Solutions for Quiz 1

Problem 1. Compute the following Limits.

a. \[ \lim_{x \to \infty} \frac{\sqrt{x+1}}{x} = 0 \]

\[ \frac{\sqrt{x+1}}{x} = \frac{\sqrt{x+1}}{x} \times \frac{1}{\frac{x}{\sqrt{x}}} = \frac{1}{\frac{x}{\sqrt{x}}} + \frac{1}{\sqrt{x}} \]

As \( x \to \infty \), the function \( \sqrt{\frac{1}{x}} + \frac{1}{x^2} \) tends to 0 because \( \lim_{x \to \infty} \frac{1}{x} = \lim_{x \to \infty} \frac{1}{x^2} = 0 \).

For an alternative explanation, the limit is 0 because \( x \) grows much faster than \( \sqrt{x+1} \).

b. \[ \lim_{x \to 5} \frac{x^2 - 6x + 5}{x^2 - 25} = \frac{2}{5} \]

\[ \frac{x^2 - 6x + 5}{x^2 - 25} = \frac{(x - 1)(x - 5)}{(x + 5)(x - 5)} = \frac{x - 1}{x + 5} \]

as long as \( x \neq 5 \) (where the original function is not defined.) The original limit is thus \( \lim_{x \to 5} \frac{x - 1}{x + 5} = \frac{4}{10} = \frac{2}{5} \).

Note that we can’t just naively plug in 5 into the original function because we would get \( \frac{0}{0} \) which doesn’t make sense.

c. \[ \lim_{x \to -\infty} (2^x - 2^{2x}) = 0 \]

The limit is zero because both terms in the limit tend to 0.

d. \[ \lim_{x \to \infty} \frac{x^2 - x + 1}{x^2 + x - 1} = 1 \]

\[ \frac{x^2 - x + 1}{x^2 + x - 1} = \frac{x^2 - x + 1}{x^2 + x - 1} \times \frac{x^2}{x^2} = \frac{1 - \frac{1}{x^2} + \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}}. \]

Thus \( \lim_{x \to \infty} \frac{x^2 - x + 1}{x^2 + x - 1} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2} + \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}} = \frac{1}{1} = 1 \)
Problem 2. A bacteria culture starts with population 20 at $t = 0$. The population doubles every 3 hours. Write a function that expresses the population of the culture as a function of time (in hours.) At what time is the population 640.

$$P(t) = 20 \times 2^{t/3}.$$ To compute the time $T$ when the population is 640, we plug in: $640 = 20 \times 2^{T/3}$. Then $32 = 2^{T/3}$ so $T/3 = 5$ and $T = 15$. 