The BNR correspondence

(Beauville-Narasimhan-Ramanan)

Goal:

- Conjecture of Simpson: rigid local systems are motivic.

$X/\mathbb{C}$, L a $\mathbb{C}$-local system

rigid $\Rightarrow$ $\exists\, U \subseteq X$, $\begin{array}{c} Y \\ \downarrow \pi \\ X \end{array}$, i

s.t. $L|_U$ summand of $R^i \pi_* \mathbb{C}$.

- Suppose L is of geo origin on X (no rigidity assumption).

$L \longleftrightarrow (V, \nabla)$

spread out: $(\mathcal{X}, \mathcal{V}, \nabla)$
$$\downarrow$$
$$S$$

($\sim$ irreducible scheme, smooth over $\mathbb{Z}$, of f.t.)

$\forall\ p \gg 0$, $\forall\ s \in S$ of residue char p,

ψ_{∇_s} is nilpotent. (Deligne / Katz)

$\Updownarrow$

$(V, \nabla)^{\wedge}_s$ / $\mathcal{X}^{\wedge}_{/s}$ p-adic formal scheme

is crystal on crystalline site $\mathrm{CRIS}(\mathcal{X}_s / \mathbb{Z}_p)$

$$\boxed{\begin{array}{l}\text{Goal: rigid flat connections on proj.}\\ \text{varieties satisfy the above; globally}\\ \text{nilpotent.}\end{array}}$$

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Higgs BNR correspondence

Let X/k smooth variety of dim d.

Let E be a vector bundle on X of rank r.

Claim:

$$\left\{\begin{matrix}\text{Higgs field } \theta \\ \text{on } E\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Action of} \\ \mathrm{Sym}^\bullet T_X \curvearrowright E\end{matrix}\right\}$$

$\theta : E \to E \otimes \Omega^1$ $\quad \mathcal{O}_X$-linear

$$\begin{tikzcd} \rightsquigarrow T_X \arrow[r] \arrow[d, hook] & \mathrm{End}_{\mathcal{O}_X}(E) \\ \mathrm{Sym}^\bullet T_X \arrow[ur] & \end{tikzcd}$$

of associative alegbras

integrability of θ $\Longleftrightarrow$ extends to

$\mathrm{Sym}^\bullet T_X \longrightarrow \mathrm{End}_{\mathcal{O}_X} E$

Given (E, θ) Higgs bundle

$$\theta \in \mathrm{End}(E) \otimes \Omega^1$$
$$\downarrow \mathrm{tr}$$
$$\mathrm{Tr}(\theta) \in H^0(X, \Omega^1_X)$$

Q: What is the char poly of θ?

Where does it live?!

$$\Lambda^i \theta \in \mathrm{End}(\Lambda^i E) \otimes \mathrm{Sym}^i \Omega_X$$

$$a_i := \mathrm{Tr}(\Lambda^i \theta) \in H^0(X, \mathrm{Sym}^i \Omega_X)$$

$\{\mathrm{Tr}$

$$\chi(\theta) := T^r + a_1 T^{r-1} + \cdots + a_r,$$

char poly of θ

T formal variable

In other words

$$(E, \theta) \rightsquigarrow \vec{a} \in \underbrace{\mathcal{A} := \bigoplus_{i=1}^{r} H^0(X, \mathrm{Sym}^i \Omega_X)}_{\text{Hitchin base.}}$$

Spectral Cover

(Eigenvalues of θ)

Note that a Higgs bundle (E, θ) gives rise to coherent sheaf $\mathcal{E}$ on T^*X.

$T^*X \simeq \underline{\mathrm{Spec}}(\mathrm{Sym}^* T_X)$

$\pi_* \mathcal{E} \simeq E$ is a vector bundle

$$\begin{array}{c} T^*X \\ \tilde{\pi} \downarrow \\ X \end{array}$$

Question: What is the natural subscheme of T^*X on which $\mathcal{E}$ is supported?!

Note that $\pi^* \Omega_X$ has a canonical section: λ. (This is "general AG": if V is a vector bundle on X, then $\pi: \mathrm{Tot}(V) \to X$, $\pi^* V$ has tautological section.)

Let's consider the formal equation:

$$\underbrace{\lambda^n}_{\mathrm{Sym}^n \pi^* \Omega^1} + \underbrace{a_1}_{\pi^* \Omega^1} \underbrace{\lambda^{n-1}}_{\mathrm{Sym}^{n-1} \pi^* \Omega^1} + \dots + a_r$$

is a section of

$\mathrm{Sym}^n \pi^* \Omega^1$.

Can consider
$$\begin{tikzcd} Y_a := Z(\lambda^r + a_1\lambda^{r-1} + \dots + a_r) \arrow[drr, "\pi_a"'] & \subseteq & T^*X \arrow[d, "\pi"] \\ & & X \end{tikzcd}$$

Note that π_a is surjective: fibers are the eigenvalues of Θ.

$Y_a \xrightarrow[\pi_a]{} X$ is called the spectral cover.

"Thm" (Higgs BNR correspondence)

$$\left\{ \begin{array}{l} \text{Higgs bundles of} \\ \text{rank } r \text{ on } X, \text{ mapping} \\ \text{to } a \in \mathcal{A} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{coherent sheaves } \mathcal{E} \\ \text{on } Y_a \text{ s.t.} \\ \pi_{a*}\mathcal{E} \text{ is loc.} \\ \text{free of rank } r \end{array} \right\}$$

Pf Cayley-Hamilton theorem.

Spectral Cover in families

"Family of Higgs bundles on X, parametrized by S":

$$\left\{ E \in \mathrm{Vect}_r(X \times S),\ \theta : E \to E \otimes \Omega^1_{X\times S/S} \;\middle|\; \theta \wedge \theta = 0 \right\}$$

$$a_i := \mathrm{Tr}(\Lambda^i \theta) \in H^0(X \times S, \mathrm{Sym}^i \Omega_{X\times S/S})$$

$$\rightsquigarrow \quad a : S \longrightarrow \mathcal{A}$$

$\rightsquigarrow$ spectral cover in families

$$\begin{tikzcd} Y_a \arrow[rd, "\pi_a"'] & \subseteq & T^*X \times S \arrow[ld] \\ & X \times S & \end{tikzcd}$$

When X/k is proper,

$$\mathcal{A} = \mathrm{Spec}\left(\mathrm{Sym}^* \left(\bigoplus_{i=1}^{r} H^0(X, \mathrm{Sym}^i \Omega^1_X) \right)\right)$$

With this notation, BNR correspondence holds in families.

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dR BNR correspondence.

X/k smooth over perfect field k of char p.

(E, ∇) vector bundle w/ flat connection

$\rightsquigarrow \psi_\nabla : E \to E \otimes F^* \Omega^1_{X'}$

$(\rightsquigarrow$ section in $H^0(X, \mathrm{End}\, E \otimes F^* \Omega^1_{X'}))$

ψ_∇ is flat $\Rightarrow$ section is flat

$\mathrm{Tr}(\psi_\nabla) \in H^0(X, F^* \Omega_{X'})$

ψ_∇ flat $\Longrightarrow \mathrm{Tr}(\psi_\nabla) \in H^0(X, F^{-1} \Omega^1_{X'})$

i.e., $\mathrm{Tr}(\psi_\nabla) \in H^0(X', \Omega^1_{X'})$

$\Lambda^i \psi_\nabla \in H^0(X, \mathrm{End}(\Lambda^i E) \otimes \mathrm{Sym}^i F^* \Omega^1_{X'})$

horizontal $\Rightarrow$ $\mathrm{Tr}(\Lambda^i \psi_\nabla)$ horizontal

$$\Rightarrow \quad a_i := \mathrm{Tr}(\Lambda^i \psi_\nabla) \in H^0(X', \mathrm{Sym}^i\, \Omega^1_{X'})$$

Another way of saying this:

$$\psi_\nabla : E \to E \otimes F^* \Omega^1_{X'}$$

$$\Leftrightarrow \quad \text{section} \quad H^0(X', \mathrm{End}(F_* E) \otimes \Omega^1_{X'})$$

$$\downarrow \mathrm{Tr}$$

$$H^0(X', \Omega^1_{X'})$$

Def The Higgs Hitchin map is

$$\chi_{Higgs} : M_{Higgs} \longrightarrow A$$

$$(E, \theta) \longmapsto \chi(E, \theta)$$

The dR Hitchin map is

$$\chi_{dR} : M_{dR} \longrightarrow \underbrace{\bigoplus_{i=1}^{r} H^0(X', \mathrm{Sym}^i(\Omega^1_{X'}))}_{A'}$$

$(E,\nabla) \mapsto$ "coeffs of char poly of ψ_0"

$=$

One more notation:

$$\begin{array}{c} \mathcal{Y} \\ \pi \downarrow \\ X \times A \end{array}$$

"universal spectral cover"

Thm (dR BNR) X/k smooth of dim d

perfect of char $p>0$

$\exists$ natural equivalence

$$\left\{ \begin{array}{c} (E,\nabla) \text{ on } X \\ \text{of rank } r \end{array} \right\} \longleftarrow \left\{ \begin{array}{l} \cdot\ a \in A' \\ \cdot\ \mathcal{E} \text{ coherent sheaf on} \\ \quad \mathcal{Y}'_a, \text{ w/ action of} \\ \quad \mathcal{D}_X \quad \text{s.t.} \end{array} \right\}$$

- $(\pi_a)_* \mathcal{E}$ is a locally free Higgs bundle on X' of rank $p^d \cdot r$, w/ char poly $(a)^{p^d}$

<u>Recall</u>

$$X \xrightarrow{F} X'$$

Start w/ D_X on X, $F_* D_X$

$$Z(F_* D_X) \xleftarrow[\text{p-curvature map}]{\sim} \mathrm{Sym}^* T_{X'}$$

$\rightsquigarrow$ get $\mathcal{D}_X$ Azumaya alg on $T^* X'$.

Now, $Y'_a \subseteq T^* X'$, if $\mathcal{E}$ is on Y'_a & $(\pi_a)_*$ is a bundle of right rank

$\rightsquigarrow$ $\mathcal{E}$ corresponds to a Higgs bundle on X'

Bruno's Q: If X sm. proj. curve:

$$\begin{tikzcd} \mathcal{M}_{dR}(X) \arrow[dr] & & \mathcal{M}_{Higgs}(X') \arrow[dl] \\ & \mathcal{A}' & \end{tikzcd}$$

Claim: étale locally over $\mathcal{A}'$, these two fibrations are isomorphic.

Pf of dR BNR

Morita: $QCoh(X, D_X) \simeq QCoh(T^*X', \mathcal{D}_X)$

$(E, \nabla) \longmapsto F_* E$, an $F_* D_X$ module

$\rightsquigarrow$ action by $\mathrm{Sym}^* T X'$

$\rightsquigarrow$ promote to $\mathcal{F}$, coh sheaf on T^*X'

$\mathrm{rank}(F_* E) = p^d \cdot r$

BNR for Higgs $\rightsquigarrow$

$F_a E \rightsquigarrow \mathcal{F}$ on T^*Y'

supported on a spectral cover

$Y'_b \subseteq T^* X'$

deg $p^d \cdot r$

<u>Goal</u>: $\boxed{\text{prove } \quad b = a^{p^d}}$

as polynomials in

$$\bigoplus_{i=0}^{\infty} H^0(X', \mathrm{Sym}^i \, T_{X'})$$

Dévissage $\rightsquigarrow$ may assume we work over an étale cover that splits Azumaya algebra into a matrix algebra over $\mathcal{O}$

Let $U' \longrightarrow X'$ be such a cover.

$$\mathcal{D}|_{U'} \simeq \mathrm{End}\left(\mathcal{O}_{U'}^{(\oplus p^d)}\right)$$

[Details more tricky: $\mathcal{D}$ is alg Azumaya on T^*X', not on X'! G uses Henselian local rings)

$$\mathcal{D}_{U'} \simeq M_{p^d}(\mathcal{O}_{U'})$$

On U', Morita theory $\Rightarrow$

$$\mathcal{E} \simeq \bigoplus_{i=1}^{p^d} - \quad \text{(split via idempotents on)}$$

$\rightsquigarrow$ pulled back to U',

$$F_* E \simeq \bigoplus_{i=1}^{p^d} \underbrace{(\tilde{E}, \tilde{\theta})}_{\text{rank } r}$$

$$\Rightarrow \left(\chi_{Higgs}(\tilde{E}, \tilde{\theta})\right)^p = b$$

But all polynomials are monic

$$\Rightarrow \chi_{Higgs}(\tilde{E}, \tilde{\theta}) = a$$

$\Rightarrow$ both $\mathcal{F}, \mathcal{E}$ supported on Y_a'.

$\square$