1. Compute the following trigonometric integrals:
   
   (i) \( \int \sin(\theta)^2 \cos(\theta)^3 \, d\theta \),
   
   (ii) \( \int \sin(\sqrt{x}) \, dx \),
   
   (iii) \( \int_{\pi/2}^{0} \sin(t)^2 \cos(t)^2 \, dt \),
   
   (iv) \( \int \tan(y)^2 \, dy \),
   
   (v) \( \int \tan(z)^3 \sec(z) \, dz \),
   
   (vi) \( \int \sin(8u) \cos(5u) \, du \).

2. In this Problem, we are going to compute the following relations: for positive integers \( n \) and \( m \),

   \[
   \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = \begin{cases} 
   0 & \text{if } m \neq n, \\
   \pi & \text{if } m = n,
   \end{cases}
   \]

   \[
   \int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \begin{cases} 
   0 & \text{if } m \neq n, \\
   \pi & \text{if } m = n,
   \end{cases}
   \]

   \[
   \int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx = 0.
   \]

   To do this, use the prosthaphaeresis formulae,

   \[
   \sin(A) \sin(B) = \frac{1}{2} \left( \cos(A - B) - \cos(A + B) \right),
   \]

   \[
   \cos(A) \cos(B) = \frac{1}{2} \left( \cos(A - B) + \cos(A + B) \right),
   \]

   \[
   \sin(A) \cos(B) = \frac{1}{2} \left( \sin(A - B) + \sin(A + B) \right),
   \]

to express the integrands as sums of individual trigonometric functions. Then show that the integrands you get end up being even or odd functions, depending on whether you have \( n \neq m \) or \( n = m \). If it is helpful to you, feel free to choose specific positive integers \( n \) and \( m \) representing the cases above in doing this computation.

   These relationships are effectively the starting point to Fourier analysis; these give you ways to tease out waves of a particular frequency in some given periodic signal!

3. Use the trigonometric substitution \( x = 3 \sin(\theta) \) to evaluate

   \[
   \int_{0}^{1} x^2 \sqrt{9 - x^2} \, dx.
   \]

   Be careful about the bounds of integration once you do your substitution: what must \( \theta \) be when \( x = 0 \) or \( x = 1 \)?

4. Sometimes you are going to have to do some manipulations before being able to perform a trigonometric substitution. Here is an example:

   (i) Write the polynomial \( 3 - 2x - x^2 \) in the form \( a - (x + b)^2 \), for some numbers \( a \) and \( b \), by completing the square.
(ii) Do the substitution $u = x + b$ followed by a trigonometric substitution to evaluate the integral

$$\int \sqrt{3 - 2x - x^2} \, dx.$$

5. Given a circle of radius $a$, its circumference is $2\pi a$ and its area is $\pi a^2$.

(i) Compute the integral $\int_0^a 2\pi r \, dr$.

(ii) Thinking about polar coordinates, try to explain how the computation in (i) is a way of computing the area of a circle of radius $r$.

As an analogy, it might be helpful to think about how the integral

$$\int_0^1 x \, dx$$

computes the area of the right triangle

where the vertices are at $(0,0)$, $(0,1)$, and $(1,1)$. 