# Introduction to local systems of <u>geometric</u> origin

<u>Notation</u>:

- $X/k$ means sm., q-proj variety over perfect field $k$
  $\left\{ k = \begin{array}{l} \cdot\ \mathbb{F}_q \\ \cdot\ \text{\#field} \\ \cdot\ \mathbb{C} \end{array} \right\}$
- $L$ stands for a ($\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$) local system on $X$

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Let $S$ be a $\mathbb{C}$-manifold

<u>Def</u> Let $A$ be a comm. ring.
An A-local system (on $S$) is a locally constant sheaf of finite free $A$-modules on $S$.

There is a "classical" correspondance:

$$\begin{tikzcd}[column sep=small]
& \{\mathbb{C}\text{-local systems on } S\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\left\{(\overset{\text{hol v.b}}{V}, \overset{\text{flat conn.}}{\nabla}) \text{ on } S\right\} \arrow[rr, leftrightarrow] & & \left\{\begin{array}{c}\text{fin. dim rep} \\ \text{of } \pi_1(S)\end{array}\right\}
\end{tikzcd}$$

$$\begin{tikzcd}
& L \arrow[dl] \arrow[dr] & \\
(V, \nabla) \arrow[rr, leftrightarrow] & & \rho
\end{tikzcd}$$

<u>Examples</u>

- $S = \mathbb{C}^\times$, $\pi_1(S) = \mathbb{Z}$

  $\rightsquigarrow$ rank $n$ local system on $S$ "is" $M \in GL_n(\mathbb{C})$

- $X/\mathbb{C}$ be $\underbrace{\text{hyperbolic}}_{g-1+r}$ curve

Corrected from lecture, where I had falsely said $\pi_1$ has a finite index free subgp?

Exercise: show that $\pi_1(X^{an})$ surjects onto finite rank free subgroup.

$\rightsquigarrow$ zillions of local systems on $X^{an}$.

- $\Gamma(2) := \{ M \in SL_2(\mathbb{Z}) \mid M \equiv I_2 \pmod 2 \}$

$$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$$

$$\rightsquigarrow \left[ \mathbb{H} / \Gamma(2) \right]$$

$$\text{IS}$$

$$Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

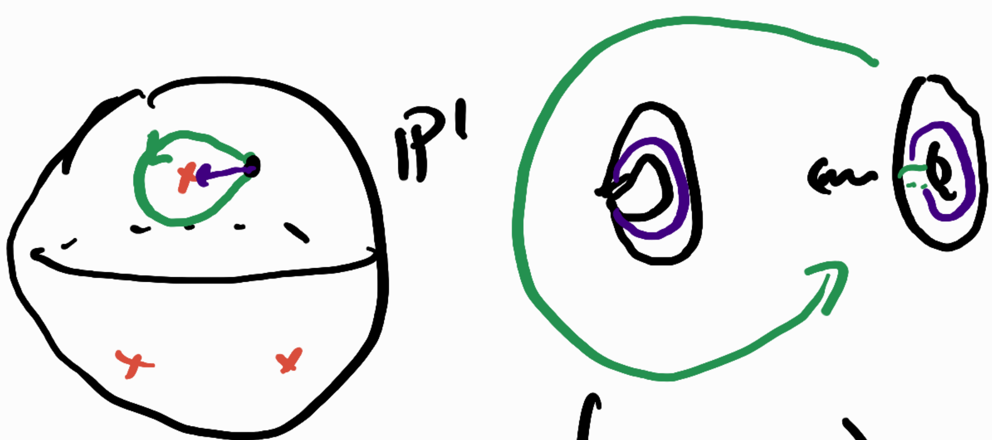
"moduli of elliptic curves w/ full level 2 structure"

$\exists$ "natural" local system on $Y(2)$

$$\begin{array}{ccc} \mathcal{E} & \hookrightarrow & z^2 = x(x-1)(x-\lambda) \\ \downarrow{\scriptstyle \pi} & & \\ Y(2) & & \end{array}$$

$\lambda$-coordinate

$\rightsquigarrow R^1\pi_*\mathbb{Z}$ is a rank 2 $\mathbb{Z}$ local system on $Y(2)$




$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

rep? $\quad \pi_1(Y(2)) = \Gamma(2)$

$$\Gamma(2) \hookrightarrow SL_2(\mathbb{Z})$$

Def Let $B/\mathbb{C}$, let $L$ be a $\mathbb{C}$-local system on $B^{an}$.

We say $L$ is of geometric origin if

- $\exists\, \mathcal{U} \subseteq B$ open dense
- $\begin{array}{c} \mathcal{Y} \\ \downarrow \pi \\ \mathcal{U} \end{array}$ sm. proj.
- $i \geq 0$

s.t. $L|_{\mathcal{U}^{an}}$ is a subquotient of $R^i \pi_* \mathbb{C}$.

Properties of Geometric local systems ____ (over $X/\mathbb{C}$)

- $\mathbb{L}$ is semi-simple
- $\mathbb{L}$ is "integral": $\exists$ # subfield $F \subseteq \mathbb{C}$ s.t.

$$\begin{tikzcd} \rho: \pi_1(X) \arrow[r] \arrow[dr] & GL_n(\mathbb{C}) \\ & GL_n(\mathcal{O}_F) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

($\overset{\Rightarrow}{\text{traces}} \in \overline{\mathbb{Z}}$)

- $\mathbb{L}$ underlies $\mathbb{C}$-PVHS

(for every $\iota \in \mathrm{Aut}(\mathbb{C})$,

$\rightsquigarrow \mathbb{L}_\iota \longleftrightarrow {}^\iota\rho: \pi_1(X) \to GL_n(\mathbb{C}) \xrightarrow{\iota} GL_n(\mathbb{C})$

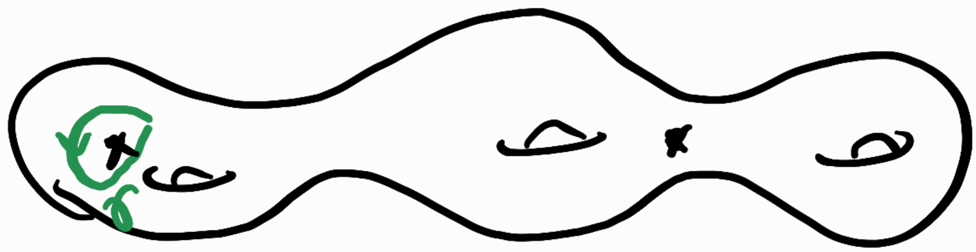
also underlies $\mathbb{C}$-PVHS)

(i.e. $\mathbb{L}$ is summand of $\mathbb{Z}$-PVHS)

- $L$ has quasi-unipotent monodromy @ $\infty$.

WLOG $X$ is a curve:

$$\begin{matrix} X \\ \cap \\ \overline{X} \end{matrix}$$




$L \rightsquigarrow \rho$

$\gamma$ is not "uniquely defined" in $\pi_1(X)$, BUT $[\gamma]$ is well defined.

$\Rightarrow \rho(\gamma)$ not well defined but $[\rho(\gamma)]$ is.

eigenvalues of $\rho(\gamma)$ are in $\mu_\infty$. (Many proofs!)

- $L$ has finite order determinant.
  ($\Leftarrow$ det trivial on a finite cover)

- $(V, \nabla)$ has vanishing p-curvature for $p \gg 0$
  ($\Leftarrow$ underlies crystal over every p-adic completion + F-structure for all $p \gg 0$)

- more: • arithmetic.
  - Simpson has some bi-algebraicity conj.,

Conj (Simpson)

Let $X/\mathbb{C}$, let $L$ be a $\mathbb{C}$-local system on $X^{an}$ s.t.

1. irr.
2. quasi-unipotent monodromy @ $\infty$
3. triv. det
4. rigid

$\rightsquigarrow$ Then $L$ is of geometric origin

① ✓

② ✓

③ ✓

④ Assume for simplicity that $X$ is proj.

$\exists$ f.t. moduli space $/\mathbb{Q}$, $\mathcal{M}$, parametrizing rank $n$ local systems (w/ triv det) on $X^{an}$.

$$\Leftrightarrow \{\pi_1(X^{an}) \to SL_n(\mathbb{C})\}/\sim$$

$$\rho \longmapsto [\rho] \in \mathcal{M}(\mathbb{C})$$

$L$ is rigid $\Leftrightarrow$ $[\rho]$ is an isolated pt of $\mathcal{M}(\mathbb{C})$.

# Evidence

Simpson
- $\rho \sim \pi_1(X) \to SL_n(K)$, $K$ # field
- $X$ proj: $L$ underlies a $\mathbb{Q}$VHS.

Esnault-Gröchenig
- $L$ is coh rigid $\rightsquigarrow \rho \sim \pi_1(X) \to SL_n(\mathcal{O}_K)$
- $X$ proj: $(V, \nabla)$ has nilpotent $p$-curvature for all $p \gg 0$, underlies $\underbrace{F^\delta \text{ structure for all } p \gg 0}$

also when $X$ not. nec. proj, w/ $L$ strongly coh. rigid

Question (Simpson)

Let $X/\mathbb{C}$ be proj of dim $\geq 2$.
Let $D \subseteq X$ be smooth ample div'
Let $\mathbb{L}$ be a local system on $X^{an}$, s.t. $\mathbb{L}|_{D^{an}}$ is motivic.

Then is $\mathbb{L}$ motivic?

(in fact, suggestion was that the motive over $D$ should extend, after an isogeny, to all of $X$)

Exercise: prove it when $\mathbb{L}|_D$ comes from a family of AVs!
(corollary of work of Simpson)

Arithmetic

If $X/k$ general $\rightsquigarrow$ $\pi_1^{\acute{e}t}(X)$, profinite group.

Def A $\mathbb{Q}_\ell$-local system on $X$ is a conti hom

$$\rho : \pi_1^{\acute{e}t}(X) \to GL_n(\mathbb{Q}_\ell)$$

Conj (Deligne) (from Weil II)
Let $X/\mathbb{F}_q$, let $L$ be a $\overline{\mathbb{Q}}_\ell$ local system on $X$ s.t.

1. irr.
2. N/A
3. triv. det
4. N/A

Then $L$ is of geom. origin (up to a Tate twist)

Question: why fewer hypotheses?

Note ② "Automatic" by Grothendieck's quasi-unipotent monodromy theorem

④ Morally "automatic" after finiteness result of Deligne, Drinfeld (~2010)

Rmk

① Conj. D is known when $\dim X = 1$ due to proof (not just statement) of Langlands, due to L. Lafforgue.

"Why"

$$L \longleftrightarrow \overline{\mathbb{Q}}_\ell \text{ cusp form}$$

$$\wedge$$

$$C^\infty\Big(G(\hat{\mathcal{O}})\backslash \underbrace{G(\mathbb{A}_{\mathbb{F}_q(X)})/G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell}\Big)$$

has no topology

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\iota} \overline{\mathbb{Q}}_{\ell'} \qquad \downarrow \wr$$

$$\text{"}\iota_L\text{"} = \underline{L}' \longleftrightarrow C^\infty( \quad , \overline{\mathbb{Q}}_{\ell'})$$

$\underline{\underline{\text{cont}}}$ rep of $\pi_1(X) \to Gl_n(\overline{\mathbb{Q}}_{\ell'})$

"compatible"

local-global compatible of Langlands

$\iff$ Frobenius eigenvalues match.

<u>Rmk</u> A conj. of dJ, proved by Gaitsgory for $\ell > 2$, implies the following:

<u>Cor (dJG)</u> Let $U/\overline{\mathbb{F}}_q$ be hyperbolic.

Let $\mathcal{S} := \{ L$ $\overline{\mathbb{Q}}_\ell$ - local systems on $U/\overline{\mathbb{F}}_q$ w/ fixed rank, bounded ramification @ $\infty$, fixed local monodromies $\}$

$U/$

$\mathcal{S}^{geo}$

Then ① $\mathcal{S}^{geo}$ is infinite

② ~ " $\mathcal{S}^{geo}$ is Zariski dense in $\mathcal{S}$ "

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<u>Rmk</u> • Using dJ's conj., Drinfeld proved semi-simplicity + Hard Lefschetz thms for perverse sheaves in char 0.

• Using dJ + companions + ... dJ-Esnault proved that

$\pi : X \to Y$ (quasi proj. normal)

$L_Y$ is s.s. $\mathbb{C}$- local system

$\Rightarrow \pi^* L_Y =: L_X$ is s.s.

Rmk Recently, several authors have generalized Conj D to include the finitely generated /Q (Litt, Petrov)

This may be thought of as a relative Fontaine-Mazur conjecture.

maybe skip

Q: Is CorAJG true for other base fields? E.g.

Question: Let $X/\mathbb{C}$ be smooth. Let $\mathrm{Char}^{B}(X)$ be a character variety (moduli of $\pi_1(X) \to GL_n$)

Is the set of pts of geo origin dense?!?

L-L prove that such a statement is false in general for low rank.

<u>Notation</u> $C$ sm. proj. curve/$\mathbb{C}$ genus $g$

$x_1, \ldots, x_N$ distinct pts of $C$.

$U := C \setminus \{x_1, \ldots, x_N\}$

<u>Thm</u> (LL'1) Let $(C, x_1, \ldots, x_N)$ be analytically very general in $\mathcal{M}_{g,n}$.
Let $\mathbb{L}$ be a local system of geo origin on $\mathcal{U}$ w/ $\infty$-monodromy. Then

$$\operatorname{rank} \mathbb{L} \geq 2\sqrt{g+1}$$

<u>Slogan</u>: "very general curve admits

<u>NO</u> low rank local systems of

geo origin"