1. Solve the second-order linear differential equations

\[ y'' = y \quad \text{and} \quad y'' = -y \]

using a series method.

The game here is to assume our differential equation has a solution \( f(x) \) which has a power series expansion:

\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{with some positive radius of convergence.} \]

Note. There is no a priori reason for this to be the case, that is, there is no reason to believe that our equation can be solved by something with a power series expansion. But let's play along and see what happens...

So say \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) solves the equation \( y'' = y \)

i.e. \( f''(x) = f(x) \) as function of \( x \).
Putting in the power series for $f(x)$ on both sides, this gives:

$$
\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{n+2} c_{n+2} x^n = f''(x) = f''(x) = \sum_{n=0}^{\infty} c_n x^n
$$

( differentiate power series term-by-term )

Now two power series are the same only when the coefficients of $x^n$ coincide for all $n$. So comparing the coefficients of $x^n$ on both sides above, we obtain the relationship:

$$
(n+2)(n+1) c_{n+2} = c_n \quad \text{for all } n \geq 0
$$

$$
\iff \quad c_{n+2} = \frac{c_n}{(n+2)(n+1)} \quad \text{for all } n \geq 0.
$$

Since this applies for all $n \geq 0$, we could iteratively apply this relation and relate $c_n$ with $c_{n-2}$, $c_{n-4}$, all the way until we cannot drop the subscript by 2, i.e.

either $c_1$ or $c_0$, depending on whether $n$ is even or odd.
That is:

\[
C_n = \frac{C_{n-2}}{n(n-1)} = \frac{C_{n-4}}{n(n-1)(n-2)(n-3)} = \ldots = \left\{ \begin{array}{ll}
\frac{C_1}{n!} & \text{if } n \text{ is odd} \\
\frac{C_0}{n!} & \text{if } n \text{ is even}
\end{array} \right.
\]

Thus all the coefficients \( C_n \) of the power series are related to either \( c_0 \) or \( c_1 \). Let's use this relationship to simplify the expression for \( f(x) \):

\[
f(x) = \sum_{n=0}^{\infty} C_n x^n = \sum_{m=0}^{\infty} C_{2m} x^{2m} + \sum_{m=0}^{\infty} C_{2m+1} x^{2m+1}
\]

\[
= \sum_{m=0}^{\infty} \frac{C_0}{(2m)!} x^{2m} + \sum_{m=0}^{\infty} \frac{C_1}{(2m+1)!} x^{2m+1}
\]

\[
= c_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} + c_1 \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}
\]

\[
= c_0 \cosh(x) + c_1 \sinh(x)
\]

where the functions \( \cosh(x) \) and \( \sinh(x) \) are those defined by

\[
\cosh(x) := \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}, \quad \sinh(x) := \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}
\]
A direct computation shows that these functions are related to the exponential function by:

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \& \quad \sinh(x) = \frac{e^x - e^{-x}}{2}. \]

This allows us to express \( f(x) \) in terms of \( e^x \) and \( e^{-x} \):

\[
\begin{align*}
    f(x) &= e_0 \cosh(x) + c_1 \sinh(x) \\
    &= e_0 \left( \frac{e^x + e^{-x}}{2} \right) + c_1 \left( \frac{e^x - e^{-x}}{2} \right) \\
    &= \left( \frac{e_0 + c_1}{2} \right) e^x + \left( \frac{e_0 - c_1}{2} \right) e^{-x} \\
    &= b_0 e^x + b_1 e^{-x} \quad \text{where} \quad b_0 := \frac{e_0 + c_1}{2} \quad b_1 := \frac{e_0 - c_1}{2}
\end{align*}
\]

Summary: The general solution to \( y'' = y \) is given by

\[
    f(x) = e_0 \cosh(x) + c_1 \sinh(x) \\
    = b_0 e^x + b_1 e^{-x}
\]

for arbitrary constants \( e_0, c_1 \) or \( b_0, b_1 \).
but's do the same thing with the equation \( y'' = -y \):

\[
\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{(n+1)} c_{n+1} x^n = y'' = -y = \sum_{n=0}^{\infty} -c_n x^n.
\]

So comparing the coefficients of \( x^n \) on both sides, we get the relation:

\[
c_n = (-1)^n \cdot \frac{c_{n-1}}{2n(n-1)} = (-1)^n \frac{c_{n-4}}{n(n-1)(n-2)(n-3)} = \ldots = \begin{cases} (-1)^{\frac{n}{2}} \frac{c_1}{(2m+1)!} & \text{if } n = 2m+1 \\
(-1)^m \frac{c_0}{(2m)!} & \text{if } n = 2m \\
& \text{even} \end{cases}
\]

Thus:

\[
f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} c_{2m} x^{2m} + \sum_{m=0}^{\infty} c_{2m+1} x^{2m+1}
\]

\[
= c_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} + c_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}
\]

\[
= c_0 \cos(x) + c_1 \sin(x)
\]

**Summary.** The general solution to \( y'' = -y \) is given by

\[
f(x) = c_0 \cos(x) + c_1 \sin(x)
\]

for arbitrary constants \( c_0 \) and \( c_1 \).
Time to find areas between curves:

\[ y = g(x) = x \]

\[ \text{lower curve} \]

\[ y = f(x) = \sin(x) \]

\[ \text{upper curve} \]

\[ A = \int_{\pi/4}^{\pi} (x - \sin(x)) \, dx \]

\[ = \left[ \frac{1}{2} x^2 + \cos(x) \right]_{\pi/4}^{\pi} \]

\[ = \left( \frac{1}{2} \pi^2 - 1 \right) - \left( \frac{1}{8} \pi^2 + 0 \right) \]

\[ A = \frac{3}{8} \pi^2 - 1. \]
\[ A = \int_{0}^{2} (4x - x^4) - x^2 \, dx \]

\[ = \int_{0}^{2} 4x - 2x^4 \, dx \]

\[ = \left[ 2x^2 - \frac{2}{3}x^5 \right]_{0}^{2} = 8 - \frac{16}{3} = \frac{8}{3}. \]

\[ \Rightarrow \quad A = \frac{8}{3}. \]
\[ A = \int_0^\infty \left| \frac{x}{1+x^2} - \frac{x^2}{1+x^2} \right| \, dx = \int_0^1 \frac{x}{1+x^2} - \frac{x^2}{1+x^2} \, dx + \int_1^\infty \frac{x^2}{1+x^2} - \frac{x}{1+x^2} \, dx \\
= \left( \frac{1}{2} \int_1^2 \frac{du}{u} - \frac{1}{3} \int_1^2 \frac{dv}{v} \right) + \lim_{t \to \infty} \left( \frac{1}{3} \int_2^{1+t^3} \frac{dv}{v} - \frac{1}{2} \int_2^{1+t^3} \frac{du}{u} \right) \\
= \left( \frac{1}{2} \log(2) - \frac{1}{3} \log(2) \right) + \lim_{t \to \infty} \frac{1}{3} \log(1+t^3) - \frac{1}{3} \log(2) \\
- \frac{1}{2} \log(1+t^2) + \frac{1}{2} \log(2) \\
= \frac{1}{3} \log(2) + \lim_{t \to \infty} \log \left( \frac{(1+t^3)^{\frac{1}{3}}}{(1+t^2)^{\frac{1}{2}}} \right) \\
= \frac{1}{3} \log(2) + \lim_{t \to \infty} \log \left( \lim_{t \to \infty} \frac{(1+t^3)^{\frac{1}{3}}}{(1+t^2)^{\frac{1}{2}}} \right) \\
= \frac{1}{3} \log(2) + \log(1) \Rightarrow A = \frac{1}{3} \log(2) \]
\[ A = \int_{0}^{1} x - \frac{x}{4} \, dx + \int_{1}^{2} \frac{1}{x} - \frac{x}{4} \, dx \]

\[ = \left[ \frac{1}{2} x^2 \right]_{0}^{1} + \left[ \log(x) \right]_{1}^{2} - \left[ \frac{1}{8} x^2 \right]_{0}^{2} \]

\[ = \frac{1}{2} + \log(2) - \frac{1}{2} \]

\[ \Rightarrow A = \log(2) \]
3. Now for some solids of revolution.

(i)

\[ V = \int_0^2 \pi (x+1)^2 \, dx = \pi \int_0^2 x^2 + 2x + 1 \, dx \]

\[ = \pi \left[ \frac{1}{3} x^3 + x^2 + x \right]_0^2 = \pi \left( \frac{8}{3} + 4 + 2 \right) \]

\[ \Rightarrow V = \frac{26}{3} \pi. \]
\[ V = \int_{-1}^{1} \pi e^{2x} \, dx \]

\[ = \frac{\pi}{2} \left[ e^{2x} \right]_{-1} = \frac{\pi}{2} (e^2 - e^{-2}) \]

\[ \Rightarrow V = \left( \frac{e^2 - e^{-2}}{2} \right) \pi \]

\[ = \pi \sinh(2) \]
\[ V = \int_0^1 \pi \cdot \left( (1-x^2)^2 - (1-x^\frac{4}{3})^2 \right) \, dx \]

\[ = \pi \int_0^1 \left( 1 - 2x^2 + x^4 \right) - \left( 1 - 2x^\frac{4}{3} + x \right) \, dx \]

\[ = \pi \int_0^1 x^4 - 2x^2 - x + 2x^\frac{4}{3} \, dx \]

\[ = \pi \left( \frac{1}{5} - \frac{2}{3} - \frac{1}{2} + 2 \cdot \frac{2}{3} \right) = \pi \left( \frac{11}{30} \right) \]

\[ \Rightarrow V = \frac{11}{30} \pi. \]
\[ V = \int_{0}^{1} \pi \left( (2 - y^{1/3})^2 - 1^2 \right) \, dy \]

\[ = \pi \int_{0}^{1} y^{2/3} - 4y^{1/3} + 3 \, dy \]

\[ = \pi \left[ \frac{3}{5} - 4 \cdot \frac{3}{4} + 3 \right] \]

\[ = \frac{3}{5} \pi \]

\[ \Rightarrow V = \frac{3}{5} \pi \]