Arithmetic Structure of rigid flat connections

<u>Goal</u>: let $X/\mathbb{C}$ be smooth projective, let (E, ∇) be a vector bundle w/ a flat connection on X. Then $\exists$ a spreading out

$$\begin{array}{c}(\mathcal{X}, \mathcal{E}, \nabla) \\ \downarrow \\ S\end{array}, \quad \text{where}$$

- $S = \mathrm{Spec}(A)$, $A \subseteq \mathbb{C}$ is a smooth $\mathbb{Z}$-algebra (of finite type)

such that

① $\forall$ closed points s of S, $(\mathcal{E}, \nabla)_s$ has nilpotent p-curvature

(Richen proved this last time)

② $\exists f = f(X, \mathcal{E}, \nabla, S)$ s.t. $\forall$ closed points s of S,

$(\mathcal{E}, \nabla)_s$ initiates an f-periodic flow (w/r/t any $W_2(\kappa(s))$ point $\tilde{s}$ of S)

Rmk This is strong arithmetic evidence that rigid flat connections are motivic.

We will first recall the argument Richer explained and the necessary background.
To slightly simplify notation, we will assume that (E, ∇) has trivial determinant.

First of all, there are finite type moduli spaces

$$M_{dR}(X/\mathbb{C}, r), \quad M_{Dol}(X/\mathbb{C}, r$$

which parametrize stable flat connections
(resp. stable Higgs bundles) of rank r.

Then $M^{rig}_{dR}(X/\mathbb{C}, r)$ $M^{rig}_{Dol}(X/\mathbb{C}, r)$ are
the subschemes of isolated points.
These are necessarily Artinian $\mathbb{C}$-schemes.

A fundamental theorem of Langer implies there
exists a spreading out

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ S \end{array}$$

such that $M_{dR}(\mathcal{X}/S, r)$, $M_{Dol}(\mathcal{X}/S, r)$
exist as finite type coarse moduli
spaces parametrizing P-stable flat connections
(resp. Higgs bundles) on $\mathcal{X}/S$ of rank r.

Moreover, the rigid locus is defined as the maximal open subscheme

$$\begin{array}{c} M_*(\mathcal{X}/S, r) \\ \downarrow \\ S \end{array}$$

that is quasi-finite, for

$$* \in \{dR, Dol\}$$

As $M_*^{rig}(X/\mathbb{C}, r)$ is Artinian, one can show that we may choose $\mathcal{X}/S$ s.t. "every rigid dR bundle (resp Higgs bundle) is defined/S".

More concretely, if

$n_* = |\mathcal{M}^{rig}_*(X|\mathbb{C}, r)|$, then

$n_{dR} = n_{Dol}$ (Simpson),

and one may choose $\mathcal{X}/S$ s.t. there are exactly n_* sections

$$\begin{array}{c} \mathcal{M}^{rig}_*(\mathcal{X}/S, r) \\ \downarrow \\ S \end{array}$$

corresponding to the rigid flat connections / Higgs bundles over $\mathbb{C}$.

(Imagine that $\mathcal{M}^{rig}_*(\mathcal{X}/S, r)$ is isomorphic to $\hat{\coprod\limits_{i=1}}$ (nilpotent thickenings of S))

Recall that by NAHT, every stable rigid Higgs bundle has nilpotent Higgs field.

Naive Proof

Pick $\mathcal{X}/S$ as above. Let (V, θ) be a rigid Higgs bundle. For

$$p := \mathrm{char}(k(s)) > r,$$

the order of nilpotency of $(V, \theta)_s$ is $\leq p-1$.

Then $C^{-1}(V, \theta)_s$ is one of my rigid dR bundles. On the other hand, as it arises from C^{-1}, the p-curvature is nilpotent. $\square$.

Problem : C^{-1} & NAHT are

NOT compatible in general. In particular,

$C^{-1}(\mathcal{V}, \theta)_s$ is not necessarily

rigid.

Key Lemma Let $\mathcal{X}/S$ as above.

$\exists\ D = D(\mathcal{X}/S)$ s.t. for any

closed point s w/ $\mathrm{char}(\kappa(s)) > D$,

and any rigid stable Higgs bundle

$(\mathcal{V}_s, \theta_s)$ on $\mathcal{X}_s$,

$C^{-1}(\mathcal{V}_s', \theta_s')$ is a rigid

stable flat connection.

(Here, $(\mathcal{V}_s', \theta_s')$ means use

$w: \mathcal{X}_s \xrightarrow{\sim} \mathcal{X}'_s$ to transport $(\mathcal{V}_s, \theta_s)$ to a rigid stable Higgs bundle on $\mathcal{X}'_s$.)

To prove, we need to recall preliminaries.

Let Z/k be smooth projective, over a perfect field of char $p>0$.

Let (E, ∇) be a flat connection on Z. Then

$$\psi_\nabla : E \longrightarrow E \otimes F^* \Omega^1_{Z'/k}$$

is the p-curvature, which is horizontal.

$$\mathrm{Tr}(\Lambda^i \psi_\nabla) \in H^0(Z, F^* \mathrm{Sym}^i \Omega^1_{Z'/k})^{\nabla = 0}$$

$$\parallel$$

$$H^0(Z', \mathrm{Sym}^i \Omega^1_{Z'/k})$$

$$\chi_{dR}(E, \nabla) = (-\mathrm{Tr}(\psi_\nabla), +\mathrm{Tr}(\Lambda^2 \psi_\nabla), \ldots, (-1)^r \mathrm{Tr}(\Lambda^r \psi_\nabla))$$

$$\cap$$

$$\mathcal{A}' := \bigoplus_{i=1}^{r} H^0(Z', \mathrm{Sym}^i \Omega^1_{Z'/k})$$

" twisted Hitchin base "

$\chi_{dR}(E, \nabla)$ is " char poly of ψ_∇ "

The construction works in families:
if (E, ∇) is an S-relative flat connection on $Z \times S$, then $\exists$

$$\chi_{dR} : S \longrightarrow \mathcal{A}'$$

Now by general nonsense, given

$$a : S \longrightarrow \mathcal{A}', \quad \exists$$

a spectral cover

$$Z'_a \subseteq T^* Z' \times S$$

cut out by

$$\lambda^r - a_1 \lambda^{r-1} + a_2 \lambda^{r-2} + \dots = 0$$

$$a = (a_1, \dots, a_r)$$

<u>Thm</u> (Classical BNR for Z'/k)
Let S/k be a scheme, let $a : S \to \mathcal{A}'$. Then

$$\left\{ \begin{array}{l} S\text{-relative Higgs bundles} \\ (E, \theta) \text{ on } Z \times S \\ \text{of rank } r \text{ w/} \\ \chi_{Higgs}(E, \theta) = a \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{sheaves } M \text{ on } T^* Z' \times S, \\ \text{supported on } Z'_a, \\ \text{w/ } \pi_* M \text{ of} \\ \text{rank } r \end{array} \right\}$$

<u>Thm</u> (dR BNR) Let S/k be a scheme.

$$a : S \longrightarrow A'.$$

Then $\exists$ equivalence of categories

$$\left\{ \begin{array}{l} S\text{-relative flat connection} \\ \text{on } Z \times S \quad (E, \nabla) \\ \text{of rank } r \text{ w/} \\ \quad \chi_{dR}(E,\nabla) = a \end{array} \right\} \longleftarrow$$

$$\longrightarrow \left\{ \begin{array}{l} \mathcal{D}_{Z'}\text{-modules } M \text{ on } T^*Z' \times S \\ \text{supported on } Z'_a, \quad \text{s.t.} \\ \quad \pi_* M \text{ has rank } p^d \cdot r \end{array} \right\}$$

<u>Rmk</u> The key point is that M is supported on a poly of rank r, not $r \cdot p^d$.

Pf uses Azumaya property of $\mathcal{D}_{Z'}$ on

T^*Z'. Note that we get a rank $p^{d}\cdot r$ Higgs bundle, which is supported on a low-deg (ind of p) spectral cover.

Before we prove the lemma we need one final def.

Def Let Z/k be smooth projective. Let T/k be a scheme. We say

$$a : T \longrightarrow Z'$$

is OV-admissible if

$$Z'_a \subseteq T^*Z' \times T \quad \text{if}$$

Z'_a factors through a $(p-1)$ nbhd of the zero-section.

<u>Prop</u> (OV + BNR)

Let Z/k be sm. proj, $\tilde{Z}/W_2(k)$

Let T/k be a scheme,

$$a : T \longrightarrow A'$$

which is OV admissible

Then $\exists$ equivalence of stacks:

$$\left\{ \begin{array}{l} \text{stable } (E,\nabla) \text{ on } Z \times T \\ \text{w/ } r < p-1, \text{ w/} \\ \quad \chi_{dr}(E,\nabla) = a \end{array} \right\} \xrightarrow{C_a}$$

$$\longrightarrow \left\{ \begin{array}{l} \text{stable } (V,\theta) \text{ on } Z' \times T \text{ w/} \\ r < p-1, \text{ w/} \\ \quad \chi_{Higgs}(V,\theta) = a \end{array} \right\}$$

Key: $\tilde{Z}$ induces a splitting of $\mathcal{D}_{Z'}$ on $(p-1)$ nbhd of zero-section of T^*Z'.

Proof of Lemma

Set $D >$ degs $\left\{ \begin{array}{l} M_{dR}^{rig}(\mathcal{X}/S) \to S \\ M_{Dol}^{rig}(\mathcal{X}/S) \to S \end{array} \right\}$

Let (V_s, θ_s) be a rigid Higgs bundle on $\mathcal{X}_s$. Then any def $(\mathcal{V}_B, \theta_B)$
$$\downarrow$$
$$B$$

has $B \to A$ landing in $\underbrace{A^{(D)}}$

order D thickening of 0.

If $C^{-1}(V_s, \theta_s) =: (E_s, \nabla_s)$ is not rigid, $\exists$

pos. dim. T-def.

$$(E_T, \nabla_T) / _{X \times T}$$

<u>Subclaim</u> $\exists$ a function $R(r, m)$ linear in m, quadratic in r, s.t.

$p > R(r, m) \Rightarrow$ any map

$a : T \longrightarrow \mathbb{A}^{1^{(m)}}$ is OV admissible

in other words, if χ_{dR} lands in a fixed order thickening of $0 \in \mathbb{A}^1$, and p is big, then (E_T, ∇_T) is OV admissible.

<u>Pf</u>: $X'_{S,a} \subseteq T^*(\mathcal{X}_S)' \times T$, given by

zero locus of

$$\lambda^r + a_1 \lambda^{r-1} + \cdots a_r = 0,$$

w/ $a_i \in \mathcal{O}(T)$, λ the canonical section of $\pi^* \Omega^1_{\mathcal{X}'_s}$.

Know: $a_i^m = 0 \quad \forall i$

Q: when is

$$(\lambda^r + a_1 \lambda^{r-1} + \cdots + a_r)^{p-1} \in (\lambda) ?$$

A: For every partition λ ~~μ~~ $\vdash p-1$ into r parts, at least one must be $\geq m$.

$\square$

Now, back to main pf. Pick $m > D$ and assume $p > R(r, m)$.

Let s be a closed point, set

$$(E_s, \nabla_s) := C^{-1}(\underbrace{V_s, \theta_s}_{\text{rigid}}).$$

Assume (E_s, ∇_s) is NOT RIGID. Then $\exists$

- $T/_{k(s)}$ scheme
- T-family $(E_T, \nabla_T)/\mathcal{X}_s \times T$, w/ (E_s, ∇_s) as one fiber s.t.
- $T \to M_{dR}(\mathcal{X}|S, r)$ has pos dim image.

TWO OPTIONS

① $\chi_{dR}: T \to \mathbb{A}'$ factors

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & \mathbb{A}' \\ & (\mathbb{A}')^{(m)} \arrow[u] \end{tikzcd}$$

② $\chi_{dR}: T \to \mathbb{A}'$ does not factor through $(\mathbb{A}')^{(m)}$.

CASES

① As $p > R(r,m)$, χ_{dR} is OV admissible. By Prop OV+BNR

$\Rightarrow \exists$ a T-family of Higgs bundles $(V_T, \theta_T) / \mathcal{X}'_s \times T$ s.t.

$$C^{-1}(V_T, \theta_T) = (E_T, \nabla_T)$$

But (V_s, θ_s) was rigid

apply ω $\rightsquigarrow$ $T \longrightarrow M^{\mathrm{rig}}_{\mathrm{Dol}}(\mathcal{X}/s, r) \subseteq M_{\mathrm{Dol}}(\mathcal{X}/s, r)$

has image in an isolated point of length $\leq D$

$\rightsquigarrow T \longrightarrow M_{dR}(\mathcal{X}/s, r)$ has zero dim image, contradicting our assumption.

(2) $\chi_{dR} : T \to \mathbb{A}' \supseteq (\mathbb{A}')^{(m)}$ does not factor through $(\mathbb{A}')^{(m)}$.

Let $T^{(n)}$ be n^{th} order nbhd of $t \in T$ corresponding to (E_s, ∇_s).

We have a restricted family
$$(E_{T^{(n)}}, \nabla_{T^{(n)}}) \,/\, X \times T^{(n)},$$

whose Hitchin invariant

$$\begin{tikzcd}
\chi_{dR} : T^{(n)} \arrow[r] \arrow[dr, "\text{factors}"'] & (\mathcal{A}') \arrow[d, phantom, sloped, "\supseteq"] \\
& (\mathcal{A}')^{(n)}
\end{tikzcd}$$

<u>Claim</u>: $\exists\, n \geq m$ s.t. χ_{dR} factors through $(\mathcal{A}')^{(m)}$ but not through $(\mathcal{A}')^{(m-1)}$

However, $p > R(r,m) \Rightarrow \chi_{dR}|_{T^{(n)}}$ is

OV admissible $\Rightarrow$ Prop OV+BNR

implies $\exists$ a map

$$T^{(n)} \longrightarrow M^{ng}_{Dol}(\mathcal{X}'_s, r)$$

with the same Hitchin invariant

$$\chi_{Dol}: T^{(n)} \longrightarrow (\mathcal{A}')^{(m)}$$

As the map χ_{Dol} does not factor through $(\mathcal{A}')^{(n)}$ for $n < m$, we obtain a strictly order m deformation of (V_s, θ_s), contradicting the choice $m > D$.

Rmk We used BNR in Prop OV BNR. The idea: given (E,∇), we get a Higgs bundle of rank $p^d \cdot r$ on Z', w/ Hitchin invariant $(a)^{p^d}$, supported on Z'_a. OV implies that a lift $\tilde{Z}/W_2$ splits this huge Higgs bundle into rank r pieces, each w/ Hitchin invariant a.

Corollary A: If s is a closed point of S s.t.

① $\mathrm{char}(\kappa(s)) > D$

② $n_{dR}(\mathcal{X}_s, r) = n_{Dol}(\mathcal{X}'_s, r)$

Then for every rigid flat connection $(E_s, \nabla_s) \in \mathcal{M}^{rig}_{dR}(\mathcal{X}_s, r)$ has nilpotent p-curvature.

Pf $C^{-1} : \mathcal{M}^{rig}_{Dol}(\mathcal{X}'_s, r) \to \mathcal{M}^{rig}_{dR}(\mathcal{X}_s, r)$ is injective, but both sets have the same size.

Main Theorem: $\exists f = f(\mathcal{X}/S, r)$ s.t. for all closed points s of S w/ sufficiently large residue characteristic $\forall (V_s, \theta_s) \in \mathcal{M}^{rig}_{Dol}(\mathcal{X}'_s, r)$, (V_s, θ_s) initiates a canonical Higgs-dR flow of length $\leq f$.

Naive Pf

"Flow operator $\mathrm{gr}_{\mathrm{Fil}^{\mathrm{simp}}} \circ C^{-1}$ is a bijection of a finite set"

<u>Problem</u>: why is $\mathrm{gr}_{\mathrm{Fil}^{\mathrm{simp}}}(C^{-1}(V_s, \theta_s))$ rigid?!

To resolve, consider the whole moduli

$$M_{\mathrm{Hodge}}(X, r)$$

of λ-connections:

(N, D), where N is a vector bundle of rank r,

$$D : N \longrightarrow N \otimes \Omega^1_X$$

w/ $D(fs) = f D(s) + \lambda s \otimes df$

$\lambda = 0 \rightsquigarrow$ Higgs

$\lambda = 1 \rightsquigarrow$ dR

$$\lambda : \mathcal{M}_{Hodge}(X, G) \longrightarrow \mathbb{A}^1$$

Fiber $|_0$ $=$ $\mathcal{M}_{Dol}$

Fiber $|_1$ $=$ $\mathcal{M}_{dR}$

Set $\mathcal{M}^{rig}_{Hodge}(X, G) \subseteq \mathcal{M}_{Hodge}(X, G)$ to be the maximal open subset on which λ is quasi-finite.

<u>Key Facts</u>

① $\mathcal{M}^{rig}_{Hodge}(X, G)$ splits $\mathbb{G}_m$-equivariantly:

$$\begin{tikzcd} \mathcal{M}^{rig}_{Hodge}(X,r) \arrow[d, "\lambda"] & \simeq & \mathcal{M}^{rig}_{Dol}(X,r) \times \mathbb{A}^1 \arrow[d] & ((V,\theta),\mu) \arrow[d, mapsto] \\ \mathbb{A}^1 & & \mathbb{A}^1 & \mu \end{tikzcd}$$

Moreover, if $(V,\theta) \in \mathcal{M}^{rig}_{Dol}$,

then the image of $((V,\theta),1)$

corresponds to a $\mathbb{C}$-PVHS

(E,∇,Fil,ψ) s.t.

$$gr_{Fil}(E,\nabla) \simeq (V,\theta)$$

② Can build relative moduli spaces of the above over the spreading out $\mathfrak{X}/S$.

May ensure that for

every rigid $(\mathcal{E},\nabla)$ on $\mathfrak{X}/S$,

the Hodge filtration spreads out

s.t. $gr_{Fil}(\mathcal{E},\nabla) \simeq (\mathcal{V},\theta)$

is one of my rigid Higgs bundles.

Using key fact 2, can fix the naive pf.

Pf of Thm

$C^{-1}(\mathcal{V}_s,\theta_s) = (E_s,\nabla_s)$ is rigid

$\rightsquigarrow$ globalizes to $(\mathcal{E},\nabla)$ on $\mathfrak{X}/S$

$\rightsquigarrow$ $\exists$ Fil on $(\mathcal{E},\nabla)$ s.t.

by choice of spreading out

$gr_{Fil}(\mathcal{E},\nabla)$ is stable rigid Higgs bundle restricting $\mathcal{F}il\big|_{\mathcal{X}_s}$, we see the associated graded is rigid. We therefore get a bijection

$$\mathcal{M}^{rig}_{Dol}(\mathcal{X}'_s, r) \longrightarrow \mathcal{M}^{rig}_{Dol}(\mathcal{X}'_s, r)$$

of finite sets.

Finally, we sketch an alternative approach to the global nilpotence.

Assume Z/k is smooth projective over perfect k.

$$\begin{tikzcd} T^*Z' \times G_m \arrow[r, "m"] \arrow[dr] & T^*Z' \arrow[d, "\pi"] \\ & Z' \end{tikzcd}$$

"conical structure"

For $V \subseteq T^*Z'$ a subscheme,

set $V_n := V \times \underbrace{\mathrm{Spec}\, k[t, t^{-1}] / (t-1)^n}_{n^{th} \text{ order nbhd of neutral elt in } G_m}$

$$\begin{tikzcd} m: V_n \times G_m \arrow[r] \arrow[dr] & T^*Z' \arrow[d] \\ & Z' \end{tikzcd}$$

$r : V_n \longrightarrow V$ is the

retraction induced from

$$k \hookrightarrow k[t, t^{-1}] / (t-1)^n$$

<u>Key Prop</u>

Suppose $\mathcal{Z}$ lifts to W_2, and fix a rank N. Let A' be the N-Hitchin base.