COHOMOLOGY OF $B_{\alpha_p}$ AND APPROXIMATION BY VARIETY (DRAFT UNFINISHED)

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ABSTRACT. We calculate the crystalline, de Rham and Hodge cohomology of $B_{\alpha_p}$ by hypercovering spectral sequence. We show the Hodge to de Rham spectral sequence fail to degenerate for $B_{\alpha_p}$ on $E_1$ page. We push such example to varieties and construct a smooth projective threefold with $h^{2,0}_{Hdg} = 1, h^{1,1}_{Hdg} = 8, h^{0,2}_{dR} = 3, h^2_{dR} = 5$. This example can be used to pin down the structure of the Hodge ring of characteristic $p$ varieties, where it is asked to find a threefold with $h^{2}_{Hdg} - h^2_{dR}$ being an odd number, see [vDdB18].

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1. INTRODUCTION

Let $k$ be a perfect field of characteristic $p$. Note that $B_{\alpha_p}$ can be identified with the quotient stack $[E^{(p)}/E]$ by the relative Frobenius. We use a proper smooth hypercovering to calculate the crystalline, de Rham and Hodge cohomology of $B_{\alpha_p}$.

2. GENERAL CALCULATION ON $B_{\alpha_p}$

[Under construction, some linear algebra calculation is missing.]

3. AN EXAMPLE

We work over $k$, an algebraically closed field of characteristic $p$. Let $E$ be a supersingular elliptic curve and $F': E \to E^{(p)}$ the relative Frobenius. Take the representation of $\alpha_p$ in $GL_5$ with only one fixed line [vDdB18, 1.2.7]. Consider the morphism $\mathbb{P}^4 \times E \to X_E = \mathbb{P}^4 \times_{\alpha_p} E$. Consider the elliptic fibration to $X = \mathbb{P}^4/\alpha_p$ and the projective bundle onto $E^{(p)}$. Let $Y$ be an ample divisor in the fixed point free locus of $X$. Our goal is to calculate the Hodge and de Rham cohomology of the threefold $Y$. Let $Y_E = Y \times_X X_E$ be the $E$ fibration over $Y$.

\[
\begin{array}{ccc}
Y_E & \longrightarrow & X_E = \mathbb{P}^4 \times_{\alpha_p} E \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X = \mathbb{P}^4/\alpha_p \\
\downarrow & & \downarrow \\
E & \longrightarrow & E^{(p)}
\end{array}
\]

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We can calculate the cohomology of $Y$ by hypercovering spectral sequence induced by $Y_E \to Y$. Let's identify the Cech nerves of the covering. In order to avoid confusion with fiber product, we denote the quotient of $E \times \mathbb{P}^4$ by diagonal $\alpha_p$ action by $\frac{E \times X_{\mathbb{P}^4}}{\alpha_p}$). Note that

$$Y_E \times_Y Y_E = \left( E \times \mathbb{P}^4 \right) \times_{\mathbb{P}^4/\alpha_p} \left( \mathbb{P}^4 \times E \right) \times_Y \left( Y \times \mathbb{P}^4/\alpha_p \right) \times_{\mathbb{P}^4/\alpha_p} \left( \mathbb{P}^4 \times \mathbb{P}^4 \right) \times_{\mathbb{P}^4/\alpha_p} \left( \mathbb{P}^4 \times E \right) \times_Y \left( E \times \mathbb{P}^4 \right) \times_{\mathbb{P}^4/\alpha_p} \left( E \times \mathbb{P}^4 \right) \times_{\mathbb{P}^4/\alpha_p} \mathbb{P}^4 \times \mathbb{P}^4$$

The second to last step equality follows because $\frac{E \times X_{\mathbb{P}^4}}{\alpha_p}$ is an $E$-torsor over $\mathbb{P}^4/\alpha_p$, after base change by itself, the diagonal provides a section, so it is a trivial $E$-torsor. Similarly, one can show the degree $n$ part of the coskeleton to $Y_E \to Y$ can be expressed as $U_n = E^{n-1} \times Y_E$.

In the following sections, we first calculate the crystalline cohomology of $Y$ using the hypercovering, then get the de Rham cohomology by derived tensor. After that we use the hypercovering to calculate the Hodge numbers of $Y$.

3.1. Crystalline cohomology.

3.1.1. Preparation.

3.1.2. $H^i_{\text{crys}}(X_E)$. The crystalline cohomology of a projective bundle is the same as that of a product [reference], we have

$$H^0_{\text{crys}}(X_E) = H^0_{\text{crys}}(X) = W$$

and for $1 \leq i \leq 9$, we have

$$H^i_{\text{crys}}(X_E) = W \oplus W.$$ 

3.1.3. $H^i_{\text{crys}}(Y_E)$. We adopt the following Lefschetz theorem in [BMS16, 2.12]

**Theorem 3.1.1.** Let $k$ be a perfect field of characteristic $p$, and $X$ be a smooth projective variety of dimension $d$ over $k$, with a line bundle $\mathcal{L}$. Let $i_L \geq 0$ be an integer such that for any coherent sheaf $\mathcal{F}$ on $X$, the cohomology group $H^i(X, \mathcal{F} \otimes \mathcal{L}^n)$ vanishes if $n$ is sufficiently large and $i > i_L$. Then there exists some integer $n_0$ such that for $n \geq n_0$, and any smooth hypersurface (not necessarily ample) $H \subset X$ with divisor $\mathcal{L}^n$, the map

$$H^j_{\text{crys}}(X/W) \to H^j_{\text{crys}}(H/W)$$

is an isomorphism for $j < d - i_L - 1$ and injective with torsion-free cokernel for $j = d - i_L - 1$.

Note that $Y_E$ is a relative hypersurface for $X_E \to E^{(p)}$. Take $i_L = 1$, we know

$$H^0(Y_E) = W, H^1(Y_E) = W \oplus W, H^2(Y_E) = W \oplus W$$

and

$$0 \to H^3(Y_E) \to H^3(X_E) = W^\oplus 2 \to W^\oplus r \to 0$$

Note that $H^2(Y_E) \cong H^2(E^{(p)}) \oplus H^0(E^{(p)})$, but perhaps non-canonically, anyway the splitting can be obtained by taking a section of the projective bundle.
3.1.4. The hypercovering spectral sequence. The $E_1$ page of hypercohomology spectral sequence for crystalline cohomology has the following form:

$$
\begin{array}{cccc}
H^0(U_4) & H^1(U_3) & H^2(U_2) & H^3(U_1) \\
d_1^{3,0} & d_1^{2,0} & d_1^{1,0} & d_1^{0,0} \\
H^0(U_3) & H^1(U_2) & H^2(U_1) & H^3(U_0) \\
\end{array}
$$

3.1.5. The maps. Let’s pin down the maps corresponding to $H^i(E \times E_Y/W) \rightarrow H^i(E_Y/W)$ induced by projection and action, we claim this is the same as induced from the fiber product diagram:

$$
\begin{array}{ccc}
E \times Y_E & \rightarrow & Y_E \\
\downarrow & & \downarrow \\
E \times E^{(p)} & \rightarrow & E^{(p)} \\
\end{array}
$$

Note that we have sections, $E \times 0$ and $0 \times E^{(p)}$. By Kunneth formula, $H^i(E \times E^{(p)}) = \wedge^i(H^1(E, \mathcal{O}) \oplus H^1(E^{(p)}, \mathcal{O}))$. For the action map, the induced map on frobenius is $\phi = 0$: $H^1(E^{(p)}/W) \rightarrow H^1(E/W)$ is given by the matrix (where $v_p(\pi) = 1$)

$$
\begin{bmatrix}
0 & \pi \\
1 & 0
\end{bmatrix}
$$

3.1.6. The first column. This column has terms

$$
H^0(Y_E) \rightarrow H^0(E \times Y_E) \rightarrow H^0(E \times E \times Y_E) \rightarrow \cdots ,
$$

where the differential is given by alternating sum of face maps. Since $Y_E$ is smooth connected, we know this column is

$$
W \xrightarrow{0} W \xrightarrow{1} W \xrightarrow{0} W \xrightarrow{1} W \xrightarrow{0} \cdots
$$

In degrees 0, 1, this can be identified with $H^i(E^{(p)})$ via pullback.

3.1.7. The second column. This column has terms

$$
H^1(Y_E) \rightarrow H^1(E \times Y_E) \rightarrow H^1(E \times E \times Y_E) \rightarrow \cdots .
$$

This can be identified via Kunneth formula with

$$
H^1(E^{(p)}) \rightarrow H^1(E) \oplus H^1(E^{(p)}) \rightarrow H^1(E)^{\oplus 2} \oplus H^1(E^{(p)}) \rightarrow H^1(E)^{\oplus 3} \oplus H^1(E^{(p)})
$$

The homology of this complex is

$$
0 \ W/p \ 0 \ 0 \ \cdots
$$
3.1.8. The third column. The column has terms
\[ H^2(Y_E) \rightarrow H^2(E \times Y_E) \rightarrow H^2(E \times E \times Y_E) \rightarrow \cdots. \]
This can be identified via Kunneth formula with
\[ H^2(Y_E) \cong H^2(E) \oplus H^1(E) \otimes H^1(Y_E) \oplus H^2(Y_E)[1]. \]
\[ H^2(E) \oplus H^2(E) \oplus H^2(Y_E) \oplus (H^1(E) \otimes H^1(E)) \oplus (H^1(E) \otimes H^1(Y_E)) \oplus (H^1(E) \otimes H^1(E^p)) \oplus (H^1(Y_E) \otimes H^1(E^p)) \oplus (H^1(Y_E) \otimes H^1(Y_E)) \oplus \cdots \]
This is a bit complicated to express the maps, but we guess this may be isomorphic to the first column direct sum with the third column, which is
\[ W \oplus W/p \Rightarrow. \]

3.2. De Rham cohomology. Hence the crystalline cohomology is
\[ H^0_{cryst}(Y) = W, H^1_{cryst}(Y) = 0, \]
\[ 0 \rightarrow W/p \rightarrow H^2_{cryst}(Y) \rightarrow W \rightarrow 0. \]

Hence
\[ H^2_{cryst}(Y) = W \oplus W/p. \]
The de Rham cohomology is
\[ H^0_{dR}(Y) = W/p \]
\[ H^1_{dR}(Y) = W/p \]
\[ H^2_{dR}(Y) = W/p \oplus W/p \]

3.3. Hodge Cohomology.

3.3.1. Preparation.

3.3.2. \( H^i(Y_E, \mathcal{O}_{Y_E}) \). Consider the Leray spectral sequence associated to the projective bundle \( f: X_E \rightarrow E^p \), by cohomology and base change, we know the \( E_2 \) page only has entry \( H^0(E^p, f_* \mathcal{O}_{X_E}) \) and \( H^1(E^p, f_* \mathcal{O}_{X_E}) \). As \( f \) is geometrically reduced and geometrically connected, we know \( H^0(X_E, \mathcal{O}_{X_E}) = H^1(X_E, \mathcal{O}_{X_E}) = k \), all other cohomologies vanish. Let \( T \) be the ideal sheaf of \( Y_E \) in \( X_E \), we have
\[ 0 = H^0(X_E, T) \rightarrow H^0(X_E, \mathcal{O}_{X_E}) \rightarrow H^0(Y_E, \mathcal{O}_{Y_E}) \rightarrow \]
\[ 0 = H^1(X_E, T) \rightarrow H^1(X_E, \mathcal{O}_{X_E}) \rightarrow H^1(Y_E, \mathcal{O}_{Y_E}) \rightarrow \]
\[ 0 = H^2(X_E, T) \rightarrow H^2(X_E, \mathcal{O}_{X_E}) \rightarrow H^2(Y_E, \mathcal{O}_{Y_E}) \rightarrow \]
\[ 0 = H^3(X_E, T) \rightarrow H^3(X_E, \mathcal{O}_{X_E}) \rightarrow H^3(Y_E, \mathcal{O}_{Y_E}) \rightarrow \]
Hence we know
\[ H^0(Y_E, \mathcal{O}_{Y_E}) = H^1(Y_E, \mathcal{O}_{Y_E}) = k, H^2(Y_E, \mathcal{O}_{Y_E}) = 0. \]

3.3.3. \( H^i(Y_E, \Omega^1_{Y_E}) \). We know the Hodge numbers of \( X_E \) are
\[ H^0(X_E, \Omega^1_{X_E}) = k, H^1(X_E, \Omega^1_{X_E}) = k + k \]
and
\[ H^2(X_E, \Omega^1_{X_E}) = k. \]
We have the short exact sequence
\[ 0 \rightarrow (\mathcal{I}/\mathcal{I}^2)|_{Y_E} \rightarrow \Omega^1_{X_E}|_{Y_E} \rightarrow \Omega^1_{Y_E} \rightarrow 0. \]
Thus the associated long exact sequence
\[ 0 \rightarrow H^0(Y_E, (\mathcal{I}/\mathcal{I}^2)|_{Y_E}) \rightarrow H^0(Y_E, \Omega^1_{X_E}|_{Y_E}) \rightarrow H^0(Y_E, \Omega^1_{Y_E}) \rightarrow \]
\[ H^1(Y_E, (\mathcal{I}/\mathcal{I}^2)|_{Y_E}) \rightarrow H^1(Y_E, \Omega^1_{X_E}|_{Y_E}) \rightarrow H^1(Y_E, \Omega^1_{Y_E}) \rightarrow \]
\[ H^2(Y_E, (\mathcal{I}/\mathcal{I}^2)|_{Y_E}) \rightarrow H^2(Y_E, \Omega^1_{X_E}|_{Y_E}) \rightarrow H^2(Y_E, \Omega^1_{Y_E}) \rightarrow \]
We apply cohomology and base change to calculate the fibers of \( \mathcal{I}/\mathcal{I}^2 \). Suppose the relative degree of \( Y_E \) is \( d \), take
\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-2d) \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \rightarrow \mathcal{O}_H(-d) \rightarrow 0 \]
This gives the long exact sequence

$$0 \to H^0(O_H(-d)) \to H^1(O_{p^*}(-2d)) \to H^1(O_{p^*}(-d)) \to H^1(O_H(-d)) \to$$

$$\to H^2(O_{p^*}(-2d)) \to H^2(O_{p^*}(-d)) \to H^2(O_H(-d)) \to$$

$$\to H^3(O_{p^*}(-2d)) \to H^3(O_{p^*}(-d)) \to H^3(O_H(-d)) \to$$

$$\to H^4(O_{p^*}(-2d)) \to H^4(O_{p^*}(-d)) \to H^4(O_H(-d)) \to$$

We see $H^i(O_H(-d)) = 0$ for $i = 0, 1, 2$, and could be large for $d = 3$. Hence

$$H^0(Y_E, \Omega^1_{X_E}|Y_E) = H^0(Y_E, \Omega^1_{X_E}|Y_E)$$

and

$$H^1(Y_E, \Omega^1_{X_E}) = H^1(Y_E, \Omega^1_{X_E}|Y_E).$$

Note that we have

$$0 \to \Omega^1_{X_E} \otimes \mathcal{I} \to \Omega^1_{X_E} \to \Omega^1_{X_E}|Y_E \to 0$$

The long exact sequence tells us

$$0 \to H^0(X_E, \Omega^1_{X_E} \otimes \mathcal{I}) \to H^0(X_E, \Omega^1_{X_E}) \to H^0(Y_E, \Omega^1_{X_E}|Y_E) \to$$

$$\to H^1(X_E, \Omega^1_{X_E} \otimes \mathcal{I}) \to H^1(X_E, \Omega^1_{X_E}) \to H^1(Y_E, \Omega^1_{X_E}|Y_E) \to$$

$$\to H^2(X_E, \Omega^1_{X_E} \otimes \mathcal{I}) \to H^2(X_E, \Omega^1_{X_E}) \to H^2(Y_E, \Omega^1_{X_E}|Y_E) \to$$

$$\to H^3(X_E, \Omega^1_{X_E} \otimes \mathcal{I}) \to H^3(X_E, \Omega^1_{X_E}) \to H^3(Y_E, \Omega^1_{X_E}|Y_E) \to$$

We calculate $H^i(X_E, \Omega^1_{X_E} \otimes \mathcal{I})$. Let $F$ be a fiber, we have

$$0 \to \mathcal{O}_F(-d) \to \Omega^1_{X_E}(-d)|_F \to \Omega^1_{X_E}(-d) \to 0,$$

hence long exact sequence

$$\cdots \to H^i(F, \mathcal{O}_F(-d)) \to H^i(F, \Omega^1_{X_E}(-d)|_F) \to H^i(F, \Omega^1_{F}(-d)) \to \cdots$$

Note Bott’s calculation that $H^q(\mathbb{P}^n, \mathcal{O}^p_{\mathbb{P}^n}(r)) = 0$ unless

$$(1)p = q, r = 0, \quad (2)q = 0, r > p, \quad (3)q = n, r < p - n.$$

We have $H^i(F, \Omega^1_{X_E}(-d)|_F) = 0$ for $i = 0, 1, 2, 3$, by cohomology and base change, we know $H^i(X_E, \Omega^1_{X_E} \otimes \mathcal{I}) = 0$ for $i = 0, 1, 2, 3$. Hence $H^i(X_E, \Omega^1_{X_E}) = H^i(Y_E, \Omega^1_{X_E}|Y_E)$ for $i = 0, 1, 2$, hence

$$H^0(Y_E, \Omega^1_{Y_E}) = k,$$

$$H^1(Y_E, \Omega^1_{Y_E}) = k^{\otimes 2},$$

and

$$h^2(Y_E, \Omega^1_{Y_E}) \geq h^2(Y_E, \Omega^1_{X_E}|Y_E) = h^2(X_E, \Omega^1_{X_E}) = 1.$$

3.3.4. $H^i(Y_E, \Omega^2_{Y_E}).$ The short exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \big|_{Y_E} \to \Omega^1_{X_E}|Y_E \to \Omega^1_{Y_E} \to 0$$

implies the short exact sequence

$$0 \to \Omega^1_{Y_E} \otimes (\mathcal{I}/\mathcal{I}^2) \big|_{Y_E} \to \Omega^2_{X_E}|Y_E \to \Omega^2_{Y_E} \to 0$$

We only care about $H^0(Y_E, \Omega^2_{Y_E})$, so look at $H^i(\Omega^1_{Y_E} \otimes (\mathcal{I}/\mathcal{I}^2))$ for $i = 0, 1$ and then $H^0(Y_E, \Omega^2_{X_E}|Y_E)$.

The first step:

For a fiber $G$ of $Y_E \to E^{(p)}$, we analyse $\Omega^1_{Y_E} \otimes (\mathcal{I}/\mathcal{I}^2)|_G$. Note that we have $0 \to \mathcal{O}_G \to \Omega^1_{Y_E}|_G \to \Omega^2_{G} \to 0$.

Hence

$$0 \to \mathcal{O}_G(-d) \to \Omega^1_{Y_E}|_G(-d) \to \Omega^2_{G}(-d) \to 0.$$
0 → \mathcal{O}_F → \Omega^1_{X_E}|_F → \Omega^1_F → 0, hence 0 → \Omega^1_F → \Omega^2_{X_E}|_F → \Omega^2_F → 0, hence 0 → \Omega^1_F(-d) → \Omega^2_{X_E}|_F(-d) → \Omega^2_F(-d) → 0. But by Bott’s calculation \( H^0(F, \Omega^1_F(-d)) = H^0(F, \Omega^2_F(-d)) = 0 \), hence \( H^1(F, \Omega^2_{X_E}|_F(-d)) = 0 \). So \( H^0(Y_E, \Omega^2_{X_E}|_F) = H^0(Y_E, \Omega^2_{X_E}|_F) = 0 \).

3.4. Calculation with hypercovering.

3.4.1. The maps. Let’s pin down the maps corresponding to \( H^i(E \times E, \mathcal{O}) → H^i(E_E, \mathcal{O}) \) induced by projection and action, we claim this is the same as induced from the fiber product diagram

\[
\begin{array}{c}
E \times Y_E \longrightarrow Y_E \\
\downarrow \downarrow \\
E \times E^{(p)} \longrightarrow E^{(p)}
\end{array}
\]

Note that we have sections, \( E \times 0 \) and \( 0 \times E \). By Kunneth formula,

\[
H^i(E \times E^{(p)}) = \bigwedge^i (H^1(E, \mathcal{O}) \oplus H^1(E^{(p)}, \mathcal{O})).
\]

For the action map, the induced map on frobenius is \( \phi = 0 : H^1(E^{(p)}, \mathcal{O}) → H^1(E, \mathcal{O}) \).

3.4.2. \( H^2(Y, \mathcal{O}) \). Consider the hypercovering spectral sequence whose \( E_1 \) page is

\[
\begin{array}{ccc}
H^0(E^3 \times Y_E, \mathcal{O}) & H^1(E^2 \times Y_E, \mathcal{O}) & H^2(E \times Y_E, \mathcal{O}) \\
d_2^0 & d_2^1 & d_2^2 \\
H^0(E^2 \times Y_E, \mathcal{O}) & H^1(E \times Y_E, \mathcal{O}) & \\
d_2^0 & \\
H^0(Y_E, \mathcal{O}) & H^1(Y_E, \mathcal{O}) & \\
& \\
& \\
\end{array}
\]

The map \( U_1 → U_0 \) is given by

\[
d_0^1 : (a, x) → x, \quad d_0^2 : (a, x) → \phi a + x = x,
\]

the map is the same as previously calculated, we have the corresponding matrix, where \( \phi = 0 \)

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & -\phi
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & \phi \\
0 & 1
\end{bmatrix}.
\]

Thus the \( E_2 \) page is

\[
\begin{array}{cccc}
& & & \\
& 0 & 0 & \? \\
& & & \\
& 0 & k & \? \\
& & & \\
k & k & 0 \\
& & & \\
\end{array}
\]

Hence

\[
H^1(Y, \mathcal{O}) = k, H^2(Y, \mathcal{O}) = k.
\]
3.4.3. $H^1(Y, \Omega^1_Y)$. Note that the hypercovering spectral sequence for $H^1(Y, \Omega^1_Y)$ is

\[
\begin{array}{c}
\begin{array}{c}
H^0(E^2 \times Y_E, \Omega^1) \\
d_{1,0}^1 \\
H^0(E \times Y_E, \Omega^1) \\
d_{2,0}^2 \\
H^0(Y_E, \Omega^1)
\end{array}
\end{array}
\begin{array}{c}
H^1(E \times Y_E, \Omega^1) \\
d_{2,1}^1 \\
H^1(Y_E, \Omega^1)
\end{array}
\begin{array}{c}
k^\oplus 3 \\
k^\oplus 2 \\
k^\oplus 4 \oplus k \\
k \\
k \oplus k
\end{array}
\]

The $E_2$ page is

\[
\begin{array}{c}
0 \\
k \\
? \\
k \\
k \oplus k
\end{array}
\]

The $d_{2,1}^1$ map is given by $[0,1,0,1]^T$. Hence

$H^1(Y, \Omega^1_Y) = 3$

3.4.4. $H^0(Y, \Omega^2_Y)$. By the lemma in previous section, $H^0(Y_E, \Omega^2_{Y_E}) = 0$, hence

$H^0(Y, \Omega^2_Y) = 0$.

3.5. **Conclusion.** In the example, we obtain a threefold $X$ with

$h^{1,0} = h^{0,1} = 1, h^{1,1}_{dR} = 1$,

and

$h^{2,0} = 0, h^{1,1} = 3, h^{0,2} = 1, h^{2}_{dR} = 2$.

If we take the sixfold $X \times X$, by Kunneth formula, we have $h^{2,0} + h^{1,1} + h^{0,2} = 1 + 8 + 3 = 12$, while $h^{2}_{dR} = 5$. Slicing with hypersurfaces of sufficiently high degree, we arrive at a threefold with $h^{2}_{Hdg} - h^{2}_{dR} = 5$, this example could be used to pin down the structure of Hodge ring of varieties, see [vDdB18].

**References**
