Let's do some practice for the midterm. Most of this amounts to practice with the techniques we have developed over the last month.

1. Evaluate the integrals
   
   (i) \( \int e^{\arctan(y)} \frac{dy}{1 + y^2} \),
   
   (ii) \( \int_0^\pi t \cos(t) \, dt \),
   
   (iii) \( \int_0^1 (1 + \sqrt{x})^8 \, dx \),
   
   (iv) \( \int \sqrt{u} e^{\sqrt{u}} \, du \),
   
   (v) \( \int \frac{\sec(\theta)^2}{\tan(\theta)^2 - 16} \, d\theta \),
   
   (vi) \( \int \phi \tan(\phi)^2 \, d\phi \),
   
   (vii) \( \int x \arctan(x) \, dx \),
   
   (viii) \( \int \frac{\sin(\psi) \cos(\psi)}{\sin(\psi)^4 + \cos(\psi)^4} \, d\psi \).

2. Compute
   
   \( \int_0^{\pi/2} \sin(x)^n \, dx \)
   
   for \( n = 0, 1, 2, 3 \). Can you guess what the value of the integral might be for arbitrary \( n \)?

3. Evaluate the integral
   
   \( \int \frac{x^2 + 1}{(x^2 - 2x + 2)^2} \, dx \)
   
   by completing the square in the denominator and doing a trigonometric substitution.

   Okay, I think if you can do all the integrals above, then you are in more than good shape for the midterm. The remainder of this set is culture in something that I think is pretty cool; I hope you have some fun with it!

4. Let's think about the calculations that we performed in Assignment 2, Question 2. A function of the form
   
   \( f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_N \sin(Nx) \)
   
   is called a \textit{finite Fourier series}. Show that
   
   \( a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx \).

   Moreover, verify that
   
   \( \int_{-\pi}^{\pi} f(x) \cos(mx) \, dx = 0 \)
   
   for all positive integers \( m \). This gives a way of extracting the coefficients \( a_i \) in a function of this form!

5. The calculation in 3. suggests a way to approximate certain functions. Consider the function
   
   \( f(x) := \begin{cases} 
   -1 & \text{if } -\pi \leq x < 0, \\
   1 & \text{if } 0 \leq x \leq \pi.
   \end{cases} \)
(i) Calculate

\[ a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx \]

for \( m = 1, 2, 3, 4, 5, 6, 7, 8 \).

(ii) Let

\[ f_i(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_{2i-1} \sin((2i-1)x) + a_{2i} \sin(2ix). \]

With the help of a computer, graph the functions \( f_1, f_2, f_3, \) and \( f_4 \) in the interval \([-\pi, \pi]\). Also draw the graph of \( f \).