There are three approaches to universal constructions:

- universal arrows (limits, colimits, initial and terminal objects)
- adjunctions
- representable functors

In fact, any construction expressed in one of the above ways can be also described in any other.

3. Limits / colimits

Let $D: I \to C$ be a diagram in $C$ of type $I$. Then:

- $\lim_I D$ = the terminal object in $\text{Cone}(D)$
- $\text{Mor}_{[I,C]}(\Delta_X, D) \cong \text{Mor}_C(X, \lim_I D)$, so $\Delta_\cdot \to \lim_I (\cdot)$
  where $Y \in \text{Ob}_C$, $\Delta_\cdot : C \to [I, C]$ is a functor given by
  $\Delta_X(i) = X$, $\Delta_X(i \to j) = 1_X$

- $\text{Mor}(A, \lim_I D) \cong \lim_I \text{Mor}(A, D(\cdot))$, so $\lim_I D$ represents the functor on the right.

Examples

1) $I = \{1, 2\}$ discrete category with two objects

$D(1) = A$, $D(2) = B$ $\Rightarrow$ $\lim_2 D = A \times B$

$\Delta_X(1) = \Delta_X(2) = X$ $\Rightarrow$ a natural transformation $\alpha : \Delta_X \Rightarrow D$ consists of two arrows $X \xrightarrow{\alpha_A} A$ and $X \xrightarrow{\alpha_B} B$

Let $F: [I, C] \to C$ be right adjoint to $\Delta_\cdot$. Then the bijectivity

$\text{Mor}(\Delta_X, D) \cong \text{Mor}(X, F(D))$

means for any two arrows $X \to A$ and $X \to B$ there is a unique arrow $X \to F(D)$, so $F(D)$ must be a product $A \times B$:

The projections $F(1) \xrightarrow{\pi_A} A$ and $F(2) \xrightarrow{\pi_B} B$ are given by

$\text{Mor}(\Delta_{FD}, D) \cong \text{Mor}(FD, FD)$ $(\pi_A, \pi_B) \mapsto 1_{FD}$
Now, representability of $FX := \text{Mor}(X,A) \times \text{Mor}(X,B)$ means

$$\text{Mor}(X,A) \times \text{Mor}(X,b) \cong \text{Mor}(X,D)$$

for some object $D \in C$.

But we know that the left is equal $\text{Mor}(X,A \times B)$ and by uniqueness theorem, $D \cong A \times B$. This also can be seen by the universality of a pair $(D, u)$, where $u \in \text{Mor}(D,A) \times \text{Mor}(D,B)$ is a pair of projections:

$$u \xrightarrow{a_a} A \leftarrow D \rightarrow B \xleftarrow{a_b} u$$

2) $I = \emptyset$ empty category

- $D$ - empty functor $\Rightarrow \lim_\Delta D = T$ a terminal object in $C$
- $\Delta X$ - empty functor $\Rightarrow$ there is only one natural transformation $\alpha : \Delta X \to D$, the empty one.

Hence, $\Delta_{X \to D} : \Delta_{(\_)} \to F$ means that:

$$|\text{Mor}_C(X,FD)| = |\text{Mor}_{\{1,0\}}(\Delta X, D)| = 1$$

$\Rightarrow$ FD is the terminal object in $C$.

2) Adjunctions

Let $C \xrightarrow{F \circ G} D$ be two functors. Then the following are equivalent:

- $F \to G$
- there is a natural transformation $\eta : \text{id} \to GF$ s.t.

$$\xymatrix{X \ar[r]^{GF} \ar[d]_\eta \ar@{..>}[rd]^{GF} & GFX \ar[d] \ar@{..>}[r] & GF \ar[d]_G \ar@{..>}[d] \ar@{..>}[r] & G\ar[d] \ar[r]_G & GY \ar[d] \ar@{..>}[r] & GY }$$

- there is a natural transformation $\varepsilon : FG \to \text{id}$ s.t.

$$\xymatrix{FX \ar[r]^{FG} \ar[d]_\varepsilon \ar@{..>}[rd]^{FG} & FGX \ar[d] \ar@{..>}[r] & GF \ar[d]_F \ar@{..>}[d] \ar@{..>}[r] & FY \ar[d]_F \ar[r]_F & FY \ar[d] \ar@{..>}[r] & FY }$$

- for any $X \in C$, $FX$ represents the functor $\text{Mor}_C(X, G(-))$
- for any $Y \in D$, $GY$ represents the functor $\text{Mor}_D(F(-), Y)$
The arrows \( \eta_x \) and \( \varepsilon_x \) can be seen as universal ones (terminal or initial) in some category:

\[ \eta_x \text{ is initial in } X \downarrow G \]
\[ \text{Ob}(X \downarrow G) = \text{arrows in } C \]
\[ X \rightarrow GY \]
\[ \text{Mor}(X \downarrow G) = \text{commutative triangles} \]
\[ f \]
\[ X \]
\[ f' \]
\[ GY \rightarrow GY' \]

\[ \varepsilon_x \text{ is terminal in } F \downarrow X \]
\[ \text{Ob}(F \downarrow X) = \text{arrows in } D \]
\[ FY \rightarrow X \]
\[ \text{Mor}(F \downarrow X) = \text{commutative triangles} \]
\[ F \phi \]
\[ FY \rightarrow FY' \]

The universality of \( \eta \) and \( \varepsilon \) is exactly what we need to reconstruct \( F \) or \( G \) knowing the other. In fact, we need only morphisms \( \eta_x \) and \( \varepsilon_y \) for every \( X \in C \), \( Y \in D \) without any assumption they form a natural transformation!

For the last two points notice:

\[ F \dashv G \Rightarrow \exists \text{ a natural bijection } \text{Mor}_C(FX, Y) \cong \text{Mor}_D(X, GY) \]

hence, both sides are obviously represented by \( GY \) or \( FX \). For the converse, say

\[ \text{Mor}_C(X, G(-)) \cong \text{Mor}_D(A_x, -) \]

is representable by some object \( A_x \in D \). We may put \( F(X) = A_x \). The value on morphisms can be defined by the universal element \( u \in \text{Mor}_C(X, G(FX)) \):

\[ \text{(more in the next week)} \]

(3) Representable functors

Let \( F : C \rightarrow \text{Set} \) be a functor. Then TFAE:

- \( F \) is representable
- \( F \) has a left adjoint
- there is a universal pair \( (A, u) \) s.t.

\[ \begin{array}{c}
\scalebox{0.7}[1]{\text{u}} & \scalebox{0.7}[1]{\text{FA}} \\
\text{x} & \text{FA} \\
\text{FX} & \end{array} \]

i.e., \( \text{u} \rightarrow \text{FA} \) is the initial object in \( X \downarrow F \)
Notice, a covariant functor is representable iff there is an initial object in some category. Dually, a contravariant functor is representable iff some terminal object in some category exists. Because people prefer limits than colimits, you can usually find a definition of representability for contravariant functors!

We will now prove, that $F$ is representable iff it has a left adjoint.

"$\Rightarrow$" Let $U \Rightarrow F$ and $\ast$ be a singleton. Then:

$$FX \cong \text{Mor}(\ast, FX) \cong \text{Mor}(U(\ast), X) \Rightarrow F \text{ is represented by } A := U(\ast)$$

any element $s$ in a set $S$ can be seen as a map

$\ast \to S$

"$\Rightarrow$" First notice, that $\text{Mor}(A, X) \cong FX \cong \text{Mor}(\ast, FX)$ if $A$ represents $F$.

So it must be $U(\ast) = A$. But for any set $S$:

$$S = \coprod_{s \in S} \ast \quad S \text{ is a disjoint sum of singletons}$$

Therefore, let $U(S) := \coprod_{s \in S} A_s$ be the coproduct of $|S|$ copies of $A$.

Then $U \Rightarrow F$ and the bijection $\varphi_{S,X} : \text{Mor}(S, FX) \cong \text{Mor}(U(S), X)$ is defined by

$$\varphi_{S,X}(f : S \to FX) := \langle f(s) \rangle_{s \in S}$$

where the right side is the unique map $\coprod_{s \in S} A_s \to X$ given by maps $A_s \xrightarrow{f(s)} X$.

"(notice that $f(s) \in FX \cong \text{Mor}(A, X)$)."

Here we assumed that $C$ has all coproducts.