Let $f: P \to S$ be a flat morphism of schemes, whose geometric fibers are all isomorphic to projective spaces of the same dimension, then we call $P/S$ a Brauer-Severi scheme [Gro95]. Given a Brauer-Severi scheme, we can always associate it with a Brauer class $\alpha \in \text{Br}(S)$. We recall how this is defined:

Consider the Leray spectral sequence $H^i(S, R^j f_* \mathcal{G}_m) \Rightarrow H^{i+j}(S, \mathcal{G}_m)$. We have the map

$$d^0_{1}: H^0(S, \text{Pic}_{P/S}) = H^0(S, \mathbb{Z}) \to H^2(X, \mathcal{G}_m).$$

The Brauer class $\alpha$ is defined to be $d^0_{1}(1) \in H^2(S, \mathcal{G}_m)$. One can show this class actually lies in the Azumaya Brauer group [Gir71, V.4]. We will not distinguish an element in $\text{Br}(S)$ or $H^2(S, \mathcal{G}_m)$, since the former is always canonically a subgroup of the later [Gro95, 2].

Let $k$ be a field, let $X, Y$ be smooth irreducible varieties over $k$. Let $f: X \to Y$ be a proper morphism. Suppose we are in the following situation: there exist a closed subset of $Z \subset Y$ (let’s denote $Y \setminus Z$ by $U$), with codimension at least two, such that $X \times_Z Y \to Z$ and $X \times_Y U \to U$ are both Brauer-Severi schemes. Let $\alpha_Z$ and $\alpha_U$ be their corresponding Brauer classes. By purity of Brauer group [Ces17], the class $\alpha_U$ extends to a unique class $\alpha_Y$. We may ask if the restriction of $\alpha_Y$ to $Z$ necessarily equal to $\alpha_Z$? If $X \to Y$ is flat, then the answer is yes, by the functoriality of Leray spectral sequence. We show in general:

**Theorem.** If $\text{char}(k) = 0$, then $\alpha_Y|_Z = \alpha_Z$.

The proof is uses the following theorem [GS17, 5.3.3]:

**Theorem 1** (Châtelet). Let $k$ be a field. Two Brauer-Severi varieties over $k$ have the same class in $\text{Br}(k)$ is and only if there exists a Brauer-Severi variety $Z$ over $k$ into which both $X$ and $Y$ can be embedded as twisted-linear subvarieties.

We will get the twisted subvarieties over the high codimensional loci by taking closure, provided we can find an ample line bundle on $X$, such that it has same degree on all the geometric fibers. The existence of such line bundles is the crucial fact we want to show. It is not clear how to show this in characteristic $p$ cases.

### 1. Some easy lemmas

Let $k$ a field, recall the Chow ring $\text{CH}^*(\mathbb{P}^n_k)$ is isomorphic to $\mathbb{Z}[H]/(H^{n+1})$, where $H$ is the hyperplane class. Let $x = cH^i \in \text{CH}^i(\mathbb{P}^n_k), y \in \text{CH}^j(\mathbb{P}^n_k)$.

**Lemma 2.** Let $k$ be a field, let $X$ be subscheme of $\mathbb{P}^n_k$. Suppose the Hilbert polynomial $P$ of $X$ has degree $m$. Then $P_X(d) \geq \binom{m+d}{d}$ (here $\geq$ means inequality holds when the variable is large enough). The equality holds if and only if $X$ is a linearly embedded $\mathbb{P}^m_k$.

**Proof.** We only need to show the lemma when $X$ is reduced and irreducible, since the Hilbert polynomial of a reduced top dimensional irreducible component has the same degree. The case $n = 1$ is trivial. For $n > 1$, if $X$ is contained in a hyperplane, then we reduce to the $n-1$ case. Suppose $X$ is not contained in a hyperplane in $\mathbb{P}^n_k$. If $X$ is the whole space, then we are done, since $P_{\mathbb{P}^n_k}(d) = \binom{n+d}{d}$. If $X$ is not the whole space, and is not contained in any linear subspace, we show $\deg(X) \neq 1$. If $X$ has codimension 1, this is clear since degree 1 hypersurfaces are hyperplanes. If $X$ has codimension $1+r$, take projection $f: \text{Bl}_{L(\mathbb{P}^n_k)} \to \mathbb{P}_{k}^{n-r}$ from a linear subspace of dimension $r$ disjoint from $X$, so that $f_* X$ is a divisor. Let $H$ be a line in $\mathbb{P}_{k}^{n-r}$. Then $1 = f_*(X \cdot f^*H) = f_* X \cdot H$, see [Ful98, 8.1.1], so $f_* X$ is a hyperplane and $X$ is contained in the hyperplane $f^{-1}(f(X))$, contradiction. Hence $X$ has degree at least 2, then the leading coefficient of $P_X$ will be at least $2/m!$, then $P_X(d) > \binom{m+d}{d}$. 

\[\square\]
Lemma 3. Let $R$ be a discrete valuation ring, let $Z = \text{Spec}(R)$. Let $f: X \to Z$ be a projective morphism such that the generic fiber $X_\eta$ and the special fiber $X_s$ are both Brauer-Severi varieties. Suppose an ample line bundle $\mathcal{L}$ on $X$ restricts to the same degree on both fibers. Then the closure $\overline{X_\eta}$ restricts to a linearly embedded subspace of $X_s$.

Proof. Since the morphism $\overline{X_\eta} \to Z$ is dominant, it is flat by [Har77, 9.7]. Hence the Hilbert polynomial of $X_\eta$ and $(\overline{X_\eta})_s$ are equal by [Har77, 9.9]. Denote $\dim_{\eta}(X_\eta) = a$, $\dim_{\eta}(X_s) = b$.

In the following calculation of intersection number, we always base change to algebraically closed fields, and take hyperplane divisors over the base change of Brauer-Severi varieties.

Let $H_s$ be the hyperplane class of $X_s$, let $H_\eta$ be the hyperplane class of $X_\eta$. By our assumption, there exists a positive integer $k$ such that $c_1(\mathcal{L}|_{X_s}) = kH_s$, $c_1(\mathcal{L}|_{X_\eta}) = kH_\eta$. Suppose the cycle class $[(\overline{X_\eta})_s] = m(H_s)^{b-a}$, we want to show $m = 1$.

Since intersection number with an ample line bundle is determined by the leading coefficient of the Hilbert polynomial, by flatness, we have $(\mathcal{L}|_{X_\eta} \cdot X_\eta)_{X_\eta} = (\mathcal{L}|_{X_s} \cdot X_s)_{X_s}$. So $k^a = (k^a H_\eta^a \cdot mH_\eta^{b-a}) = k^a \cdot m$. Thus $m = 1$. Let $P_{X_s}(d)$ be the Hilbert polynomial of $(\overline{X_\eta})_s$ for the hyperplane class $H_s$, then $P_{(\overline{X_\eta})_s}(kd) = P^\mathcal{L}_{\overline{X_\eta}}(d) = P^\mathcal{L}_{X_\eta}(d) = (\frac{a+kd}{d})$, so $P_{X_s}(d) = (\frac{a-d}{d})$. Thus $(\overline{X_\eta})_s$ is a subspace linearly embedded in $X_s$, by 2. \hfill \Box

Lemma 4 (Cohomological purity). Let $S$ be a scheme, let $i: Z \to X$ be a closed immersion of smooth $S$-schemes, and let $\mathcal{F}$ be a locally constant torsion sheaf on $X_{\text{et}}$ whose torsion is prime to $\text{char}(X)$. Denote $c = \text{codim}_X(Z)$. Then

$$R^n i^* \mathcal{F} = \begin{cases} 0, & n \neq 2c \\ i^* \mathcal{F}(-c), & n = 2c. \end{cases}$$

Let $U = X \setminus Z$, let $j: U \to X$ be the open immersion, the above implies that

$$(R^n j_*) j^* \mathcal{F} = \begin{cases} \mathcal{F}, & n = 0 \\ 0, & n \neq 0, 2c - 1. \end{cases}$$

Proof. See [Mil80, 5.1]. \hfill \Box

2. Some crucial lemmas

Lemma 5. Let $X, Y$ be smooth varieties over $k$. Let $f: Y \to X$ be a projective morphism. Let $Z \subset X$ be a smooth closed subvariety, let $U = X \setminus Z$.

$$\begin{array}{c}
Y_Z \xrightarrow{i'} Y \xleftarrow{j'} Y_U \\
\downarrow f_Z \quad \quad \downarrow f \\
Z \xrightarrow{i} X \xleftarrow{j} U
\end{array}$$

Let $i: Z \to X$ be the closed immersion and $j: U \to X$ be the open immersion. Denote their base change along $f$ by $i', j'$. Let $f_Z, f_U$ be the base change of $f$ along $i, j$. Suppose $Y_U \to U, Y_Z \to Z$ are both Brauer-Severi schemes of positive relative dimension. Suppose the codimension of $Z$ in $X$ and $Y_Z$ in $Y$ are both at least 2. Then for any $n \in \mathbb{Z}_{>0}$ prime to $\text{char}(k)$, we have

$$R^2 f_* \mu_n = \mathbb{Z}/n X.$$ 

Proof. We use $Rf_*$ to denote functors between derived categories, we use $R^i f_*$ to denote the $i$-th cohomology of $Rf_*$, etc. When there’s a canonical isomorphism between two objects, and the canonical isomorphism is clear from context, we write $=$ for simplicity.

Consider the short exact sequence

$$0 \to j_* \mathbb{Z}_U \to \mathbb{Z}_X \to i_* \mathbb{Z}_Z \to 0.$$ 

Apply $R \text{Hom}_X(-, Rf_* \mu_n)$ to this sequence, we get a distinguished triangle in $D^b(X)$:

$$R \text{Hom}_X(i_* \mathbb{Z}_Z, Rf_* \mu_n) \to R \text{Hom}_X(\mathbb{Z}_X, Rf_* \mu_n) \to R \text{Hom}_X(j_* \mathbb{Z}_U, Rf_* \mu_n).$$
By Yoneda lemma and properties of adjoint functors, we know
\[ R\mathcal{H}om_X(i_*, \mathbb{Z}/n, Rf_*\mu_n) = i_*Ri^!Rf_*\mu_n, \]
\[ R\mathcal{H}om_X(\mathbb{Z}/n, Rf_*\mu_n) = Rf_*\mu_n, \]
\[ R\mathcal{H}om_X(j_*\mathbb{Z}/n, Rf_*\mu_n) = Rj_*j^*Rf_*\mu_n. \]
So we have a distinguished triangle
\[ i_*Ri^!Rf_*\mu_n \rightarrow Rf_*\mu_n \rightarrow Rj_*j^*Rf_*\mu_n. \]
Taking cohomology we get an exact sequence
\[ H^2(i_*(Ri^!Rf_*\mu_n)) \rightarrow R^2f_*\mu_n \xrightarrow{\phi} H^2(Rj_*Rf_{U,*}\mu_n). \]
(1)
We’ll show the map \( \phi \) in (1) is an isomorphism and prove the lemma.

**Step 1.** \( \phi \) is injective.

By cohomological purity for the pair \( (Y_Z, Y) \), we know \( R^nG^i\mu_n = 0 \) for \( n = 0, 1, 2 \). By the spectral sequence \( R^pZ, R^qG^i\mu_n \Rightarrow (Rf_*R^qG^i\mu_n)^{p+q} \), we know
\[ H^2(Rf_*R^qG^i\mu_n) = 0. \]
We know \( R^qG^i \) is right adjoint to \( f^*i_* \). By proper base change we have \( f^*i_* = i'_*f_Z^* \). Note that \( Rf_*R^qG^i \) is right adjoint to \( i'_*f_Z^* \). By uniqueness of adjoint functor, we know
\[ R^qG^i = Rf_*R^qG^i. \]
Together with exactness of \( i \), we know
\[ H^2(i_*(R^qG^iRf_*\mu_n)) = i_*H^2(R^qG^iRf_*\mu_n) = i_*H^2(Rf_*R^qG^i\mu_n) = 0 \]
hence the map \( \phi \) in (1) is injective.

**Step 2.** \( \phi \) is surjective. We already know \( R^2f_*\mu_n \xrightarrow{\phi} H^2(Rj_*Rf_{U,*}\mu_n) \) is injective. It suffices to show the cardinality of stalk on the right hand side is no greater than cardinality of stalk on the left hand side.

By assumption \( f_U : Y_U \rightarrow U \) is a Brauer-Severi scheme, so
\[ f_U,*\mu_n = \mu_n, \]
\[ R^1f_U,*\mu_n = 0, \]
\[ R^2f_U,*\mu_n = \mathbb{Z}/n, \]
Consider the spectral sequence
\[ R^pf_J,R^qf_{U,*}\mu_n \Rightarrow H^{p+q}(Rj_*Rf_{U,*}\mu_n). \]
By cohomological purity for \( (Z, X) \), the \( E_2 \) page can be explicitly written down as
\[
\begin{array}{c@{}ccc}
\multicolumn{4}{c}{j_*R^2f_{U,*}\mu_n = \mathbb{Z}/n_X} \\
\underbrace{d_2^{0,2}} & j_*R^1f_{U,*}\mu_n = 0 & R^1j_*R^1f_{U,*}\mu_n = 0 & R^2j_*R^1f_{U,*}\mu_n = 0 \\
\multicolumn{4}{c}{} \\
\multicolumn{4}{c}{j_*f_{U,*}\mu_n = \mathbb{Z}/n_X} \\
\end{array}
\]
Thus \( H^2(Rj_*Rf_{U,*}\mu_n) = \ker(d_2^{0,2}) \). Note that \( \ker(d_2^{0,2}) \) is a subsheaf of \( \mathbb{Z}/n \), so the cardinality of stalk of \( H^2(Rj_*Rf_{U,*}\mu_n) \) is at most \( n \). By proper base change, the fibers of \( R^2f_*\mu_n \) all has cardinality \( n \). So the injection \( \phi \) in (1) is indeed an isomorphism and forces \( \ker(d_2^{0,2}) = \mathbb{Z}/n \). Hence
\[ R^2f_*\mu_n = H^2(Rj_*Rf_{U,*}\mu_n) = \mathbb{Z}/n. \]
Remark 6. In the proof above, we need \((n,\text{char}(k)) = 1\) so that the assumptions on cohomological purity for étale cohomology holds.

Lemma 7. In the same setting as 5, assume furthermore that \(\text{char}(k) = 0\). Then any line bundle \(L\) on \(Y\) restricts to the same degree on each geometric fiber.

Proof. Apply \(f_*\) to the Kummer sequence on \(Y\):

\[
0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0.
\]

We get a morphism of sheaves \(R^1f_*\mathbb{G}_m \xrightarrow{\psi} R^2f_*\mu_n \cong \mathbb{Z}/n\).

For any geometric point \(\overline{x}\) of \(X\), the map on stalk \(\psi_{,\overline{x}}\) sends a germ represented by line bundle \(L\) to \(\deg(L_{\overline{x}}) \in \mathbb{Z}/n\). Consider the following diagram of cospécialization maps.

\[
\begin{array}{ccc}
(R^1f_*\mathbb{G}_m)_{\overline{\tau}} & \xrightarrow{\psi_{,\overline{x}}} & (R^2f_*\mu_n)_{\overline{\tau}} \\
\begin{array}{ccc}
cosp_1 & \downarrow & \downarrow \\
\end{array} & & \begin{array}{ccc}
cosp_2 & \sim & h \\
\end{array} \\
(R^1f_*\mathbb{G}_m)_{\overline{\tau}} & \xrightarrow{\psi_{,\overline{x}}} & (R^2f_*\mu_n)_{\overline{\tau}} \\
\end{array}
\]

We know the cosp\(_p\) is an isomorphism as \(R^2f_*\mu_n\) is a constant sheaf, see 5. It is not clear if cosp\(_1\) is also isomorphism, or equivalently, if \(h\) is the identity map. If this is true, then the degree of line bundle over \(\overline{x} \in \mathbb{Z}\) and \(\overline{\tau}\) are the same modulo any \(n\), take \(n\) large enough, we are done. But this is not true in general, see 8.

However, since cosp\(_2\) is isomorphism, for any \(n\) prime to \(\text{char}(k)\), we know \(n|\deg(L_{\overline{x}})\) if and only if \(n|\deg(L_{\overline{x}})\). If \(\text{char}(k) = 0\), the above implies \(\deg(L_{\overline{x}}) = \pm \deg(L_{\overline{x}})\). So \(\deg(L_{\overline{x}}) = \deg(L_{\overline{x}})\) holds for an ample line bundle \(L\). For a general line bundle, we can write it as the difference of two ample line bundles. \(\square\)

Remark 8. Let’s consider the family \(V(tT_0^2 + tT_1^2 + tT_2^2) \subset \mathbb{P}^2 \times \mathbb{A}^1\), here \(t\) is coordinates of \(\mathbb{A}^1\) and \(T_i\) are homogeneous coordinates of \(\mathbb{P}^2\). The map \(h\) in above proof is not identity. Take \(n = 3\), as \(\mathcal{O}(1)\) restricts to degree 1,2 on the special fiber and generic fiber, we know \(h\) is multiplication by 2. In this case, we know \(R^2f_*\mu_3\) is still a constant sheaf, but the cospécialization map does not preserve degree. This is not an counterexample to our assumptions on smoothness and codimension are not satisfied.

Remark 9. If \(\text{char}(k) = p > 0\), by the same argument, we can only claim \(\deg(L_{\overline{x}}) = np \cdot \deg(L_{\overline{x}})\). But the lemma could be true in general.

Lemma 10. Let \(X\) be a regular scheme, \(k(X)\) be its function field, then the restriction map \(\text{Br}(X) \to \text{Br}(k(X))\) is an injection.

Proof. See [Mil80, 2.22]. \(\square\)

3. The proof

Theorem 11. In the same setting as Lemma 5, denote Brauer classes of \(Y_U \to U\) and \(Y_Z \to Z\) by \(\alpha_U, \alpha_Z\). By purity of Brauer group, there exists a unique class \(\alpha \in \text{Br}(X)\) such that \(\alpha|_U = \alpha_U\). If \(\text{char}(k) = 0\), then \(\alpha|_Z = \alpha_Z\).

Proof. By shrinking \(X\) and \(Z\), we can find a filtration \(Z \subset Z_1 \subset X\) with \(Z_1\) smooth and \(\text{codim}_{Z_1}(Z) = 1\). Let \(L\) be an ample line bundle on \(Y\). By 5 it has same degree on each geometric fiber.

Let \(\eta, \eta_1\) be the generic point of \(Z, Z_1\). Let’s denote the Brauer class of \(Y_\eta \to \eta, Y_{\eta_1} \to \eta_1\) by \(\alpha_\eta, \alpha_{\eta_1}\). By 3, we can extend the Brauer-Severi scheme \(Y_{\eta} \to \eta\) to a Brauer-Severi scheme over \(\text{Spec}(\mathcal{O}_{Z_1, Z})\) by taking closure. The Brauer-Severi scheme obtained by taking closure gives a class \(\alpha'_{\eta_1} \in \text{Br}(\text{Spec}(\mathcal{O}_{Z_1, Z}))\).

Since \(\alpha'_{\eta_1}|_{\eta_1} = \alpha_{\eta_1} = \alpha_U|_{\eta_1} = \alpha|_{\eta_1} = (\alpha|_{\text{Spec}(\mathcal{O}_{Z_1, Z})})|_{\eta_1}\), we see \(\alpha'_{\eta_1} = \alpha|_{\text{Spec}(\mathcal{O}_{Z_1, Z})}\) by 10. Thus \(\alpha'_{\eta_1}|_\eta = (\alpha|_{\text{Spec}(\mathcal{O}_{Z_1, Z})})|_\eta = \alpha|_\eta\). By 3 and 5, on the special fiber obtained by taking closure, we have linear embedding. By Theorem 1, we know \(\alpha_\eta = \alpha'_{\eta_1}|_\eta\). Thus \(\alpha_Z|_\eta = \alpha_\eta = \alpha'_Z|_\eta = \alpha|_\eta = (\alpha|_Z)|_\eta\). By 10, we know \(\alpha_Z = \alpha|_Z\). \(\square\)

Example 12. The class \(\alpha\) maps to 0 by pullback along section \([\omega_C]\) to \(\text{Br}(k)\).
By [Bos90, 8.2.7], the class $\alpha|_{\omega_C}$ is represented by $\mathbb{P}(\pi_\ast\omega_C)$. In case $\text{char}(k) = 0$, the claim directly follows from proposition above. In case $\text{char}(k) \neq 0$, we can lift the family to characteristic 0, since the construction involved in $\alpha$ can be carried out over $\text{Spec}(W(k))$.

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