A GALOIS–DYNAMICS CORRESPONDENCE FOR
UNICRITICAL POLYNOMIALS

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Abstract. For any nonlinear polynomial \( \phi \) with rational coefficients such that its \( N \)-th dynatomic polynomial \( \Phi_N \) is irreducible, the action of a subgroup of the Galois group of \( \Phi_N \) on the periodic points of \( \phi \) of exact period \( N \) coincides with the action of \( \phi \) by iteration. For unicritical polynomials \( \phi_{d,c}(z) := z^d + c \) with rational coefficients, this coincidence of actions reduces finding periodic points over \( K \) to finding rational points of dynatomic curves over \( \mathbb{Q} \). This coincidence always occurs if \( c \) is not contained in a density zero exceptional subset \( \Sigma_{d,N} \) of the rational numbers that can be explicitly described for quadratic polynomials and small \( N \). For any quadratic number field \( K \), this is sufficient to obtain the non-existence of 5-cycles in \( K \) for all but finitely many rational \( c \) and the complete determination of all 6-cycles in \( K \) for all but finitely many rational \( c \).

1. Introduction

Let \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map over a field \( k \) and let \( \phi^N \) denote the \( N \)-th iterate of \( \phi \), so \( \phi^N := \phi \circ \phi^{N-1} \). Let \( \text{Per}_{N,K}(\phi) \) denote the periodic points of \( \phi \) of exact period \( N \) in any extension \( K/k \). In order to understand how the dynamics of \( \phi \) interact with the action of Galois groups, we introduce the following relation.

Definition 1.1. The rational map \( \phi \) satisfies the Galois–dynamics correspondence (GDC) for period \( N \) and nontrivial Galois extension \( K/k \) if for every periodic point \( z \in \text{Per}_{N,K-k}(\phi) \), there is a positive integer \( i < N \) and a nontrivial \( \sigma \in \text{Gal}(K/k) \) such that \( \phi^i(z) = \sigma z \).

Remark 1.2. For convenience, we may write that \( (\phi, N, K/k) \) satisfies the Galois–dynamics correspondence. If we wish to specify a particular periodic point \( z \) then we may say that \( (\phi, N, K/k, z) \) satisfies the Galois–dynamics correspondence.

The Galois–dynamics correspondence describes a relation between the dynamical action of \( \phi \) and the Galois action of \( \text{Gal}(K/k) \). Let \( T_z \) be the rooted \( d \)-ary tree obtained from the backwards orbit of a point \( z \in K \) under \( \phi \). In terms of the arboreal Galois representation associated to \( \phi \) and a point \( z \in K \),

\[ \rho_{\phi,z} : \text{Gal}(K/k) \to \text{Aut}(T_z), \]

the Galois–dynamics correspondence specifies that the image of \( \rho_{\phi,z} \) contains a nontrivial subgroup of \( \mathbb{Z}/N\mathbb{Z} \) for every periodic point \( z \) of \( \phi \) of exact period \( N \).

This definition is closely related to the automorphism polynomial defined by Morton–Patel [MP94] for polynomials \( \phi \in k[z] \). If the dynatomic polynomial \( \Phi_N(z) \) is irreducible over \( k \) with splitting field \( K \) and \( \phi \) is an automorphism polynomial.

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of $\Phi_N(z)$ for a fixed positive integer $N$, then $\phi$ satisfies the Galois–dynamics correspondence for $K$ and $N$ with $i = 1$ for each periodic point $z$.

If $\phi$ is a polynomial with coefficients in $k$, then its iterative action commutes with the action of $\text{Gal}(K/k)$. In this setting, $\phi$ satisfies the Galois–dynamics correspondence for $N$, $K$, and $z$ if and only if the $N$-cycle of $z$ is not disjoint from its conjugates by some element of the Galois group, i.e.

$$\{z, \ldots, z^{N-1}\} \cap \{\sigma z, \ldots, \sigma z^{N-1}\} \neq \emptyset,$$

for some nontrivial $\sigma \in \text{Gal}(K/Q)$.

The Galois–dynamics correspondence does not hold in general. Even if we restrict our attention to unicritical polynomials, which can be written as $\phi_{d,c}(z) := z^d + c$, there are instances in which the Galois–dynamics correspondence might not be satisfied.

**Example 1.3.** If $K$ is the splitting field of $\phi_{2,t}^3(z) - z$ over the rational function field $C(t)$, then the directed graph of periodic points of $\phi_{2,t}$ of exact period $N = 3$ consist of two disjoint 3-cycles. A result of Bousch [Bou92, Chapter 3, Theorem 3] implies that the Galois group $\text{Gal}(K/C(t))$ properly contains the full automorphism group of this graph and, in particular, contains an element $\sigma$ of order 2 that interchanges the two 3-cycles. If $k$ is the fixed field $K^\sigma$, then $[K : k] = 2$. For any $z \in K$ of exact period 3, the Galois($K/k$)-orbit of $z$ is $\{z, \sigma z\}$ so $z \in K - k$, but $\sigma z$ is not in the forward orbit of $z$ with respect to $\phi_{2,t}$.

**Remark 1.4.** The Galois–dynamics correspondence only needs to be considered for intermediate Galois extensions of the splitting field of the dynatomic polynomial $\Phi_N$ because each periodic point of $\phi$ in the algebraic closure $\overline{k}$ is contained in the splitting field of $\Phi_N$. Furthermore, if the Galois–dynamics correspondence is true over a given Galois extension then it is also true for all of its intermediate Galois extensions.

For the remainder of this paper, we will assume that $k = Q$. The aim of this paper is to characterize the occurrence of the Galois–dynamics correspondence for unicritical polynomials $\phi_{d,c} \in Q[z]$ of arbitrary degree $d \geq 2$. We first relate the Galois–dynamics correspondence to the rationality of periodic points in Section 2.

**Theorem 1.5.** Let $c$ be a rational number and let $N$ be an integer greater than 1. If $\phi_{d,c}$ satisfies the Galois–dynamics correspondence for $N$ and a quadratic number field $K$, then:

1. For any $N$-cycle $\{z_0, \ldots, z_{N-1}\}$ of $\phi_{d,c}$ in $K - Q$, its trace $\sum z_i$ is rational;

2. Furthermore, each $N$-periodic point $z_i$ is rational if $N$ is odd.

Let $C_1(N)$ denote the dynatomic modular curve parametrizing periodic points of $\phi_{d,c}$ of exact period $N$ and let $C_0(N)$ denote the dynatomic modular curve parametrizing $N$-cycles. Then Theorem 1.5 can be restated as saying that the fiber $\pi^{-1}_{c}(\mathbb{P}^1(K))$ of the projection $\pi_c : C_0(N) \to \mathbb{P}^1$ is contained in $C_0(N)(Q)$. Theorem 1.5 reduces the problem of finding quadratic periodic points of $\phi_{d,c}$, which are a priori correspond to elements of $C_1(N)(K)$, to finding rational points in $C_0(N)(Q)$ for all $N$ and furthermore to finding rational points in $C_1(N)(Q)$ for odd $N$. For $d = 2$ and $N = 5$ or 6, this implies the following results.
Corollary 1.6. Let $N = 5$. For quadratic number fields $K$ and rational numbers $c$ such that $\phi_{2,c}$ satisfies the Galois–dynamics correspondence, there are no periodic points of $\phi_{2,c}$ of exact period $N$ in $K$.

Corollary 1.7. Let $N = 6$ and let $J$ be the Jacobian of $C_6(6)$. Suppose that the $L$-series $L(J,s)$ extends to an entire function, $L(J,s)$ satisfies the standard functional equation, and the weak Birch and Swinnerton-Dyer conjecture is valid for $J$.

For quadratic number fields $K$ and rational numbers $c$ such that $\phi_{2,c}$ satisfies the Galois–dynamics correspondence, there are no periodic points of $\phi_{2,c}$ of exact period $6$ that are defined over $K$ unless $K = \mathbb{Q}(\sqrt{33})$ and $c = -\frac{11}{33}$, in which case there is exactly one 6-cycle:

$$z_0 = -1 + \frac{\sqrt{33}}{12}, z_1 = -\frac{1}{4} - \frac{\sqrt{33}}{6}, z_2 = -\frac{1}{2} + \frac{\sqrt{33}}{12}, z_{i+3} = z_i.$$

In Section 3 the Galois–dynamics correspondence is connected to the irreducibility and Galois groups of the dynatomic polynomial $\Phi_N$. The initial work of Vivaldi–Hatjispyros showed that the action of the Galois group of $\Phi_N(z)$ over $\mathbb{Q}$ on periodic points of any nonlinear polynomial $\phi_d$ mimics the dynamical action of iteration of $\phi$ if $\Phi_N(z)$ is irreducible in $\mathbb{Q}[z]$.

Proposition 1.8. Let $d$ and $N$ be integers greater than 1 and let $c$ be a rational number. If the dynatomic polynomial $\Phi_N(z,c)$ is irreducible in $\mathbb{Q}[z]$ then $\phi$ satisfies the Galois–dynamics correspondence for $N$ and all Galois extensions $K/\mathbb{Q}$.

The dynatomic polynomial $\Phi_N(z,t)$ is known to be irreducible over $\mathbb{Q}[z,t]$ by the work of Bousch [Bou92, Chapitre 3, Théorème 1], Gan–On [GO14, Theorem 1.2], and Morton [Mor96, Corollary 1]. Therefore, we can consider Proposition 1.8 to be a statement about its specializations to $c \in \mathbb{Q}$. As an application of Hilbert’s irreducibility theorem,

Proposition 1.9. For each integer $d$ and $N$ greater than 1, there exists a thin set $\Sigma_{d,N}$ of $\mathbb{Q}$ such that the unicritical polynomial $\phi_{d,c}$ satisfies the Galois–dynamics correspondence for all $c \in \mathbb{Q}$ not contained in $\Sigma_{d,N}$ and all Galois extensions $K/\mathbb{Q}$.

In particular, the exceptional set $\Sigma_{d,N}$ is a density 0 subset of $\mathbb{Q}$ that can be studied as the set of rational $c$ such that the Galois group $G_N$ of $\Phi(z,c)$ over $\mathbb{Q}$ is not isomorphic to the Galois group $G_N,c$ of $\Phi(z,c)$ over $\mathbb{Q}$. The exceptional set is well-understood for $d = 2$ and small $N$ by the work of Morton [Mor92] and Krumm [Kru18]. After adding what is known in Section 4 about the Galois–dynamics correspondence for small $N$, the quadratic polynomials $\phi_{2,c}$ satisfies the Galois–dynamics correspondence in the following cases.

Theorem 1.10. The quadratic polynomial $\phi_{2,c}(z) = z^2 + c$ with rational coefficients satisfies the Galois–dynamics correspondence for all nontrivial Galois extensions $K/\mathbb{Q}$ in the following cases:

- $N = 2$: all $c \in \mathbb{Q}$;
- $N = 3$: all $c \in \mathbb{Q}$;
- $N = 4$:
  - $[K : \mathbb{Q}] = 2$: all $c \in \mathbb{Q}$;
  - $[K : \mathbb{Q}] > 2$: all $c \notin \{-\frac{1}{2}\} \cup \left\{-\frac{s^2-2s+4}{4s^2} \mid s \in \mathbb{Q} \right\}$;
- $N = 5, 6, 7, \text{ or } 9$: all but finitely many $c \in \mathbb{Q}$.
A consequence of Corollary 1.6 and Theorem 1.10 is the following corollaries.

**Corollary 1.11.** Let $K$ be a quadratic number field. For all but finitely many $c \in \mathbb{Q}$, the polynomial $\phi_{2,c}$ has no periodic points of exact period 5 in $K$.

**Corollary 1.12.** If the $L$-series $L(J, s)$ extends to an entire function, $L(J, s)$ satisfies the standard functional equation, and the weak Birch and Swinnerton-Dyer conjecture is valid for $J$, then for all but finitely many $c \in \mathbb{Q}$, $\phi_{2,c}$ has no periodic points of exact period 6 in $K$.

Both cases are completely determined up to the exceptional sets $\Sigma_{2,5}$ and $\Sigma_{2,6}$.

### 2. Rational Points on Dynamatic Curves

An $N$-periodic point $z$ of $\phi_{d,c}$ satisfies the polynomial equation $\phi_{d,c}^N(z) - z = 0$ for all multiples $n$ of $N$. By the Möbius inversion formula, we have a factorization in terms of the $N$-th dynamatic polynomial $\Phi_N(z, c)$ in indeterminates $z$ and $c$,

$$
\phi_{d,c}^N(z) - z = \prod_{N|n} \Phi_N(z, c)
$$

$$
\Phi_N(z, c) := \prod_{m|N} (\phi_{d,c}^m(z) - z)^{\mu(N/m)} \in \mathbb{Q}[z, c],
$$

where $\mu$ is the Möbius function. The zero locus of $\Phi_N(z, c)$ defines an affine curve with smooth projective model $C_1(N)$ (by abuse of notation we sometimes also refer to the affine curve as $C_1(N)$) that carries an action induced by the iteration of $\phi_{d,c}$.

We can then define the quotient $C_0(N)$ of $C_1(N)$ by this action.

The study of periodic points of exact period $N$ in a number field $K$ is related to the study of $K$-points on the curves $C_1(N)$ and $C_0(N)$: $K$-points on $C_1(N)$ correspond to pairs $(z, \phi_{d,c})$ for a point $z \in K$ and a map $\phi_{d,c}(z) = z^d + c$ with $c \in K$ such that $z$ is a periodic point of period $N$ of $\phi_{d,c}$, and $K$-points on $C_0(N)$ correspond to pairs $(\mathcal{O}, \phi_{d,c})$ of a Gal($K/K$)-stable $N$-cycle $\mathcal{O}$ and a map $\phi_{d,c}(z) = z^d + c$ with $c \in K$; these include all pairs $(\mathcal{O}, \phi_{d,c})$ where elements of $\mathcal{O}$ are contained in $K$, and hence contain full information about periodic points in $K$.

In order to prove Theorem 1.5 we first specify the dynamical indices that the Galois action can assume through the Galois–dynamics correspondence. The following lemma is true in greater generality for any rational map $\phi$ and any nontrivial finite Galois extension $K/k$, but we state it in the unicritical polynomial and number field setting.

**Lemma 2.1.** Let $K$ be a nontrivial finite Galois extension of $\mathbb{Q}$ of degree $D$, let $N \geq 2$, and denote $g := \gcd(N, D)$. Let $\{z_0, \ldots, z_{N-1}\}$ be an exact $N$-cycle of $\phi_{d,c}$ contained in $K - \mathbb{Q}$.

If there exists a nontrivial $\sigma \in \text{Gal}(K/\mathbb{Q})$ and a positive integer $i < N$ such that

$$
\sigma z_0 = \phi_{d,c}^i(z_0),
$$

then $g > 1$ and $i = mN/g$ with $1 \leq m \leq g - 1$.

**Proof.** Suppose that there exists a nontrivial $\sigma \in \text{Gal}(K/\mathbb{Q})$ such that $\phi_{d,c}^i(z_0) = \sigma(z_0)$ for some $i \in \{1, \ldots, N - 1\}$. Since $\sigma$ commutes with $\phi_{d,c}$, and the action of $\phi_{d,c}$ is transitive on the $N$-cycle, we have that $\sigma \equiv \phi_{d,c}^i$ on the entire cycle.
Since $K$ is Galois, the order of the Galois group $\text{Gal}(K/\mathbb{Q})$ is $D$. Thus,
\[ z_0 = \sigma^D(z_0) = (\phi^i_{d,c})^D(z_0) = z_iD. \]

If $iD$ is not divisible by $N$, then $(z_0, \ldots, z_{D-1})$ forms a cycle of $\phi_{d,c}$ of length
less than $N$, contradicting the assumption that \{z_0, \ldots, z_{N-1}\} is an exact $N$-cycle.
Therefore, $N$ divides $iD$ and $i$ is a multiple of $\frac{N}{g}$. In particular, there is a con-
tradiction if $g = 1$ since $0 < i < N$ by assumption. \(\square\)

In other words, $\phi_{d,c}$ satisfies the Galois–dynamics correspondence for $N$, $K,$
and $z_0 \in \mathbb{K} - \mathbb{Q}$ if and only if there exists a nontrivial $\sigma \in \text{Gal}(K/\mathbb{Q})$ and $m \in$
\{1, \ldots, g-1\} such that $\phi^m_{d,c}(z_0) = \sigma z_0$. In particular, a necessary condition for $\phi_{d,c}$
to satisfy the Galois–dynamics correspondence is that $N$ and $[K : \mathbb{Q}]$ are coprime.

Let \{z_0, \ldots, z_{N-1}\} be an $N$-cycle of $\phi_{d,c}$ defined over $K - \mathbb{Q}$. If $\phi_{d,c}$ satisfies
the Galois–dynamics correspondence for a quadratic number field $K$ and an even
$N$, then the trace $\sum_{i=0}^{N-1} z_i$ must lie in $\mathbb{Q}$ because the $N$-cycle consists of paired
Galois conjugates $(z_i, z_i + \frac{N}{2})$. If $\phi_{d,c}$ satisfies the Galois–dynamics correspondence
for a quadratic number field $K$ and an odd $N$, then the set of periodic points
$\text{Per}_{N,K-Q}(\phi_{d,c})$ must be empty because otherwise, $i \in \{1, \ldots, N - 1\}$ would be a
multiple of $N$ by virtue of the fact that $g = \gcd(N, 2) = 1$. Hence any periodic point
of $\phi_{d,c}$ of exact odd period $N$ contained in $K$ must itself be rational. Therefore we
have Theorem 1.5.

Combined with the existing literature on the rational points of dynatomic curves,
Theorem 1.5 immediately yields Corollaries 1.6 and 1.7.

Corollary 1.6. Similarly by Theorem 1.5, a 5-cycle of $\phi_{2,c}$ necessarily corresponds
to a $\mathbb{Q}$-point on $C_0(5)$ and a rational point on $C_1(5)$. By the work of Flynn–Poonen–
Schaefer [FPS97], the genus 2 dynatomic curve $C_0(5)$ has only six rational points,
three of which correspond to $c = \infty$ and three of which ultimately correspond to
5-cycles in quintic cyclic extensions of $\mathbb{Q}$. \(\square\)

Corollary 1.7. By the work of Stoll [Sto08] that is dependent on the usual conjec-
tures for $L(J, s)$, there are only ten points in $C_0(6)(\mathbb{Q})$. Five of the points corre-
spond to cusps on $C_1(6)$ and the other five points correspond to explicitly-described
6-cycles, of which only the cycle
\[ z_0 = -1 + \frac{\sqrt{33}}{12}, z_1 = -\frac{1}{4} - \frac{\sqrt{33}}{6}, z_2 = -\frac{1}{2} + \frac{\sqrt{33}}{12}, z_{i+3} = z_i. \]
for $c = -\frac{71}{32}$ and $K = \mathbb{Q}(\sqrt{33})$ is defined over a quadratic field. By Theorem 1.5, the
trace of a 6-cycle of $\phi_{2,c}$ in a quadratic number field $K$ would necessarily correspond
to a point in $C_0(6)(\mathbb{Q})$, so there are no other such 6-cycles. \(\square\)

Remark 2.2. In these situations, the irreducibility criterion of Proposition 1.8 is
not sufficient because $\Phi_6(z, c)$ is often reducible as a polynomial in $\mathbb{Q}[z]$, such as
when $c = -2$.

For periods 5 and 6, no such periodic points are believed to exist, as sug-
gested by extensive numerical evidence by Doyle–Faber–Krumm [DFK14], Hutz–
Ingram [HI13], and Wang–Zhang [WZ15, Section 5], in addition to the theoretical
results of Doyle [Doy14] and Krumm [Kru16].
3. Irreducibility criteria

The dynamical mimicry of the action of the Galois group of $\Phi_N$ was previously considered by Vivaldi–Hatjispyros for any nonlinear polynomial $\phi$. They established that if the dynatomic polynomial $\Phi_N(z)$ is irreducible as a polynomial in $\mathbb{Q}[z]$ for a given $N$, then there is a subgroup $H_O$ of the Galois group $G$ of $\Phi_N(z)$ for each orbit $O$ that acts on $O$ in the same way as $\phi$ [VH92, Section 3]. After modifying the proof to work for any Galois extension $K/\mathbb{Q}$, we have the following criterion.

**Proposition 3.1.** Let $d$ and $N$ be integers greater than 1 and let $\phi$ be any nonlinear polynomial in $\mathbb{Q}[z]$. If the dynatomic polynomial $\Phi_N(z)$ is irreducible in $\mathbb{Q}[z]$ then $\phi$ satisfies the Galois–dynamics correspondence for $N$ and all Galois extensions $K/\mathbb{Q}$.

**Proof.** Vivaldi–Hatjispyros proved that if the dynatomic polynomial $\Phi_N(z)$ is irreducible in $\mathbb{Q}[z]$, then $\phi$ satisfies the Galois–dynamics correspondence for $N$ and the splitting field $K$ of $\Phi_N$ over $\mathbb{Q}$. Let $K$ be any nontrivial intermediate Galois extension of $K'/\mathbb{Q}$ and let $z$ be a periodic point in $\text{Per}_{N,K'}(\phi)$. Since $z$ is also an element of $\text{Per}_{N,K-\mathbb{Q}}(\phi)$, there is a positive integer $i < N$ and a nontrivial $\sigma \in \text{Gal}(K'/\mathbb{Q})$ such that $\phi^i(z) = \sigma z$ by the definition of the Galois–dynamics correspondence.

Notice that $\sigma$ cannot be in the subgroup $\text{Gal}(K'/K)$, since $\sigma$ does not fix $z$, which is an element of $K$. Then $\sigma$ corresponds to a nontrivial element $\tau$ of the quotient group $\text{Gal}(K'/\mathbb{Q})$ such that $\phi^i(z) = \tau z$. \qed

For unicritical polynomials $\phi_{d,c}$, Vivaldi–Hatjispyros [VH92, Section 3] remark that $\Phi_N(z, c)$ being irreducible in $\mathbb{Q}[z]$ “appears to be the typical situation”. meaning that it is reducible for $c$ in a density 0 subset of $\mathbb{Q}$. They proved their hypothesis for $N \leq 3$, but the irreducibility of $\Phi_N(z, c)$ fails to hold in general. Even for $d = 2$ and small $N$, Vivaldi–Hatjispyros demonstrated that $\Phi_N(z, c)$ is never irreducible in $\mathbb{Q}[z]$ for $N > 2$ when $c = -2$, and when $N = 3$ for an infinite family of $c$, and for $N$ such that $2^N - 1$ is not a Mersenne prime when $c = 0$.

To study the irreducibility of $\Phi_N(z, c) \in \mathbb{Q}[z]$ for $c \in \mathbb{Q}$, we can view it as a specialization of $\Phi_N(z, t) \in \mathbb{Q}[z, t]$. By the results of Bousch [Bou92, Chapitre 3, Théorème 1], Gan–Ou [GO14, Theorem 1.2], and Morton [Mor06, Corollary 1], it is known that $\Phi_N(z, t)$ is irreducible as a polynomial in $\mathbb{Q}[z, t]$. Let $G_N$ denote the Galois group of $\Phi_N(z, t)$ over $\mathbb{Q}(t)$ and let $G_{N,c}$ denote the Galois group of its specialization $\Phi_N(z, c)$ over $\mathbb{Q}$. We can immediately apply Hilbert’s irreducibility theorem.

**Corollary 3.2.** For each integer $d$ and $N$ greater than 1, there exists a thin set $\Sigma_{d,N}$ of $\mathbb{Q}$ such that for all rational $c$ not in $\Sigma_{d,N}$,

1. $\Phi_N(z, c)$ is irreducible in $\mathbb{Q}[z]$, and
2. $G_N \cong G_{N,c}$.

If we define the naive height of a rational number $\frac{a}{b}$ (in reduced form) to be

$$h\left(\frac{a}{b}\right) := \max(|a|, |b|),$$

then the proportion of rational numbers of naive height at most $H$ in a thin set $\Sigma$ is $O\left(\frac{1}{H}\right)$. Therefore, thin sets have density 0 [Ser08, Proposition 3.4.2] and
Corollary 3.2 implies that, for all $N$, the set of rational $c$ such that $\Phi_N(z,c)$ is reducible in $\mathbb{Q}[z]$ is a density 0 subset of $\mathbb{Q}$.

As a consequence of Proposition 1.8 with Corollary 3.2 we obtain the exceptional set criterion in Proposition 1.9 for the Galois–dynamics correspondence.

**Proposition 3.3.** For each integer $d$ and $N$ greater than 1, there exists a thin set $\Sigma_{d,N}$ of $\mathbb{P}_Q^1$ such that the unicritical polynomial $\phi_{d,c}$ satisfies the Galois–dynamics correspondence for all $c \in \mathbb{P}_Q^1$ not contained in $\Sigma_{d,N}$ and all Galois extensions $K/\mathbb{Q}$.

3.1. The exceptional set. By Hilbert’s irreducibility theorem, understanding $G_N \neq G_{N,c}$ allows one to determine the exceptional set $\Sigma_{d,N}$. On the one hand, the structure of the Galois group $G_N$ of $\Phi_N(z,t)$ over $\mathbb{Q}(t)$ is well-understood. Bousch [Bon92, Chapitre 3] (c.f. [MP94, Theorem 4.2]) showed that the Galois group $G_N$ of $\Phi_N(z,t)$ over $\mathbb{Q}(t)$ is isomorphic to a wreath product

$$G_N \cong (\mathbb{Z}/N\mathbb{Z}) \wr S_r,$$

where $r$ is an integer such that $rN = \deg \Phi_N$, for $\phi_{2,c}$. This result was extended by Lau and Schleicher [Sch94, LS94, Mor98b] for $\phi_{d,c}$ to all $d \geq 2$.

However, the structure of the Galois group $G_{N,c}$ of the specialization $\Phi_N(z,c)$ over $\mathbb{Q}$ is not known in general. For $d = 2$, one can show that $G_{N,c}$ is not isomorphic to $G_N$ for any positive integer $N$ if $c = 0$ or $-2$, i.e. that $0, -2 \in \Sigma_{2,N}$ for all $N$, since the Galois group $G_{N,c}$ is abelian for those values of $c$ [Kru19, Proposition 9.1]. Otherwise, there are only explicit descriptions of $\Sigma_{2,N}$ for small $N$.

By the work of Morton [Mor92] and Krumm [Kru18], the exceptional sets $\Sigma_{2,3}$ and $\Sigma_{2,4}$, respectively, are infinite and have explicit descriptions. For $N \in \{5, 6, 7, 9\}$, Krumm [Kru19] showed that $\Sigma_{2,2}$ is finite. Collectively, these provide the latter cases of Theorem 1.10.

For $N = 3$, Morton [Mor92, Theorem 8] showed that there is a rational number $r \neq -7, -11$ such that

$$\Sigma_{2,3} = \left\{0, -\frac{7}{2}, -\frac{r^3 + 29r^2 + 243r + 559}{16(r + 7)(r + 11)} \right\} \cup \left\{-\frac{3^2 + 7}{4} \mid s \in \mathbb{Q}\right\}.$$

In fact, Morton showed that while $G_{N,c}$ is not isomorphic to the full wreath product when $c = 0, -\frac{7}{2}$, or $-\frac{r^3 + 29r^2 + 243r + 559}{16(r + 7)(r + 11)}$, the dynatomic polynomial $\Phi_3(z,c)$ is still irreducible in these cases.

For $N = 4$, Krumm [Kru18, Theorem 1.2] showed that

$$\Sigma_{2,4} = \left\{-5, -\frac{155}{72}, -\frac{5}{2}, -2, -\frac{5}{4}, 0, \frac{19}{16} \right\} \cup \left\{\frac{s^2 + 2s - 4}{8s} \mid s \in \mathbb{Q}^\times\right\} \cup \left\{-\frac{s^3 - 2s + 4}{4s} \mid s \in \mathbb{Q}^\times\right\},$$

with explicit presentations of the exceptional Galois groups as subgroups of $S_{12}$. In fact, $\Phi_4(z,c)$ is still irreducible when $c \in \left\{-5, -\frac{155}{72}, -2, -\frac{5}{4}, 0, \frac{19}{16}\right\} \cup \left\{\frac{s^2 + 2s - 4}{8s} \mid s \in \mathbb{Q}^\times\right\}$, but generates a different Galois group.

For the cases $N \in \{5, 6, 7, 9\}$, Krumm [Kru19] demonstrates the finiteness of $\Sigma_{2,N}$ by viewing elements of the exceptional set as rational points on high genera curves. While an explicit description of those rational points is not yet known, Krumm provides an algorithm for computing the elements of $\Sigma_{2,N}$ of bounded naive height.
4. Exceptional cases

In the exceptional cases $c \in \Sigma_{d,N}$ where $\Phi_N(z,c)$ is reducible over $\mathbb{Q}$ or the Galois groups $G_N$ and $G_{N,c}$ are not isomorphic, one cannot use the irreducibility criterion of Section 3. At least for $d = 2$, there are small $N$ where the Galois–dynamics correspondence can be checked directly. Collectively, this section proves the first few cases of Theorem 1.10.

For $d = 2$ and $N = 2$, there can only be at most one 2-cycle for a given $\phi_{2,c}$ due to an explicit parametrization given by Walde–Russo [WR94 Theorem 1], so $\phi_{2,c}$ always satisfies the Galois–dynamics correspondence for $N = 2$ and any Galois number field $K/\mathbb{Q}$.

For $d = 2$ and $N = 3$, the dynatomic polynomial $\Phi_3(z,c) \in \mathbb{Q}[z]$ for a choice of $c \in \mathbb{Q}$ can either be irreducible, factor into two (possibly reducible) cubic factors, or factor into three irreducible quadratic factors. Morton [Mor92 Theorem 8] showed that $\Phi_3(z,c)$ is irreducible unless $c \in \left\{ -\frac{s^2+7}{2} \mid s \in \mathbb{Q} \right\}$. By Proposition 1.8, $\phi_{2,c}$ satisfies the Galois–dynamics correspondence for any Galois number field $K/\mathbb{Q}$ if $\Phi_3(z,c)$ is irreducible in $\mathbb{Q}[z]$, so we only need to check this special case. However, Vivaldi–Hatjispyros [VH92 Section 5] showed that this is precisely when $\Phi_3(z,c)$ factors into two (possibly reducible) cubic polynomials.

Lemma 4.1. Let $c$ be a rational number. If $\Phi_3(z,c)$ factors into two cubic factors in $\mathbb{Q}[z]$, then $\phi_{2,c}$ satisfies the Galois–dynamics correspondence for all Galois number fields $K/\mathbb{Q}$.

Proof. Walde–Russo [WR94 Theorem 3] showed that $c$ and each complex 3-cycle $\{z_1, z_2, z_3\}$ of $\phi_{2,c}$ are explicitly parametrized by a complex number $\tau$, which satisfies the relations

$$
\tau := z_1 + z_2, \quad -\frac{\tau + 1}{\tau} = z_2 + z_3, \quad -\frac{1}{\tau + 1} = z_1 + z_3.
$$

Consequently, $c$ and the $z_i$ are simultaneously rational if and only if $\tau$ is rational. Furthermore, the $z_i$ all lie in the same minimal field of definition over $\mathbb{Q}$ since $\phi_{2,c}$ has rational coefficients.

First, observe that if $\Phi_3(z,c)$ has a reducible quadratic or cubic factor, then $\Phi_3(z,c)$ has a linear factor corresponding to a periodic point in $\mathbb{Q}$. A 3-cycle of $\phi_{2,c}$ containing a rational number must necessarily be entirely contained in $\mathbb{Q}$, so $\phi_{2,c}$ has a 3-cycle entirely contained in $\mathbb{Q}$. We can disregard such a 3-cycle, since the Galois–dynamics correspondence only concerns irrational periodic points.

If $\Phi_3(z,c)$ has an irreducible cubic factor, then two of its roots must lie in the same 3-cycle. Then $\phi_{2,c}$ satisfies the Galois–dynamics correspondence for that 3-cycle and any Galois number field $K/\mathbb{Q}$ containing it since the 3-cycle is not disjoint from its $\sigma$-conjugates for some nontrivial $\sigma \in \text{Gal}(K/\mathbb{Q})$. Alternatively, observe that if $\phi_{2,c}(z) = \sigma z$ then $\phi_{2,c}^2(z) = \phi_{2,c}(\sigma z) = \sigma \phi_{2,c}(z) = \sigma^2 z$.

Furthermore Walde–Russo [WR94 Corollary 2] and Vivaldi–Hatjispyros [VH92 Section 5] showed that the Galois group of any irreducible cubic factor of $\Phi_3(z,c)$ is isomorphic to $A_3$. \hfill \qed
Remark 4.2. Since $\Phi_3(z, c)$ can only factor into two cubic polynomials or three irreducible quadratic polynomials if it is reducible, the results of Morton [Mor92, Theorem 8] and Vivaldi–Hatjispyros [VH92, Section 5] collectively show that $\Phi_3(z, c)$ cannot factor into three irreducible quadratic factors if $c$ is rational. As a consequence, there are no periodic points of $\phi_{2, c}$ of exact period 4 in $K - \mathbb{Q}$ for any quadratic number field $K$.

Remark 4.3. The parametrization of 3-cycles by Walde–Russo [WR94, Theorem 3] implies that such a factorization occurs if and only if a particular degree 9 polynomial with integers coefficients has a rational root. Since $\Phi_3(z, c)$ cannot factor into three irreducible quadratic factors if $c$ is rational, this implies that if $u$ and $v$ are coprime nonzero integers then

$$f_{u, v}(z) = -v^4 z^9 - 4v^4 z^8 + (4uv^3 - 6v^4)z^7 + (12uv^3 - 3v^4)z^6$$

$$\quad + (-6u^2v^2 + 14uv^3 + 3v^4)z^5 + (-12u^2v^2 + 5uv^3 + 3v^4)z^4$$

$$\quad + (4u^3v - 10u^2v^2 - 6uv^3 - 2v^4)z^3 + (-4u^3v - u^2v^2 - 6uv^3 - 3v^4)z^2$$

$$\quad + (-u^4 + 2u^3v + 3u^2v^2 - 2uv^3 - v^4)z + (-u^3v + 3u^2v^2 - uv^3)$$

has no rational roots.

For $d = 2$ and $N > 3$, fewer details are known about how $\Phi_N(z, c)$ can factor in general. At least for $N = 4$, there is still a parametrization of 4-cycles of $\phi_{2, c}$. Building on the work of Netto, Morton, and Erkama [Net00, Mor98a, Erk09], Panaksa [Pan11] used the factorization of $\Phi_4(z, c)$ to prove the following fact in his thesis.

**Theorem 4.4** ([Pan11, Theorem 1.5.1.]). Let $\{z_1, z_2, z_3, z_4\}$ be a 4-cycle for the quadratic polynomial $\phi_{2, c}$ where $c \in \mathbb{Q}$. If the $z_i$ lie in a quadratic number field, then $z_1$ and $z_3$ are Galois conjugates.

Then by Lemma 2.1 we have the following corollary.

**Corollary 4.5.** For all $c \in \mathbb{Q}$, the quadratic polynomial $\phi_{2, c}(z) = z^2 + c$ satisfies the Galois–dynamics correspondence for $N = 4$ and any quadratic number field $K$.

For $d = 2$ and $N \geq 5$, it is not known whether any class of pairs $(\phi_{2, c}, N)$ satisfies the Galois–dynamics correspondence besides those satisfying the irreducibility criteria and those explicitly found by computational searches for periodic points of exact period 5 and 6 [FPS97, HI13]. Even when $K/\mathbb{Q}$ is required to be quadratic, there is an active field of research on understanding the structure of the dynamics of $\phi_{2, c}$ [Doy14, DFK13, HI13, Kru16].

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References


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