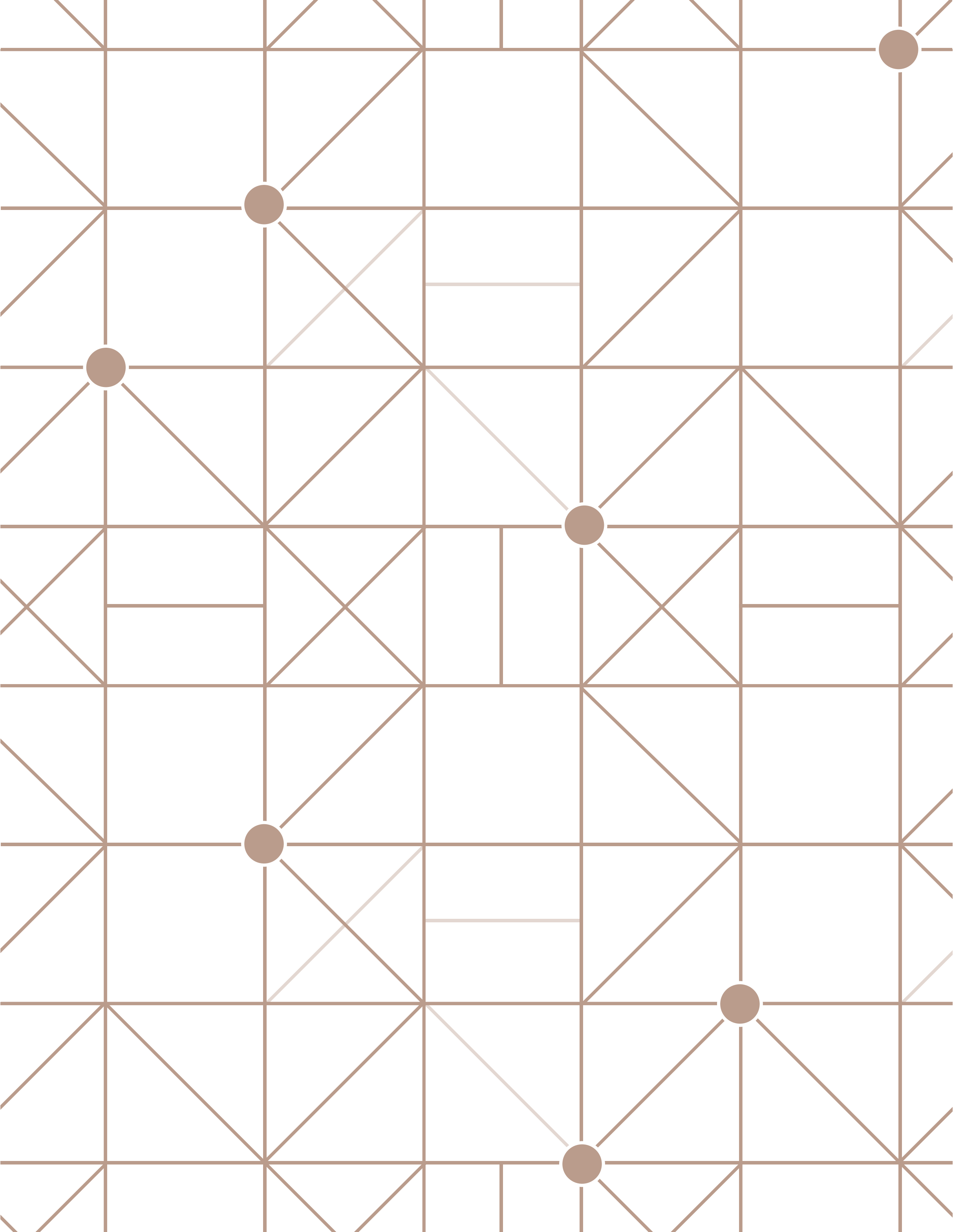

Recall: (C, D) n-pointed hyperbolic curve of genus g.
(E, ∇) flat v.b. on C with reg. sing. along D.

- universal isomonodromic deformation $(\mathcal{E}, \tilde{\nabla})$ of (E, ∇) on $\mathcal{C} \xrightarrow{\pi} \Delta = T_{g,n}$

- An isomonodromic deformation to a general nearby curve (E', ∇') of (E, ∇) is the restriction $(\mathcal{E}, \tilde{\nabla})|_{(C', D')}$ for an analytically general (C', D').

- Question (Biswas, Heu, Hurtubise):
Let (E', ∇') be an isomonodromic deformation to a general nearby curve of (E, ∇). Is E' semi-stable?

- Answer (L-L): In general not true.

- Cor (1.3.6) Notation as above. If $\mathrm{rk}(E) < 2\sqrt{g+1}$, then the isomonodromic deformation to a general nearby curve of E is s.s.

Today's Goal: Prove the previous result.

Thm (1.3.4)

Let (C, D), (E, ∇), $(\mathcal{E}, \tilde{\nabla})$ be as above, and E has irred. monodromy (i.e., $\rho_E : \pi_1(C \setminus D) \to GL_n(\mathbb{C})$ irred).

let (E', ∇') be an isom. deformation to a general nearby curve (C', D').

- $N_\bullet : 0 = N_0 \subseteq \cdots \subseteq N_n = E'$ the HN filtration on E'.
- $gr_i^N E' := N_i / N_{i-1}$, $\mu_i =$ slope of $gr_i^N E'$

Then if E' not s.s.,

1) $\forall i \neq 0, n$, $\exists$ $j < i < k$ s.t.

$$rk\, gr_{j+1}^N E' \cdot rk\, gr_k^N E' \geqslant g+1$$

2) $0 < \mu_i - \mu_{i+1} \leq 1$, $\forall$ $0 \leq i < n$.

- If (C, D) non-hyperbolic, i.e., $(g, n) = (0,0), (0,1), (0,2)$ or $(1,0)$, then $\pi_1(C \setminus D)$ abelian $\Rightarrow$ $rk E = 1$. the thm also holds.
- If E' has many non-zero graded pieces, then $rk E'$ is big.

Proof of Cor 1.3.6. using Thm 1.3.4.:

① If E has irred monodromy and E' not s.s,
then by theorem 1.3.4., $\exists\ 0 \leq j < i < k \leq n$ s.t.

$$g+1 \leq \operatorname{rk} gr^N_{j+1} E' \cdot \operatorname{rk} gr^N_k E'.$$

On the other hand,

$$\begin{aligned} &\operatorname{rk} gr^N_{j+1} E' \cdot \operatorname{rk} gr^N_k E' \\ \leq\ &\frac{1}{4}\left(\operatorname{rk} gr^N_{j+1} E' + \operatorname{rk} gr^N_k E'\right)^2 \\ \leq\ &\frac{1}{4} \operatorname{rk} E^2 \end{aligned}$$

$\Longrightarrow \quad \operatorname{rk}(E) \geq 2\sqrt{g+1}$, contradiction.

② If E not irred, use induction on the rk. Because ext. of s.s bundles are s.s. $\square$

It suffices to prove theorem 1.3.4.

- "If (E', ∇') does not satisfy 1) & 2) in thm 1.3.4, then H-N filtration cannot deform to a n.b.d. of (C', D') in $\Delta = T_{g,n}$"

Idea of the proof of thm 1.3.4. :

For $i<j$, we denote

$$\boxed{E_{i,j} = \underline{\mathrm{Hom}}(gr_i^N E', gr_j^N E')} \Rightarrow \mu(E_{i,j}) < 0$$

$\boxed{\text{Claim}}$: If E' not s.s., $\forall i$, $\exists\, j<i<k$ s.t. $E_{j+1,k}^\vee \otimes \omega_C$ is not G.G.G.

1) claim + not GGG lemma $\Longrightarrow$ $\forall i$, $\exists\, j<i<k$ s.t.

$$\mathrm{rk}\, gr_{j+1}^N E' \cdot \mathrm{rk}\, gr_k^N E' = \mathrm{rk}(E_{j+1,k}^\vee \otimes \omega_C) \geq g+1$$

$$\left(\mu(E_{j+1,k}^\vee \otimes \omega_C) = -\mu(E_{j+1,k}) + \mu(\omega_C) > 2g-2 \right)$$

2) $E_{j+1,k}^\vee \otimes \omega_C$ not GGG

$$\Rightarrow 2g-2 < \mu(E_{j+1,k}^\vee \otimes \omega_C) \leq 2g-1$$

(lem 5.1.2. $\mu(V) > 2g-1 \Rightarrow V$ globally generated)

$$\Rightarrow -1 < \mu(gr_k^N E') - \mu(gr_{j+1}^N E') \leq 0$$

$$\Longrightarrow -1 \leq \mu(gr_k^N E') - \mu(gr_{j+1}^N E')$$

(slopes of gr_i are decreasing)

$$\leq \underbrace{\mu(gr_{i+1}^N E')}_{\mu_{i+1}} - \underbrace{\mu(gr_i^N E')}_{\mu_i} < 0$$

Setting for the proof of thm 1.3.4.

- (C, D), (E, ∇), E has irred monodromy.
- A nontrivial filtration $N_\bullet$
$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = E$$

which extends to a filtration on $(\mathcal{E}, \tilde{\nabla})$ to a 1st order n.b.d. of (C, D). (take N = Harder-Narasimhan filtration in the proof)

- $N_\cdot$ induces a filtration on $\underline{\mathrm{End}}(E)$

with $N_p \underline{\mathrm{End}}(E) = \bigoplus_{j-i \leq p} \underline{\mathrm{Hom}}(N_i, N_j)$,

and $N_0 \mathrm{End}(E) = \mathrm{End}_{N_\cdot}(E)$

$\rightsquigarrow$ filtration on $\mathrm{End}(E)/\mathrm{End}_{N_\cdot}(E)$

$$\Rightarrow \bigoplus_p \mathrm{gr}_p^N \mathrm{End}(E)/\mathrm{End}_{N_\cdot}(E) = \bigoplus_{1 \leq i < j \leq n} \underline{\mathrm{Hom}}(\mathrm{gr}_i^N E, \mathrm{gr}_j^N E)$$

- $(E, \nabla) \rightsquigarrow$ a non-zero map (prop 2.1.8)

$$T_C(-D) \xrightarrow{q^\nabla} \mathrm{At}_{(C,D)}(E) \to \underline{\mathrm{End}}(E)/\underline{\mathrm{End}}_{N_\cdot}(E)$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{End}_{N_\cdot}(E) \arrow[r] \arrow[d, hook] & \mathrm{At}_{(C,D)}(E, N_\cdot) \arrow[r] \arrow[d, hook] & T_C(-D) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{End}(E) \arrow[r] \arrow[d] & \mathrm{At}_{(C,D)}(E) \arrow[r] \arrow[d] & T_C(-D) \arrow[r] \arrow[l, bend left, "q^\nabla"] & 0 \\
& \mathrm{End}(E)/\mathrm{End}_{N_\cdot}(E) \arrow[r, "\sim"] & \mathrm{At}_{(C,D)}(E)/\mathrm{At}_{(C,D)}(E, N_\cdot) & &
\end{tikzcd}$$

- $H^1(C, T_C(-D)) \longleftrightarrow \mathrm{Def}_{(C,D)}(k[\varepsilon]/\varepsilon^2)$

  $H^1(C, At_{(C,D)}(E,P^\bullet)) \longleftrightarrow \mathrm{Def}_{(C,D,E,\nabla,P^\bullet)}(k[\varepsilon]/\varepsilon^2)$

lemma 1: the induced map

$$H^1(C, T_C(-D)) \xrightarrow{q^\nabla_*} H^1(C, At_{(C,D)}(E)) \to H^1(C, \mathrm{End}(E)/\mathrm{End}_{N_\cdot}(E))$$

is identically 0.

Pf: $\forall\ s \in H^1(C, T_C(-D))$, i.e., a 1st order deformation $(\mathcal{C}, \mathcal{D})$ of (C,D), $q^\nabla(s)$ corresponds to $(\mathcal{C}, \mathcal{D}, \mathcal{E})$.

By the assumption, $N_\cdot$ extends to $\mathcal{E}$. By lemma 2.3.8

$\Rightarrow q^\nabla(s) \in \ker\left(H^1(C, At_{(C,D)}(E)) \to H^1(C, \mathrm{End}(E)/\mathrm{End}_{N_\cdot}(E))\right)$

- $q^\nabla(s) \in H^1(C, At_{(C,D)}(E,N^\bullet))$
- $H^1(C, At_{(C,D)}(E,N^\bullet)) \to H^1(C, At_{(C,D)}(E)) \to H^1(C, \mathrm{End}(E)/\mathrm{End}_{N_\cdot}(E))$

  long exact sequence induced from

  $0 \to At_{(C,D)}(E,N^\bullet) \to At_{(C,D)}(E) \to \mathrm{End}(E)/\mathrm{End}_{N_\cdot}(E) \to 0$

$\Rightarrow$ the composition is 0. $\square$

• lemma 2. $\forall\ 0 < i < n$, $\exists\ j < i < k$ s.t

$$T_C(-D) \to \underline{\mathrm{End}}(E)/\underline{\mathrm{End}}_{N.}(E)$$

induces a nonzero map

$$\phi_{j+1,k} : T_C(-D) \to E_{j+1,k}.$$

Proof: $$T_C(-D) \longrightarrow \underline{\mathrm{End}}(E)/\underline{\mathrm{End}}_{N.}(E)$$

is non-zero (E irred monochromy, Prop 2.1.8.)

* j = maximal m s.t. $\nabla(N_j) \subseteq N_i \otimes \Omega^1_C(\log D)$

$\Rightarrow$ $j < i$ (E has irred. monochromy)

* k = minimal m s.t. $\nabla(N_{j+1}) \subseteq N_k \otimes \Omega^1_C(\log D)$

$\Rightarrow$ $i+1 \leq k$ (the choice of j)

$$\Rightarrow\ N_{j+1}/N_j \to N_k/N_i \otimes \Omega^1_C(\log D)$$

$$\to N_k/N_{k-1} \otimes \Omega^1_C(\log D)$$

is a non-zero $\mathcal{O}_C$-linear map.

$$\Longrightarrow \quad \phi'_{j+1,k} \quad T_C(-D) \longrightarrow \underline{\mathrm{Hom}}(gr^N_{j+1}, gr^N_k) = E'_{j+1,k}$$

non-zero $\square$

• lemma 3: $N^\bullet = HN$, fix i, let j, k be as in lemma 2. $\phi_{j+1,k}$ non-zero,

then $\phi_{j+1,k}$ induces an identically 0 map

$$H^1(C, T_C(-D)) \xrightarrow{(\phi_{j+1,k})_*} H^1(C, E_{j+1,k}).$$

proof: 3 steps, diagram chasing. We show that the maps ① ② ③ induces the 0 map on H^1.

$\nabla N_j \subseteq N_j \otimes \Omega^1(D)$

$\nabla N_{j+1} \subseteq N_k \otimes \Omega^1(D)$

$$\begin{tikzcd}
T_C(-D) \arrow[r] \arrow[rr, bend left, "①"] \arrow[drr, "②"] \arrow[ddrr, "③"'] & \mathrm{End}(E)/\mathrm{End}_{N}(E) \arrow[r, two heads] & \underline{\mathrm{Hom}}(N_{j+1}, E/N_{k-1}) \\
 & & \underline{\mathrm{Hom}}(gr^N_{j+1} E, E/N_{k-1}) \arrow[u] \\
 & & \underline{\mathrm{Hom}}(gr^N_{j+1} E, gr^N_k E) = E_{j+1,k} \arrow[u]
\end{tikzcd}$$

Detail of proof for lemma 3.

① $\underline{End}(E) \twoheadrightarrow \underline{Hom}(N_{j+1}, E)$ natural map (sheaf surj.)

$\rightsquigarrow$ $\underline{End}(E)/\underline{End}_{N.}(E) \twoheadrightarrow \underline{Hom}(N_{j+1}, E/N_{k-1})$

$\rightsquigarrow$ By lemma 1, $T_C(-D) \to \underline{End}(E)/\underline{End}_{N.}(E) \twoheadrightarrow \underline{Hom}(N_{j+1}, E/N_{k-1})$
induces the 0 map on H^1

② the short exact sequence

$$0 \to N_j \to N_{j+1} \to gr^N_{j+1}E \to 0$$

$\rightsquigarrow$

$$0 \to \underline{Hom}(gr^N_{j+1}E, E/N_{k-1}) \to \underline{Hom}(N_{j+1}, E/N_{k-1}) \to \underline{Hom}(N_j, E/N_{k-1}) \to 0$$

long. e.s.

$\rightsquigarrow$ $\underbrace{H^0(C, \underline{Hom}(N_j, E/N_{k-1}))}_{\substack{\parallel \\ 0}} \to H^1(C, \underline{Hom}(gr^N_{j+1}E, E/N_{k-1}))$

$\hookrightarrow H^1(C, \underline{Hom}(N_{j+1}, E/N_{k-1}))$

$\mu(\underline{Hom}(N_j, E/N_{k-1})) < 0$

Since $T_C(-D) \to \underline{Hom}(gr^N_{j+1}E, E/N_{k-1}) \to \underline{Hom}(N_{j+1}, E/N_{k-1})$
induce the 0 map on H^1 (by step ①)

$$T_C(-D) \to \underline{Hom}(gr^N_{j+1}E, E/N_{k-1})$$

induces the 0 map on H^1 too.

③ the same as ②.

$$0 \to gr^N_k E \to E/N_{k-1} \to E/N_k \to 0$$

$$\rightsquigarrow \quad 0 \to \underline{Hom}(\underbrace{gr^N_{j+1}, gr^N_k E}_{=E_{j+1,k}}) \to \underline{Hom}(gr^N_{j+1}E, E/N_{k-1}) \to \underline{Hom}(gr^N_{j+1}E, E/N_k) \to 0$$

long e.s. $\rightsquigarrow \underbrace{H^0(C, \underline{Hom}(gr^N_{j+1}E, E/N_k))}_{=0} \to H^1(C, E_{j+1,k})$

$\mu(\underline{Hom}(gr^N_{j+1}E), E/N_k) < 0$ $\quad \hookrightarrow H^1(C, \underline{Hom}(gr^N_{j+1}E, E/N_{k-1}))$

Since $T_C(-D) \xrightarrow{\phi_{j+1,k}} E_{j+1,k} \to \underline{Hom}(gr^N_{j+1}E, E/N_{k-1})$

induces the 0 map on H^1 (by step ②),

$\phi_{j+1,k}$ also induces the 0 map on H^1 $\square$

- lemma 4: i, j, k, $\phi_{j+1,k}$ as above. then

$$V_{j,k} := E_{j+1,k}^{\vee} \otimes \omega_C \text{ is not G. G. G.}$$

proof: $\phi_{j+1,k}: T_C(-D) \longrightarrow E_{j+1,k}$ is non-zero

Serre Duality

$\Longrightarrow$ $E_{j+1,k}^{\vee} \otimes \omega_C \xrightarrow{\varphi} \omega_C^{\otimes 2}(D)$ is non-zero.

and φ induces the 0 map

$$\boxed{H^0(C, E_{j+1,k}^{\vee} \otimes \omega_C) \xrightarrow{0} H^0(C, \omega_C^{\otimes 2}(D))}$$

$\Longrightarrow$

$$\begin{tikzcd} H^0(C, E_{j+1,k}^{\vee} \otimes \omega_C) \otimes \mathcal{O}_C \arrow[r, "0"] \arrow[d, "e.v."] & H^0(C, \omega_C^{\otimes 2}(D)) \otimes \mathcal{O}_C \arrow[d] \\ E_{j+1,k}^{\vee} \otimes \omega_C \arrow[r, "\varphi", "\text{non-zero}"'] & \omega_C^{\otimes 2}(D) \end{tikzcd}$$

$\Longrightarrow$ e.v. factors through $\ker(\varphi) \subsetneq E_{j+1,k}^{\vee} \otimes \omega_C$
(corank ≥ 1)

$\Longrightarrow$ $E_{j+1,k}^{\vee} \otimes \omega_C$ is not G.G.G.. $\square$

Proof of Thm 1.3.4. Assume that E' not s.s.

* the locus of non s.s. fibers of $(\mathcal{E}, \nabla)$ is a closed analytic subset of $\Delta = T_{g,n}$
* A general fiber E' is assumed to be not s.s.

$\Longrightarrow$ each fiber of $(\mathcal{E}, \tilde{\nabla})$ is not s.s.

+ $N_\bullet$ extends to an open analytical subset of Δ containing (C', D').
$\overset{\shortparallel}{HN}$

So we can assume that <u>$N_\bullet$ extends to 1st order neighborhood of C'.</u>

(E', ∇') satisfies the conditions for lemmas 1 2 3 4.

$\xrightarrow{\text{lem } 1,2,3,4}$ $\forall\, i,\ \exists\ j < i < k$ s.t.

$$E_{j+1,k}^{\vee} \otimes \omega_{C'} \text{ is not GG}$$

$\square$