, S) \) with a universal element \( \omega \in \mathbb{R} \to S^1 \cdot \omega(t) = e^{2\pi it} \)

Prop: Let \( F : C \to \text{Set} \) be a representable functor, \( \text{if } (A, w) \text{ and } (A', w') \) are universal pairs for \( F \), then there is a unique iso \( h : A \cong A' \) s.t. \( F h(w) = w' \).

Fact: \( F \)-representable, then it preserves limits. If \( C \) is small, then the converse is also true.