Introduction to p-curvature.

Recall setup from the end of last lecture:

- S/k smooth, $\mathrm{char}(k) = p$.

- $$\begin{tikzcd}
S \arrow[dr, dashed, "F_{S/k}"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& S' \arrow[r, "\pi"] \arrow[d] & S \arrow[d] \\
& \mathrm{Spec}(k) \arrow[r, "\mathrm{Frob}_k"'] & \mathrm{Spec}(k)
\end{tikzcd}$$

<u>Observation</u>: for any $E \in \mathrm{Coh}(S')$, $\exists$ a "canonical" connection on $F_{S/k}^*(E) \in \mathrm{Coh}(S)$

Indeed, $$F_{S/k}^*(E) \simeq F_{S/k}^{-1}(E) \otimes_{F_{S/k}^{-1}(\mathcal{O}_{S'})} \mathcal{O}_S$$

$$\nabla^{can}(s \otimes f) := s \otimes df$$

Why is this well-defined?

Let $\lambda \in \mathcal{O}_{S'}$. Then

$$\nabla^{can}(\lambda s \otimes f) := \lambda s \otimes df$$

$$\overset{?}{=} \qquad\qquad \overset{?}{=}$$

$$\nabla^{can}(s \otimes F_{S/k}^{*}(\lambda) f) := s \otimes d(F_{S/k}^{*}(\lambda) f)$$

$$= s \otimes df \ \checkmark$$

Rmk In char 0, the following are true.

- $(E, \nabla) \in MIC(S/k) \Rightarrow E$ locally free
- S/k proper, $(E, \nabla) \in MIC(S/k) \Rightarrow c_i(E) = 0$

Both are false in char p!

<u>Upshot</u> If S/k is smooth and $\mathrm{char}(k)=p$, then $\exists$ a "canonical" subcategory of $\mathrm{MIC}(S/k)$.

<u>Questions</u>

- How do we describe this subcategory "intrinsically"? (i.e., when is an object of $\mathrm{MIC}(S/k)$ a Frobenius pullback?)
- What are properties of this abelian subcategory? E.g., is it thick?

To answer these questions, we return to the ring of crystalline differential operators.

The Ring $\mathcal{D}_S$

[learned from M. Gröchenig]

Setup • S/k smooth,

• If $\mathbb{Q} \subseteq k$, say has "char ∞"

Def The ring $\mathcal{D}_S$ of crystalline differential operators is the quasi-coherent sheaf of $\mathcal{O}_S$ algs, described as the following quotient:

$$\bigoplus_{i \geq 0} \big(\underbrace{T_S^{\otimes i}}_{T_S \otimes_k T_S \otimes \cdots \otimes T_S}\big) \Big/ \mathcal{J},$$

where $\mathcal{J}$ is the 2-sided ideal generated by relations:

• $\partial \otimes f - f \otimes \partial - (\partial f)$

• $\partial_1 \otimes \partial_2 - \partial_2 \otimes \partial_1 - [\partial_1, \partial_2]$

Rmk

- $\mathcal{D}_S$ is "universal enveloping algebra" of the Lie algebroid $T_{S/k}$

- $\mathcal{D}_S$ is a filtered algebra: $\mathcal{D}_S^{\leq d} := \mathrm{Im}\left(\bigoplus_{i \leq d} T_S^{\otimes i} \to \mathcal{D}_S \right)$

- MIC_S = "$\mathcal{O}_S$-coherent $\mathcal{D}_S$ modules"

slogan: a flat connection is equivalent to the data of a $\mathcal{D}_S$-action

Ex Let $S = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$.

Then $\mathcal{D}_S$ is generated, as an algebra$/k[t]$, by the symbols $t, \frac{\partial}{\partial t}$

w/ the relation $\left[\frac{\partial}{\partial t}, t \right] = 1$.

"Weyl alg": $k\langle t, \partial \rangle$. Note when $\mathrm{char}(k) = p$,
$Z(k\langle t, \partial \rangle) = k[t^p, \partial^p]$.

Obs $D_S \longrightarrow \mathrm{End}_k(\mathcal{O}_S)$

Q: When is this faithful?

Lemma 1

The map $D_S^{\leq \mathrm{char}(k)-1} \longrightarrow \mathrm{End}_k(\mathcal{O}_S)$

is faithful

Pf

- First, consider the case that $S = \mathbb{A}^n_k = \mathrm{Spec}\, k[t_1, \dots, t_n]$.

Then we claim:

Let $\delta \in D^{\leq d}_{\mathbb{A}^n}$, w/ $d \leq \mathrm{char}(k)-1$.

Then $\exists\, f \in k[t_1, \dots, t_n]$ of $\deg \leq d$

s.t. $\delta f \neq 0$.

Pf: Induction: $\delta = \delta_{<d} + \delta_d$,

where $\delta_{<d}$ is in D_S^{d-1}

and $$\delta_d = \sum_{|J|=d} a_J \, \partial^J$$

$$\frac{\partial^d}{\partial t_1^{j_1} \partial t_2^{j_2} \cdots \partial t_n^{j_n}}$$

If $\delta_{<d} \neq 0$, then done by induction.

Else, $\delta = \delta_d$. Suppose

$k[t_1, \dots, t_n] \ni a_J = a_{j_1 \cdots j_n} \neq 0$.

Then $f = \prod_{i=1}^{n} t_i^{j_i}$ is of deg d,

and $\delta f = \delta_d f = j_1! \cdots j_n! \, a_J$

$\neq 0$ $\square$

In general, use étale descent to reduce to this case. $\square$

$$D_S \longrightarrow \operatorname{gr} D_S$$

$$\delta \longmapsto \text{“Symbol}(\delta)\text{”}$$

<u>Thm</u> (PBW)

There is an iso, natural in pullback along étale map,

$$\operatorname{gr}(D_S) \simeq \operatorname{Sym} T_S = \pi_* \mathcal{O}_{T^*S},$$

$\pi : T^*S \to S$

<u>Idea of pf</u>

"Étale coordinates" $\Rightarrow$ reduce to (open subset of) $\mathbb{A}^n$

$$\delta \in D_{\mathbb{A}^n}^{\leq d}$$

$$\delta = \sum_{i=0}^{d} \left(\sum_{|J|=i} a_J \partial^J \right),$$

w/ $a_J \in k[t_1, \dots, t_n]$

- $\delta \longmapsto \sum_{|J|=d} a_J y^J \in \mathrm{Sym}^d(T_S)$

where "$y_i = \partial_i$" is the corresponding vector field, gives a map

$$\mathcal{D}^d_{\mathbb{A}^n} / \mathcal{D}^{d-1}_{\mathbb{A}^n} \longrightarrow \mathrm{Sym}^d(T_{\mathbb{A}^n})$$

$$\rightsquigarrow \mathcal{D}_{\mathbb{A}^n} \longrightarrow \mathrm{Sym}^*(T_{\mathbb{A}^n})$$

$$=$$

Recall that T^*S is canonically "symplectic": there exists a closed, non-degen 2-form ω:

$\omega = d\eta$, where η is the

tautological 1-form on T^*S:

$$p : T(T^*S) \to T^*S$$

$$\rotatebox{90}{$\in$}$$

$$\alpha_s$$

Given $\xi_{\alpha_s} \in T_{\alpha_s}(T^*S)$

η is determined by the tautological map:

$$T_{\alpha_s}(T^*_s S) \longrightarrow k$$

$$\xi_{\alpha_s} \longmapsto \alpha_s(p_* \xi_{\alpha_s})$$

$=$

Given (local) function $f \in \mathcal{O}_S$,

ω induces a "Hamiltonian vector field" ξ_f:

$$\omega : T^*S \xrightarrow{\sim} TS \qquad \text{(non degen)}$$

$$df \dashrightarrow X_f$$

Then ω induces a Poisson bracket

on $\pi_* \mathcal{O}_{T^*S}$

- bilinear
- skew-symmetric
- $\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0$
- Leibniz: $\{fg, h\} = f\{g,h\} + g\{f,h\}$

$$\{f,g\}_\omega := \omega(X_f, X_g)$$

"Example"

$$S = \mathbb{A}^n_k = \operatorname{Spec}(k[t_1, \ldots, t_n])$$

$$T^* S = \operatorname{Spec}(k[t_1, \ldots, t_n, y_1, \ldots, y_n])$$

"$y_i = \frac{\partial}{\partial t_i}$"

- $\omega = \sum dt_i \wedge dy_i$

- $\{f,g\} = \sum \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_i}$

<u>Lemma 3</u>

Let $\delta_1 \in D_s^{\leq d_1}$, $\delta_2 \in D_s^{\leq d_2}$

Then

- $[\delta_1, \delta_2] \in D_s^{\leq d_1 + d_2 - 1}$

- $[\delta_1, \delta_2] \bmod D_s^{d_1 + d_2 - 2}$
$= \{\delta_1 \bmod D^{\leq d_1 - 1}, \delta_2 \bmod D^{\leq d_2 - 1}\}$

If $[\delta_1, \delta_2] \notin D_s^{d_1 + d_2 - 2}$, then

$$\sigma[\delta_1, \delta_2] = \{\sigma(\delta_1), \sigma(\delta_2)\}$$

<u>Idea of Pf</u>: Coordinates

Lemma 3

Let $f \in \pi_* \mathcal{O}_{T^* S} = \mathrm{Sym}^{\bullet} T_S$ be a local section s.t. $\{f, -\} = 0$

(i.e., f is in the Poisson center).

Then $\exists\, g \in \pi_* \mathcal{O}_{T^* S'}$ ↖ Frobenius twist

s.t. $f = F_{S/k}^* g$.

Pf Coordinates: $S = \mathbb{A}^n_k$,

$T^* \mathbb{A}^n_k = \mathrm{Spec}(k[t_1, \ldots, t_n, y_1, \ldots, y_n])$

Suppose $f \in k[t_1, \ldots, t_n, y_1, \ldots, y_n]$ has at least one exponent that is not a p^{th} power. Then $\exists\, i$ s.t.

$\dfrac{\partial f}{\partial t_i}$ OR $\dfrac{\partial f}{\partial y_i}$ is non-zero.

If $\frac{\partial f}{\partial t_i} \neq 0$, $\{f, y_i\} \neq 0$

$$\{f, y_i\} = \frac{\partial f}{\partial t_i}\frac{\partial y_i}{\partial y_i} - \cancel{\frac{\partial y_i}{\partial t_i}\frac{\partial f}{\partial y_i}}$$

If $\frac{\partial f}{\partial y_i} \neq 0$, $\{f, t_i\} \neq 0$

Hence: $\{f, -\} = 0 \Rightarrow$ all exponents are <u>multiples</u> of p.

<u>Birth of p-curvature</u>

<u>Notation</u>: Let ∂ be a local section of T_S.

Then $\underbrace{\partial \circ \cdots \circ \partial}_{p}$ is again a <u>derivation</u> by binomial theorem

$$\mathcal{O}_S \longrightarrow \mathcal{O}_S$$

$\rightsquigarrow$ corresponds to a vector field,

$\partial^{[p]}$

<u>Def</u> • $\iota : T_S \to D_S^{\leq p}$

$\partial \mapsto \underbrace{\partial^p}_{\text{order } p} - \underbrace{\partial^{[p]}}_{\text{order } 1}$

• If $(E, \nabla) \in MIC(S/k)$,

$\rightsquigarrow D_S \curvearrowright E$

$\psi_\nabla(\partial) :=$ action of $\iota(\partial)$ on E

is the p-curvature.

A priori, ι is just a map of sheaves.

<u>Example</u> $S = \mathbb{A}^1_{\mathbb{F}_2} = \mathrm{Spec}(\mathbb{F}_2[t])$

$\partial = t\frac{\partial}{\partial t}$ $\partial^{[2]} : \mathcal{O}_S \to \mathcal{O}_S$

$f \mapsto t\frac{\partial}{\partial t}\left(t\frac{\partial f}{\partial t}\right)$

$= t\frac{\partial f}{\partial t}$

$$\partial^2 = t\frac{\partial}{\partial t}\left(t\frac{\partial}{\partial t}\right) = t\left(\left(1 + t\cdot\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)$$

$$= t\frac{\partial}{\partial t} + t^2\frac{\partial^2}{\partial t^2}$$

$$\iota(\partial) = t^2\frac{\partial^2}{\partial t^2} \in D^{\leq 2}_{\mathbb{A}'_{\mathbb{F}_2}}$$

<u>Lemma 4</u> $\iota : T_S \to D_S^{\leq p}$ is

p-linear: $\iota(f\partial_1 + \partial_2) = f^p\iota(\partial_1) + \iota(\partial_2)$

<u>Pf</u> Consider the symbols

$$\sigma(\iota(f\partial_1 + \partial_2)) = (f\partial_1 + \partial_2)^p$$

$$\sigma(f^p\iota(\partial_1) + \iota(\partial_2)) = f^p\partial_1^p + \partial_2^p$$

(b/c RHS is in a commutative ring!)

$$\Rightarrow \iota(f\partial_1 + \partial_2) - f^p\iota(\partial_1) - \iota(\partial_2) \in D_S^{\leq p-1}$$

Moreover, for any $g \in \mathcal{O}_S$,

$$\iota(f\partial_1 + \partial_2)(g) = (f^p \iota(\partial_1) + \iota(\partial_2))\, g$$
$$= 0!$$

Lemma 1 $\Rightarrow$ $\square$.

$\rightsquigarrow$ ι induces $\mathcal{O}_S$-linear map $\mathrm{Frob}_S^* T_S \longrightarrow D_S$,

which induces:

- an $\mathcal{O}_S$-linear map

$$\iota : F_{S/k}^* T_{S'} \longrightarrow D_S,$$

- an $\mathcal{O}_{S'}$-linear map

$$\iota : T_{S'} \longrightarrow (F_{S/k})_* D_S$$

Example $S = \mathrm{Spec}(k[t])$

$$\iota : \underbrace{k[t] \cdot y}_{\text{free rank 1 module}} \longrightarrow (F_{S/k})_{\partial}\ k\langle t, y\rangle$$

$$f \cdot y \longmapsto (f\partial)^p - (\underbrace{f\partial \circ \cdots \circ f\partial}_{p \text{ times}}) = f^p \partial^p$$

Lemma 5: Let ∂' be a local section of $T_{S'}$. Then

$$\iota(\partial') \in Z\left(\left(F_{S|_k}\right)_* D_S\right)$$

Pf Let $\partial \in T_S$

$$[\underbrace{\iota(\partial')}_{\deg p}, \partial] \bmod D^{\leq p-1}$$

$$= \{\iota(\partial') \bmod D^{\leq p-1}, \partial\}$$

$$= \{(\sigma(\partial'))^p, \partial\}$$

$(\sigma(\iota(\partial')) = \sigma(\partial'^p) = \sigma(\partial')^p$ b/c σ has values in a commutative ring.)

$$= 0 \bmod D^{\leq p-1}$$

$\Rightarrow [\iota(\partial'), \partial] \in D^{p-1}$

But $\iota(\partial') \curvearrowright \mathcal{O}$ is trivial

$\Rightarrow [\iota(\partial'), \partial] \curvearrowright \mathcal{O}$ trivial

As $D^{\leq p-1} \curvearrowright \mathcal{O}$ is faithful, it follows that $[\iota(\partial'), \partial] = 0$.

As this is true $\forall\, \partial \in T_S$.

$$\Rightarrow \iota(\partial') \in Z(F_{S/k*} D_S) \qquad \boxtimes$$

$\leadsto$ ι induces a map

$$\tilde{\iota} : \mathrm{Sym}^* T_{S'} \longrightarrow Z(F_{S/k*} D_S)$$

<u>Lemma 6</u> $\tilde{\iota}$ is an isomorphism.

- $$\begin{tikzcd} \mathrm{Sym}^* T_{S'} \arrow[r] \arrow[dr] \arrow[dr, bend right, "F^*_{T_{S/k}}"'] & Z(F_{S/k*} D_S) \arrow[d, "\sigma"] \\ & F_{S/k*} \mathrm{Sym}^* T_S \end{tikzcd}$$

$$T_S \xrightarrow{F_{T_{S/k}}} (T_S)'$$ as $\sigma = $ "$gr_{F.e}$" kills lower order terms.

$\leadsto$ • $\tilde{\iota}$ is injective ($T_{S'}$ is reduced)

• image $\underline{\text{is}}$ the Poisson center of $(F_{S/k})_* \mathrm{Sym}^* T_S$ (Lemma 3)

. Lemma 2 : $\mathrm{Im}(\sigma) \subseteq$ Poisson center

of $(F_{S/k})_* \mathrm{Sym}^* T_S$

$\Rightarrow$ $\mathrm{Im}(\sigma) =$ Poisson center

$\Rightarrow$ σ isomorphism

$\Rightarrow$ $\tilde{\iota}$ isomorphism! $\square$.

=

Lemmas 4 & 5 elegantly reprove "classical" facts about the p-curvature.

Let S/k be a smooth var, $\mathrm{char}(k) = p$.

<u>Thm 7</u>

$(E, \nabla) \in \mathrm{MIC}(S/k)$

$\rightsquigarrow \psi_\nabla : F^*_{S/k} T_{S'} \longrightarrow \mathrm{End}_{\mathcal{O}_S}(E) \subseteq \mathrm{End}_k(E)$

Pf Lemma 5: $[c(\partial), f] = 0$

Thm 8 ψ_∇ induces a map:

$$\psi: E \longrightarrow E \otimes F^*_{S/k}\Omega^1_{S'/k}$$

Then this map is flat (here,

$F^*_{S/k}\Omega^1_{S'/k}$ is equipped w/ the canonical connection.)

Pf Lemma 5: $[\iota(\partial), \partial'] = 0$. $\square$

To see the utility of Lem 6, we need:

Lemma 6) The ring $F_{S/k *}D_S$ is an Azumaya algebra over its center

$$\mathrm{Sym}^* T_{S'} \xrightarrow[\tilde{\iota}]{\sim} Z(F_{S/k *}D_S)$$

derived from the p-curvature.

"Explanation" of Lemma 6' for $\mathbb{A}^1_k$

$$\text{rank } p^2 \left\{ \begin{array}{c} k\langle t, \partial \rangle \\ \cup \\ k[t^p, y^p] \end{array} \right. \quad \text{w/} \begin{array}{l} \bullet\ \partial = y^p \\ \bullet\ [\partial, t] = 1 \end{array}$$

We claim this is an Azumaya alg.

$\iff$ it splits after a flat cover

$\uparrow$ thm of Grothendieck

$$\begin{array}{ccc} k\langle t, \partial \rangle & \longrightarrow & k\langle t, \partial \rangle \otimes_{k[t^p, y^p]} k[s, y^p] \\ \cup & & \uparrow \\ k[t^p, y^p] & \longrightarrow & k[s, y^p] \end{array}$$

Key Claim:

$$k[s, y^p] \otimes_{k[t^p, y^p]} k\langle t, \partial \rangle \simeq M_{p \times p}(k[t, y^p])$$

$$k[s, y^p] \otimes_{k[t^p, y^p]} k\langle t, \partial \rangle \to \mathrm{End}_{k[t^p, y^p]}\left(\boxed{k\langle t, \partial \rangle}\right)$$

mult on left, s acts as t.

mult on right

as a module

Rmk In fact, the Azumaya alg

$F_{S/k *} \mathcal{D}_S$ over $T^* S'$

splits over the zero section.

Concretely: $S = \mathbb{A}^1_k$

Restatement

$$k\langle t, \partial \rangle / (\partial^p = 0, [\partial, t] = 1)$$

$$\cup\!\!\mid$$

$$k[t^p]$$

isomorphic to $M_{p \times p}(k[t^p])$

Abstract pf: $F_{S/k *} \mathcal{D}_S$ is Azumaya / $T^* S'$ of rank p^{2n}. On the other hand, $(F_{S/k})_* \mathcal{O}$ has rank p^n over S' and has an action of $(F_{S/k})_* \mathcal{D}_S$ (from d)

$$\Rightarrow (F_{S/k})_* \mathcal{D}_S \Big|_{\text{zero section}} \text{ has action}$$

on rank p^n vector bundle

$\Rightarrow (F_{S/k})_* \mathcal{D}_S$ splits.

In other words,

$$\left(F_{S/k}\right)_* D_S \Big|_{\substack{\text{zero section} \\ S' \hookrightarrow T^*S'}} \xrightarrow{\sim} \mathrm{End}\left(F_{S/k*} \mathcal{O}_S\right)$$

In coordinates,

$$k\langle t, \partial\rangle \Big|_{\substack{\partial^p = 0 \\ [\partial, t] = 1}} \xrightarrow{\sim} \mathrm{End}_{k[t^p]} k[t]$$

Thm ((Cartier Descent)

$$\mathrm{Coh}(S'/k) \longrightarrow \mathrm{MIC}(S/k)$$

$$M \longmapsto (F_{S/k}^{*}M, \nabla^{can})$$

has essential image precisely those

(E, ∇) s.t. $\psi_\nabla \equiv 0$.

"pf" $(E,\nabla) \rightsquigarrow E$ is a D_S-module

Morita theory:

$F_{S/k*}D_S$ is Azumaya over $\mathcal{O}_{T^*S'}$

$\Rightarrow$

Morita: $\mathrm{Coh}(S, D_S) \longrightarrow \mathrm{Coh}(T^*S', F_{S/k*}D_S)$

equivalence

$$\pi_*(\mathrm{Morita}(E)) \simeq F_{S/k*}E$$

in $\mathrm{Coh}(S')$

- ψ_∇ induces an action of

$$Z(F_{S/k*} D_S) \simeq \pi'_* \mathcal{O}_{T^*S'}$$

on $F_{S/k*} E$.

$\psi_\nabla \equiv 0 \iff$ Morita(E) supported on the zero-section $S' \hookrightarrow T^*S'$

$(\iota(\partial) = 0 \ \forall \partial \in T_S)$

- $F_{S/k*} D_S$ splits over zero-section $S' \hookrightarrow T^*S'$

(induced by $(0, d)$)

May use these two facts to construct $E'|_{S'}$ s.t.

$$E^{\nabla = 0} \xrightarrow{\sim} E'$$