# Crystals II

Let $S|k$ be smooth, w/ $\mathrm{char}(k) = 0$.

Let $\mathcal{E}$ be a crystal on $S_{\mathrm{inf}}$

**Claim:** $\mathcal{E}$ gives rise to an object $(M, \nabla) \in \mathrm{MIC}(S|k)$

Pf Recall:

$$S \times S \supseteq S(1) = (S, \mathcal{O}_{S\times S}/\mathcal{J}^2)$$

$$\begin{tikzcd} & S(1) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S & & S \end{tikzcd}$$

$$\begin{tikzcd} S(1) \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & S \arrow[l, "\Delta"'] \end{tikzcd}$$

Set $M := \mathcal{E}(S \hookrightarrow S)$
$\uparrow$
$S_{\mathrm{inf}}$

Then

$$p_1^* M \xrightarrow{\sim} \mathcal{E}(S \hookrightarrow S(1)) \xleftarrow{\sim} p_2^* M$$

by virtue of being a crystal.

$\rightsquigarrow$ $\exists$ a natural isomorphism

$$\varphi_{12} : p_1^* M \longrightarrow p_2^* M$$

$\rightsquigarrow$ there is a natural connection $\nabla$ on $M$.

Again using the crystal property, one checks $\nabla$ is integrable.

Claim: $(M, \nabla) \in MIC(S/k)$ gives rise to a crystal $\mathcal{E}$ on $S_{inf}$.

Pf Let $(\mathcal{U} \hookrightarrow T)$ be an affine object in $S_{\inf}$, corresponding to the ring map $A \twoheadleftarrow B$

Subclaim:

$\exists$ a commutative diagram

$$\begin{tikzcd}
A & B \arrow[l, two heads] \\
P := k[T_1,\dots,T_N] \arrow[u, two heads] \arrow[r, hook] & k[T_1,\dots,T_N]^{\wedge}_{J} =: D \arrow[u, "\varphi"']
\end{tikzcd} \quad \bigstar$$

$\ker = J$

$$\mathcal{E}(\mathcal{U} \hookrightarrow T) := M \otimes_{D,\varphi} B$$

Idea: " $j_*$ extension, then restrict"

"trivial along vertical directions"

Question: why is this well-defined?!

I.e., given two maps $\varphi_1, \varphi_2$ as in $\bigstar$, how do the outputs relate?

subclaim:

$\exists$ a natural $B$-isomorphism

$$\varphi_{12} : M \otimes_{D,\varphi_1} B \longrightarrow M \otimes_{D,\varphi_2} B,$$

induced from $\nabla$:

$$m \otimes 1$$

$$\Big\downarrow$$

$$\sum_{\vec{v} \in \mathbb{Z}^N_{\geq 0}} \left( \left( \prod_{i=1}^{N} \left(\nabla_{\partial_{T_i}}\right)^{v_i} \right) m \otimes \prod \frac{\left(\overset{\overset{\text{in } I}{\widehat{\xi}}}{\varphi_1(T_i) - \varphi_2(T_i)}\right)^{v_i}}{(v_i)!} \right)$$

Q: Why does this sum "converge"?

A: $A \twoheadleftarrow B$ is a nilimmersion,

hence the sum is finite.

Q: What does the symbol $\nabla^M_{\frac{\partial}{\partial T_i}}$ mean?

$$\Leftrightarrow \theta_i : M \to M$$

$$\begin{array}{c} A = k[T_1, \dots, T_N]/J \\ \uparrow\!\!\!\uparrow \\ P = k[T_1, \dots, T_N] \end{array}$$

$$0 \to T_U \to T_{\mathbb{A}^N}|_U \to N_{U \subseteq \mathbb{A}^n} \to 0$$

replacing $U$ by a cover, this sequence splits $\rightsquigarrow$ there is a (non-canonical) notion of $\frac{\partial}{\partial T_i} \in T_U$. However,

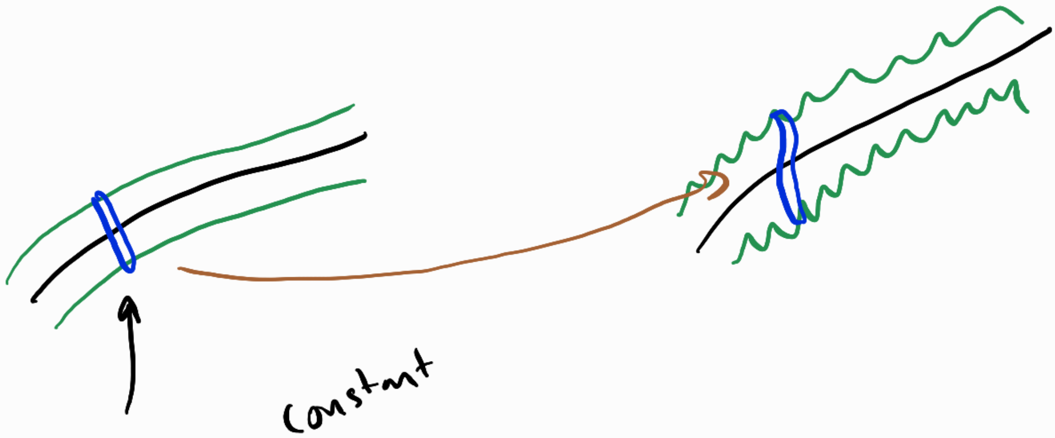
$\theta_i = \nabla_{\frac{\partial}{\partial T_i}}$ depends only on $P \twoheadrightarrow A$.

Subrmk

- $$\frac{(\varphi_1(T_i)-\varphi_2(T_i))^k}{k!} = \gamma_k(\varphi_1(T_i)-\varphi_2(T_i)) =: (\varphi_1(T_i)-\varphi_2(T_i))^{[k]}$$

- In crystalline setting, sum will converge either if $\nabla$ is topologically quasi-nilpotent ( $\nabla \circ \dots \nabla$ is divisible by $p$) <u>OR</u> if $\gamma_k \to 0$ as $k \to \infty$. Former corresponds to crystalline site, latter to nilpotent crystalline site. (Note that $p \mid \varphi_1 - \varphi_2$)

## Picture



$M \otimes_{D,v,} B$ constant along vertical slices

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Now, one can play an analogous game in the crystalline site.

Let $S|k$ smooth $|$ $k$ perfect of char $p$.

We work w/ the sites

$$CRYS(S/W(k))$$

(resp.) $$NCRYS(S/W(k))$$

Let $\hat{S}/W(k)$ be a p-adic formal scheme w/ an iso

$$\hat{S}\big|_k \simeq S$$

__Fact__

- Crystals on $\mathrm{NCRYS}(S/W(k))$ $\iff$ formal flat connections on $\hat{S}$
- Crystals on $\mathrm{CRYS}(S/W(k))$ $\iff$ formal flat connections on $\hat{S}$ that are top. quasi-nilpotent.

What does this look like?

$S = \mathrm{Spec}(\mathbb{F}_p[T])$ $\hat{S} = \mathrm{Spf}(\mathbb{Z}_p\langle T\rangle)$

$$\mathbb{Z}_p\langle T\rangle = \left\{ \sum_{i\geq 0} a_i T^i, \lim_{i\to\infty} a_i = 0 \right\}$$

"functions" on unit disk."

$$\Omega^1_{\hat{S}/\mathbb{Z}_p} \hookrightarrow \mathbb{Z}_p\langle T\rangle\{dT\}$$

a formal flat connection:

p-adic completion

$$\hat{\nabla} : \hat{M} \longrightarrow \hat{M} \hat{\otimes} \Omega^1_{\hat{S}/\mathbb{Z}_p},$$

where $\hat{M}$ is a p-adically complete + separated quasi-coherent sheaf on $\mathbb{Z}_p\langle T\rangle$.

Suppose $\hat{S}/W(k)$ is projective, i.e., that $\hat{S}$ is the p-adic completion of some $\tilde{S}/W(k)$.

Although $\hat{M}$ is algebraic, there is $\boxed{\text{No}}$ a priori reason that $\nabla$ is algebraic, i.e., comes from a connection

$$\nabla : \tilde{M} \longrightarrow \tilde{M} \otimes \Omega^1_{\tilde{S}/\mathbb{Z}_p}$$

Last time, we briefly discussed functoriality of the topos.

$$f : S \to T \quad \rightsquigarrow$$

$$\begin{array}{ccc} \mathrm{Sheaves}(\mathrm{CRYS}(T/W)) & \to & \mathrm{Sheaves}(\mathrm{CRYS}(S/W)) \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathrm{Crystals}(\mathrm{CRYS}(T/W)) & \to & \mathrm{Crystals}(\mathrm{CRYS}(S/W)) \end{array}$$

What about when $T = S$, $f = \mathrm{Frob}_S$?

Suppose $S$ has a formal lift $\hat{S}/W$ s.t. $\mathrm{Frob}_S$ lifts to $\widehat{\mathrm{Frob}_S}$.

Then the pulled back formal connection is the "evaluation of the crystal on $\hat{S}$".

This glues! (See Esnault's lecture notes, §8.

the argument is the same Taylor formula: assume there are two lifts of $\mathrm{Frob}_S$ and canonically construct a map btwn the corresponding pullbacks.)

Example of earlier

Let $E \mid \mathbb{F}_p$ be an elliptic curve

Let $\hat{E} \mid \mathbb{Z}_p$ be a lift w/out CM

(e.g., w/ transcendental $j$-invariant)

Then

$$\begin{array}{ccc} \text{Crystals}(\mathrm{NCRYS}(E/\mathbb{Z}_p)) & \longleftrightarrow & \text{Formal flat connections on } \hat{E} \\ \cup & & \cup \\ \text{rank 1 crystals} & \longleftrightarrow & \text{rank 1 formal flat connections on } \hat{E} \end{array}$$

The RHS has a Frobenius action,

even though $\{\text{rank } 1 \text{ flat connections on } \tilde{E}/\mathbb{Z}_p\}$

DOES NOT!!