THE GALOIS–DYNAMICS CORRESPONDENCE FOR
UNICRITICAL POLYNOMIALS

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Abstract. For the polynomial $\phi_{2,c}(z) := z^2 + c$ with $c \in \mathbb{Q}$, Vivaldi and Hatjispyros demonstrated that a subgroup of the Galois group of the $N$-th dynatomic polynomial $\Phi_N(z, c) \in \mathbb{Q}[z, c]$ mimics the action of $\phi_c$ on periodic points of exact period $N$ whenever specializations of $\Phi_N$ are irreducible in $\mathbb{Q}[z]$. We formalize this mimicry into a general Galois–dynamics correspondence between Galois actions and dynamical actions induced by iteration and characterize its occurrence for unicritical polynomials $\phi_{d,c}(z) := z^d + c$ of arbitrary degree $d \geq 2$ with $c \in \mathbb{Q}$.

We generalize the irreducibility criterion of Vivaldi–Hatjispyros to higher degrees, give another more general criterion for the occurrence of the Galois–dynamics correspondence, and give two proofs that the Galois–dynamics correspondence holds outside of a thin set of $P_1$. Furthermore, we show that if $\{z_0, \ldots, z_{N-1}\}$ is an $N$-cycle of $\phi_{d,c}$, then its trace $\sum z_i$ is rational. Furthermore, each $N$-periodic point $z_i$ is rational if $N$ is odd. Finally, we demonstrate that the non-existence of $K$-rational 5-cycles and complete determination of $K$-rational 6-cycles of $z^2 + c$ with $c \in \mathbb{Q}$ are consequences of the Galois–dynamics correspondence when $K$ is a quadratic number field.

1. Introduction

Let $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a rational map over an arbitrary base field $k$ and let $\phi^N$ denote the $N$-th iterate of $\phi$, i.e. let $\phi^N := \phi \circ \phi^{N-1}$. In order to understand how the dynamics of $\phi$ interact with the Galois action of $\text{Gal}(\overline{k}/k)$, we introduce the following relation.

Definition 1.1. The rational map $\phi$ satisfies the Galois–dynamics correspondence for period $N$ over $k$ if for every nontrivial finite Galois extension $K/k$ and every periodic point $z \in K - k$ of $\phi$ of exact period $N$, there is a positive integer $i < N$ and a nontrivial $\sigma \in \text{Gal}(K/k)$ such that $\phi^i(z) = \sigma z$.

Remark 1.2. For convenience, we may write that $(\phi, N)$ satisfies the Galois–dynamics correspondence and omit $k$ if it is implicitly known. If we wish to specify a particular finite Galois extension $K/k$, a particular periodic point $z$, or both then we may say that $(\phi, N, K/k)$, $(\phi, N, z)$, or $(\phi, N, K/k, z)$ satisfies the Galois–dynamics correspondence respectively.

The Galois–dynamics correspondence does not hold in general. If we restrict our attention to unicritical polynomials, which are defined via $\phi_{d,c}(z) := z^d + c$, there are many instances in which the Galois–dynamics correspondence might not be satisfied.
Example 1.3. If $K$ is the splitting field of $\phi_{2,c}^3(z) - z$ over the rational function field $k = \mathbb{C}(c)$, then the directed graph of periodic points of $\phi_{2,c}$ of exact period $N = 3$ consist of two disjoint 3-cycles. Then a result of Bousch [Bou92] Chapter 3, Theorem 3 implies that the Galois group $\text{Gal}(K/C(c))$ properly contains the full automorphism group of this graph and, in particular, contains an element $\sigma$ of order 2 that interchanges the two 3-cycles. If $k$ is the fixed field $K^\sigma$, then $[K : k] = 2$.

For any $z \in K$ of exact period 3, the $\text{Gal}(K/k)$-orbit of $z$ is $\{z, \sigma z\}$ so $z \in K - k$, but $\sigma z$ is not in the forward orbit of $z$ with respect to $\phi_{2,c}$.

We would like to identify settings in which the Galois–dynamics correspondence holds. The Galois–dynamics correspondence was never previously formalized nor stated in this generality, but the phenomenon has previously been studied by Vivaldi–Hatjispyros [VH92], who proved an irreducibility criterion for $(\phi_{2,c}, N)$ satisfying the Galois–dynamics correspondence over $\mathbb{Q}$ when $c \in \mathbb{Q}$, and by Panraksa [Pan11], who proved that $(\phi_{2,c}, 4, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence for all quadratic number fields $K/\mathbb{Q}$ when $c \in \mathbb{Q}$.

1.1. Main results and organization. For the remainder of this paper, we will assume that $k = \mathbb{Q}$. The aim of this paper is to characterize the occurrence of the Galois–dynamics correspondence for unicritical polynomials $\phi_{d,c}$ of arbitrary degree $d \geq 2$ with $c \in \mathbb{Q}$. Let $C_1(N)$ denote the dynatomic modular curve parametrizing periodic points of $\phi_{d,c}$ of exact period $N$ and let $C_0(N)$ denote the dynatomic modular curve parametrizing $N$-cycles. One of the two main theorems of this paper is the following new criterion for the occurrence of the Galois–dynamics correspondence for any polynomial $\phi_c \in \mathbb{Q}[z]$ parametrized by $c \in \mathbb{Q}$.

**Theorem 1.4.** Let $d$ and $N$ be integers greater than 1 and let $c$ be a rational number. Denote $\mathcal{F}_{1}^{\text{gal}}(N)$ to be the Galois closure of the function field of $C_1(N)$ over the algebraic function field of one variable $\mathbb{Q}(t)$. Let $C_1^{\text{gal}}(N)$ be the normalization of $\mathbb{P}^1_Q$ in $\text{Spec} \mathcal{F}_{1}^{\text{gal}}(N)$.

The Galois–dynamics correspondence holds for $(\phi_c, N)$ if $C_1(N)$ is both smooth and irreducible and if the $c$-fiber of the map $C_1^{\text{gal}}(N) \to \mathbb{P}^1_Q$ is a single closed point.

Restricting our attention to $\phi_{d,c}$ for the remainder of this section, we know that the smoothness and irreducibility conditions of Theorem 1.4 are already satisfied. The dynatomic modular curve $C_1(N)$ corresponding to $\phi_{d,c}$ is known to be smooth and (geometrically) irreducible for $d = 2$ due to Douady–Hubbard [DH85] Exposé XIV and Bousch [Bou92] Chapitre 3, Théorème 1 respectively, and for general $d$ due to Gao–Ou [GO14] Theorem 1.1, Theorem 1.2 and Morton [Mor96 Corollary 1].

We first prove the following statement about rational points of dynatomic curves in Section 2.

**Theorem 1.5.** Suppose that $c \in \mathbb{Q}$ and that $(\phi_{d,c}, N, K)$ satisfies the Galois–dynamics correspondence for $K$ a quadratic number field. If $\{z_0, \ldots, z_{N-1}\}$ is an $N$-cycle of $\phi_{d,c}$, then its trace $\sum z_i$ is rational. Furthermore, each $N$-periodic point $z_i$ is rational if $N$ is odd.

Using Theorem 1.5, we then establish the non-existence of $K$-rational $N$-cycles of $\phi_{2,c}$ with $c \in \mathbb{Q}$ for quadratic number fields $K$ and $N \in \{4, 5, 6\}$ as easy consequences. In particular, these are situations where the irreducibility criterion of
Vivaldi–Hatjispyros does not apply. Furthermore, we check that Theorem 1.4 holds for the case $N = 4$ and gives a new proof of Theorem 2.3 the result of Panraksa that $(\phi_{2,c}, 4, K/Q)$ satisfies the Galois–dynamics correspondence for all quadratic number fields $K/Q$ when $c \in Q$. In this way, one may view Proposition 1.6 and Theorem 1.4 as criteria for checking the equality of sets of rational points of $C_0(N)$ or as criteria for checking the non-existence of such $N$-cycles. In Section 3 we generalize the irreducibility criterion of Vivaldi–Hatjispyros from degree 2 to arbitrary degree.

**Proposition 1.6.** Let $d$ and $N$ be integers greater than 1 and let $c$ be a rational number. If the dynatomic polynomial $\Phi_N(z, c)$ is irreducible in $Q[z]$ then $(\phi_{d,c}, N)$ satisfies the Galois–dynamics correspondence.

We mention the known cases of the occurrence of the Galois–dynamics correspondence and demonstrate the following result as a simultaneous corollary of Proposition 1.6 and Theorem 1.4.

**Corollary 1.7.** For each integer $d$ and $N$ greater than 1, there exists a thin set $\Sigma_{d,N}$ of $P^1_Q$ such that the Galois–dynamics correspondence holds for $(\phi_{d,c}, N)$ for all $c \in P^1_Q$ not contained in $\Sigma_{d,N}$.

Finally, we prove Theorem 1.4 through a study of function fields of dynatomic curves in Section 4.

### 2. Rational Points on Dynatomic Curves

An $N$-periodic point $z$ of $\phi_{d,c}$ satisfies the polynomial equation $\phi_{d,c}^n(z) - z = 0$ for all multiples $n$ of $N$. By the Möbius inversion formula, we have a factorization in terms of the $N$-th dynatomic polynomial $\Phi_N(z, c)$ in indeterminates $z$ and $c$,

$$\phi^n(z) - z = \prod_{N|n} \Phi_N(z, c)$$

$$\Phi_N(z, c) := \prod_{m|N} (\phi^m(z) - z)^{\mu(N/m)} \in Q[z, c],$$

where $\mu$ is the Möbius function. The zero locus of $\Phi_N(z, c)$ defines an affine curve $C_1(N)$ that carries an action induced by the iteration of $\phi_{d,c}$. We can then define the quotient $C_0(N)$ of $C_1(N)$ by this action.

**Remark 2.1.** The study of periodic points of exact period $N$ in a number field $K$ (where we also allow $c \in K$) is related to the study of $K$-points on the curves $C_1(N)$ and $C_0(N)$: $K$-points on $C_1(N)$ correspond to pairs $(z, \phi_{d,c})$ for a point $z \in K$ and a map $\phi_{d,c}(z) = z^d + c$ with $c \in K$ such that $z$ is a periodic point of exact period $N$ of $\phi_{d,c}$, and $K$-points on $C_0(N)$ correspond to pairs $(\mathcal{O}, \phi_{d,c})$ of a Gal$(\overline{K}/K)$-stable $N$-cycle $\mathcal{O}$ and a map $\phi_{d,c}(z) = z^d + c$ with $c \in K$; these include all pairs $(\mathcal{O}, \phi_{d,c})$ where elements of $\mathcal{O}$ are contained in $K$, and hence contain full information about periodic points in $K$.

We can prove Theorem 1.5 by the following specification of how elements of the Galois group Gal$(K/Q)$ mimic the dynamics of $\phi_{d,c}$ when the Galois–dynamics correspondence holds.
Lemma 2.2. Let $K$ be a nontrivial finite Galois extension of $\mathbb{Q}$ of degree $D$, let $N \geq 2$, and denote $g := \gcd(N, D)$. Let $(z_0, \ldots, z_{N-1})$ be an exact $N$-cycle of $\phi_{d,c}$ in $K - \mathbb{Q}$ with $N > 1$.

Then $(\phi_{d,c}, N, K, z_0)$ satisfies the Galois–dynamics correspondence if and only if there exists an $m \in \{0, \ldots, g - 1\}$ and a nontrivial $\sigma \in \text{Gal}(K/\mathbb{Q})$ such that

$$
(\phi_{d,c})^{mN}(z_0) = \sigma z_0.
$$

Proof. Suppose that $(\phi_{d,c}, N, K, z_0)$ satisfies the Galois–dynamics correspondence. Then we have some non-trivial $\sigma \in \text{Gal}(K/\mathbb{Q})$ such that $\phi_{d,c}^i(z_i) = \sigma(z_i)$ for some $i, j \in \{0, \ldots, N\}$. Since $\sigma$ commutes with $\phi_{d,c}$ and the action of $\phi_{d,c}$ is transitive on the $N$-cycle, we have that $\sigma \equiv \phi_{d,c}^i$ on the entire cycle.

Now recall that $K$ is Galois, so the order of the Galois group $\text{Gal}(K/\mathbb{Q})$ is $D$.

Thus,

$$
z_0 = \sigma^D(z_0) = (\phi_{d,c}^i)^D(z_0) = z_{jD}.
$$

If $jD$ is not divisible by $N$, then $(z_0, \ldots, z_{r-1})$ forms a cycle of $\phi_{d,c}$ with length less than $N$, contradicting the assumption that $(z_0, \ldots, z_{N-1})$ is an exact $N$-cycle. Therefore, $N$ divides $jD$ and $j$ is a multiple of $\frac{N}{g}$.

Observe that if $K$ is a quadratic number field and $(\phi_{d,c}, N, K)$ satisfies the Galois–dynamics correspondence and $(z_0, \ldots, z_{N-1})$ is any $N$-cycle of $\phi_{d,c}$ defined over $K - \mathbb{Q}$, then the trace $\sum_{i=0}^{N-1} z_i$ must lie in $\mathbb{Q}$. In particular, if $N$ is odd then any periodic point of $\phi_{d,c}$ of exact period $N$ contained in $K$ must itself be rational. Therefore we have demonstrated Theorem 1.5.

For $N = 4$, Panakosa proved that $(\phi_{2,c}, 4, K)$ satisfies the Galois–dynamics correspondence if $K$ is a quadratic number field with the following result.

Theorem 2.3 ([Pan11] Theorem 1.5.1.) Let $\{x_1, x_2, x_3, x_4\}$ be a 4-cycle for the quadratic polynomial $\phi_{2,c}$ where $c \in \mathbb{Q}$. If the $x_i$ lie in a quadratic number field, then $x_1$ and $x_3$ are Galois conjugates.

Then we know that $C_0(4)(K) = C_0(4)(\mathbb{Q})$ by Theorem 1.5. Indeed, $C_0(4)$ is birationally equivalent to the rational curve $X_0(16)$ by a result of Morton [Mor98 Proposition 3].

Remark 2.4. In fact, we may apply Theorem 1.4 to give a new proof of Theorem 2.3. The dynatomic modular curve $C_1(4)$ is a genus 2 curve and its function field $\mathcal{F}_1(4)$ is of the form $\mathbb{Q}(z,c)$ of degree 12 over $\mathbb{Q}(c)$. The generator $z$ explicitly described by Morton [Mor98 Proposition 2]. In particular, $\mathcal{F}_1(4) = \mathcal{F}_1(4)$ (and hence $C_1(4) = C_1(4)$) since $\Phi_4(z,c)$ is irreducible in $\mathbb{Q}(c)[z]$ by Gauss’s lemma.

By the descriptions of $C_0(5)(\mathbb{Q})$ and $C_0(6)(\mathbb{Q})$ by Flynn–Poonen–Schaefer [FPS97] and a result of Stoll [Sto08], we have the following corollaries of Theorem 1.5.

Corollary 2.5. For quadratic number fields $K$ and rational numbers $c$ such that $(\phi_{2,c}, 5, K)$ satisfies the Galois–dynamics correspondence, there are no periodic points of $\phi_{2,c}$ of exact period 5 that are defined over $K$.

Corollary 2.6. Let $J$ be the Jacobian of $C_0(6)$ and suppose that the $L$-series $L(J,s)$ extends to an entire function, $L(J,s)$ satisfies the standard functional equation, and the weak Birch and Swinnerton-Dyer conjecture is valid for $J$. 

For quadratic number fields $K$ and rational numbers $c$ such that $(\phi_{2,c}, 6, K)$ satisfies the Galois–dynamics correspondence, there are no periodic points of $\phi_{2,c}$ of exact period 6 that are defined over $K$ unless $K = \mathbb{Q}(\sqrt{33})$ and $c = -\frac{71}{48}$, in which case there is exactly one 6-cycle:

$$z_0 = -1 + \frac{\sqrt{33}}{12}, \ z_1 = \frac{1}{4} - \frac{\sqrt{33}}{6}, \ z_2 = -\frac{1}{2} + \frac{\sqrt{33}}{12}, \ z_{i+3} = z_i.$$

**Remark 2.7.** Again, $\Phi_N(z, c)$ is sometimes reducible as a polynomial in $\mathbb{Q}[z]$ so we cannot use the irreducibility criterion of Vivaldi–Hatjispyros in this situation.

As a result, one may directly use Proposition 1.6 and Theorem 1.4 as criteria for the non-existence of periodic points of $\phi_{2,c}$ of exact period 5 or 6 defined over any quadratic number field. At least with regards to period 5, no such periodic points are believed to exist, as suggested by extensive numerical evidence by Doyle–Faber–Krumm [DFK14], Hutz–Ingram [HI13], and Wang [Wan] in addition to partial results by Doyle [Doy14] and Krumm [Kru16].

3. **Irreducibility criterion**

The Galois–dynamics correspondence was previously considered by Vivaldi–Hatjispyros for $d = 2$; they established that if the dynatomic polynomial $\Phi_N(z, c)$ is irreducible as a polynomial in $\mathbb{Q}[z]$ for a given $c$, then $(\phi_{2,c}, N)$ satisfies the Galois–dynamics correspondence [VH92, Section 3]. Proposition 1.6 follows as a generalization of the criterion of Vivaldi–Hatjispyros from $d = 2$ to any degree $d \geq 2$ with the same method of proof.

Vivaldi–Hatjispyros conjectured that $\Phi_N(z, c)$ is typically irreducible in $\mathbb{Q}[z]$ (i.e. occurs with probability 1 of $c \in \mathbb{Q}$) for all $N$ and demonstrated it for $N \leq 3$. Using the irreducibility of $\Phi_N(z, c)$ over $\mathbb{Q}[z, c]$ established by Bousch [Bou92], Chapitre 3, Théorème 1], Gan–Ou [GO14, Theorem 1.2], and Morton [Mor96, Corollary 1], a direct application of Hilbert’s irreducibility theorem to Proposition 1.6 yields Corollary 1.7 which prescribes the thin set $\Sigma_{d, N}$ outside which $c$ give $(\phi_{d,c}, N)$ satisfying the Galois–dynamics correspondence. Note that Corollary 1.7 also follows from Theorem 1.4. We can apply Serre’s version of Hilbert irreducibility to the condition of Theorem 1.4 to show that the Galois–dynamics correspondence typically holds (see Lemma 1.4).

If we define the naive height of a rational number (written in reduced form) to be $h(\frac{a}{b}) := \max(|a|, |b|)$, then the proportion of rational numbers in $\Sigma$ of naive height at most $H$ is $O(\frac{1}{H})$ and hence occurs with density 0 [Ser08, Proposition 3.4.2].

In this case, the set $\Sigma$ depending on $(\phi_{d,c}, N)$ can be described explicitly by the recent work of Krumm–Sutherland [KSI17, Theorem 1.1] and is known to be infinite for $(\phi_{2,c}, N)$ when $N \leq 4$ and finite when $N \in \{5, 6, 7, 9\}$ ([Kru18, Theorem 1.4]).

Nonetheless, the irreducibility of $\Phi_N(z, c)$ fails to hold in general. For instance, Vivaldi–Hatjispyros demonstrated that $\Phi_N(z, c)$ is never irreducible in $\mathbb{Q}[z]$ for an infinite family of cases of $(\phi_{2,c}, 3)$, all cases of $(\phi_{2,-2}, N)$ when $N > 2$, and for $(\phi_{2,0}, N)$ when $2N - 1$ is not a Mersenne prime.

However, the Galois–dynamics correspondence can be shown for a few cases when $d = 2$ even when $\Phi_N(z, c)$ is not irreducible in $\mathbb{Q}[z]$. For instance, there can only be at most one 2-cycle for a given $\phi_{2,c}$ so the Galois–dynamics correspondence holds for $(\phi_{2,c}, 2)$. For $(\phi_{2,c}, 3)$, we can also deduce that the Galois–dynamics correspondence is satisfied as a consequence of Vivaldi–Hatjispyros’s analysis of the
factorization of $\Phi(z, c)$ \cite{VH92} Section 5.3. For $N > 3$, fewer details are known
about how $\Phi_N(z, c)$ can factor in general. Building on the work of Netto, Morton,
and Erkama \cite{Erk06, Mor98, Net00}, Panraksa is able to use the factorization
of $\Phi(z, c)$ to directly prove Theorem \ref{thm:main}, i.e. that $(\phi_{2,c}, 4, K)$ always satisfies the
Galois–dynamics correspondence when $K$ is a quadratic number field \cite{Pan11, Theorem 1.5.1}.

For $N \geq 5$, it is not known whether any class of cases of $(\phi_{2,c}, N)$ satisfies the
Galois–dynamics correspondence besides those satisfying the irreducibility crite-
ron of Vivaldi–Hatjispyros and those found by computational searches for periodic
points of exact period 5 and 6 \cite{PPS97}. Even when $K/Q$ is required to be quadratic,
there is an active field of research making considerable progress on understanding
the structure of the dynamics of $\phi_{2,c}$ \cite{Doy14, DFK14, HIT13, Krm16}.

4. Function fields of dynatomic curves

For this section, we consider families of polynomials $\phi_c \in Q[z]$ parametrized
by $c \in Q$ such that the corresponding $C_1(N)$ is smooth and irreducible. Since $\phi_c$
internally permutes its $N$-cycles, the map $\tau : (z, c) \to (\phi_c(z), c)$ is an automorphism
of the curve $C_1(N)$ and generates a group $\langle \tau \rangle$ of order $N$. Take the quotient curve
$C_1(N)/\langle \tau \rangle$, and denote the (smooth) quotient curve by $C_0(N)$. Note that for a
given number field $K$, the $K$-points on $C_0(N)$ do not necessarily arise from $K$-points
on $C_1(N)$; rather, they correspond to $\Gal(K/K)$-stable $\tau$-orbits on $C_1(N)$.

Let $p : C_1(N) \to C_0(N)$ be the quotient map. Since $\tau : C_1(N) \sim \to C_1(N)$
leaves the dominant projection $\{\Phi_N = 0\} \to P^1$ invariant, the composite map
$\pi_1 : C_1(N) \to P^1$ factors uniquely through $p$ via a map $\pi_0 : C_0(N) \to P^1$. For each
$c \in P^1$ and $i \in \{0, 1\}$, define $F_{i,c} := \pi_i^{-1}(c)$ to be the fiber of $c$ on $C_i(N)$.

Using Galois theoretic language, we can now reformulate the Galois–dynamics
 correspondence. For the remainder of this section, we fix integers $d$ and $N$ at least
2 and a rational number $c$. Let $Q(x)^{\text{gal}}$ denote the Galois closure of the residue
field extension $Q(x)/Q(p(x))$.

**Proposition 4.1.** Choose $x \in F_{1,c}(\overline{Q})$ such that $x$ is not rational over $Q$. Let $z$
be the periodic point of $\phi_{d,c}$ associated to $x$. Then the following are equivalent:

(1) The Galois–dynamics correspondence holds for $(\phi_c, N, z)$.
(2) $Q(x)^{\text{gal}} \neq Q(p(x))$.
(3) Either $Q(x) \neq Q(p(x))$ or $Q(x)$ is not Galois over $Q(p(x))$.

**Proof.** The equivalence of (2) and (3) is immediate by inspecting the tower $Q(p(x)) \subset
Q(x) \subset Q(x)^{\text{gal}}$. Let us consider the equivalence of (1) and (2). Let $z_j := \phi_{d,c}(z)$
for integers $j$.

Observe that if the Galois–dynamics correspondence is not satisfied for $(\phi_c, N, z)$
in particular, recall that $z \in K - Q$, then

$\{z_0, \ldots, z_{N-1}\} \cap \{\sigma(z_0), \ldots, \sigma(z_{N-1})\} = \emptyset$

for all nontrivial $\sigma \in \Gal(K/Q)$, where $K/Q$ is any nontrivial finite Galois extension
such that $x$ is $K$-rational. Thus, we only need to show the equivalence of the
following two conditions:

(1) for every nontrivial finite Galois extension $K/Q$ such that $K$ contains $Q(x)$,
there is a nontrivial $\sigma \in \Gal(K/Q)$ such that $\sigma p(x) = p(x)$
(2) $Q(p(x)) \neq Q(x)^{\text{gal}}$
Assume (1), and take $K = \mathbb{Q}(x)_{\text{gal}}$. Then $\text{Gal}\left(\mathbb{Q}(x)_{\text{gal}}/\mathbb{Q}(p(x))\right)$ is nontrivial by our given $\sigma$ and so $\mathbb{Q}(p(x)) \neq \mathbb{Q}(x)_{\text{gal}}$.

Assume (2). Then for any nontrivial finite Galois extension $K$ of $\mathbb{Q}$ containing $\mathbb{Q}(x)$, $K$ also contains $\mathbb{Q}(x)_{\text{gal}}$, so $K \neq \mathbb{Q}(p(x))$. Thus, the subgroup $\text{Gal}(K/\mathbb{Q}(p(x)))$ contains a non-trivial element $\sigma$. Then $\sigma p(x) = p(x)$ and so we have (1). $\square$

We would like to rephrase Proposition 4.1 in terms of Galois groups. For each $x \in F_{1,c}$, define

$$I_x := \text{Gal}\left(\mathbb{Q}(x)_{\text{gal}}/\mathbb{Q}(p(x))\right)$$

$$J_x := \text{Gal}\left(\mathbb{Q}(x)_{\text{gal}}/\mathbb{Q}(x)\right)$$

Then notice that (2) in Proposition 4.1 says that $I_x \neq 1$ and (3) says that either $J_x \neq 1$ or $I_x \neq J_x$. Thus, we have another reformulation of the Galois–dynamics correspondence:

**Corollary 4.2.** Choose $x \in F_{1,c}(\overline{\mathbb{Q}})$ such that $x$ is not rational over $\mathbb{Q}$. Let $z$ be the periodic point of $\phi_{d,c}$ associated to $x$. Then the following are equivalent:

1. The Galois–dynamics correspondence holds for $(\phi_c, N, z)$.
2. $I_x \neq 1$.
3. Either $J_x \neq 1$ or $I_x \neq J_x$.

The idea of the remainder of the section is to use the Galois closures of the function fields of the $C_i$ and study their Galois actions on the $c$-fibers of the $C_i$.

Let $\mathcal{F}_i$ be the function field of $C_i$ and let $\mathbb{Q}(t)$ be the function field of $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{Q}}$. The projections $C_i \to C_0 \xrightarrow{\pi_0} \mathbb{P}^1$ induce the inclusions $\mathbb{Q}(t) \subset \mathcal{F}_0 \subset \mathcal{F}_1$, with $\mathcal{F}_1/\mathcal{F}_0$ cyclic of degree $N$.

If we let $\mathcal{F}_{1,\text{gal}}$ be the Galois closure of $\mathcal{F}_1$ over $\mathbb{Q}(t)$ and let $C_{1,\text{gal}}$ be the normalization of $\mathbb{P}^1$ in $\mathcal{F}_{1,\text{gal}}$ (so $C_{1,\text{gal}} = \text{Spec } R$ where $R$ is the integral closure of $\mathbb{Q}[t]$ in $\mathcal{F}_{1,\text{gal}}$), then we have the projections $C_{1,\text{gal}} \to C_1 \to C_0 \to \mathbb{P}^1$.

For $c \in \mathbb{P}_1$, let $F_{1,c}$ denote the $c$-fiber of the map $C_1 \to \mathbb{P}_1$ and $F_{1,c,\text{gal}}$ to be the $c$-fiber of the map $C_{1,\text{gal}} \to \mathbb{P}_1$.

Define $\Gamma := \text{Gal}(\mathcal{F}_{1,\text{gal}}/\mathbb{Q}(t))$. Then $\Gamma$ acts on $C_{1,\text{gal}}$ over $\mathbb{P}^1$ and $C_{1,\text{gal}}/\Gamma = \mathbb{P}^1$. Away from a $\Gamma$-stable finite set of closed points in $C_{1,\text{gal}}$, the $\Gamma$-action on $C_{1,\text{gal}}$ is free and thus $\Gamma$ has a free and transitive action on the fibers $F_{1,c,\text{gal}} \subset C_{1,\text{gal}}$.

For $c \in \mathbb{Q}$, we have an action of $G := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $F_{1,c,\text{gal}}(\overline{\mathbb{Q}})$ and this action is transitive if and only if the $c$-fiber $F_{1,c,\text{gal}}$ is a single closed point. The action of $\Gamma$ on $F_{1,c,\text{gal}}(\overline{\mathbb{Q}})$ commutes with the action of $G$. Then the action of $G$ on $F_{1,c,\text{gal}}$ is given by a homomorphism $\rho_c : G \to \Gamma$.

Let $\Gamma_0 := \text{Gal}(\mathcal{F}_{1,\text{gal}}/\mathcal{F}_0)$, a subgroup of $\Gamma$ such that $C_{1,\text{gal}}/\Gamma_0 = C_0$. Note that $\Gamma_0$ admits $\text{Gal}(\mathcal{F}_1/\mathcal{F}_0)$ as a quotient, so $\Gamma_0 \neq 1$ because $N > 1$. Let $F_y$ be the fiber of
each \( y \in F_{0,c}(\overline{Q}) \) in \( F_{1,c}^{\text{gal}}(\overline{Q}) \) under the map induced by the projection \( C^{\text{gal}}_1 \to C_0 \).

We see that \( \Gamma_0 \) similarly acts freely and transitively on \( F_y \) away from a finite set.

The action of \( \Gamma_0 \) on \( F_y \) commutes with the natural action of \( G_y := \text{Gal}(\overline{Q}/Q(y)) \) and so we have a homomorphism

\[
\tau_y : G_y \to \Gamma_0.
\]

**Lemma 4.3.** Fix \( c \in Q \) such that \( \Gamma \) acts simply transitively on \( F_{1,c}^{\text{gal}}(\overline{Q}) \) and \( \Gamma_0 \) acts simply transitively on \( F_y \) for all \( y \in F_{0,c}(\overline{Q}) \). If \( F_{1,c}^{\text{gal}} \) is a closed point, then \( I_x \) is nontrivial for all \( x \in F_{1,c}(\overline{Q}) \).

**Proof.** Since \( c \in Q \), we have \( \tau_y = \rho_c|_{G_y} \) when \( c = \pi_0(y) \). But \( \rho_c \) is surjective since \( F_{1,c}^{\text{gal}} \) is a closed point. Since the action of \( \Gamma_0 \) on \( F_y \) is simply transitive and \( \Gamma_0 \) is a subgroup of \( \Gamma \), we have the surjectivity of \( \tau_y \) for all \( y \in F_{0,c}(\overline{Q}) \). Thus, \( F_y \) is irreducible over \( Q(y) \) with nontrivial Galois group \( \Gamma_0 \) for all \( y \in F_{0,c}(\overline{Q}) \). Then this implies that \( I_x := \text{Gal}(\overline{Q}(x)^{\text{gal}}/Q(p(x))) \) is nontrivial for all \( x \in F_{1,c}(\overline{Q}) \). \( \square \)

Together, Lemma 4.3 and Corollary 4.2 imply Theorem 1.4. Through Serre’s version of Hilbert irreducibility, we may show that the necessary condition of Theorem 1.4 happens away from a thin set. With the following lemma, we have given the second proof of Corollary 1.7.

**Lemma 4.4.** The \( c \)-fiber \( F_{1,c}^{\text{gal}} \) is a closed point for all \( c \in Q \) away from a thin set.

**Proof.** By Hilbert’s irreducibility theorem in the form on specializations of Galois groups \([\text{Ser}90] \) Chapter 9, Proposition 2], there is a thin set \( \Sigma \subset P_1^Q \) (in the sense of Serre) such that \( F_{1,c}^{\text{gal}} \) consists of a single orbit under the action of \( G \) for all \( c \notin \Sigma \). Thus, \( F_{1,c}^{\text{gal}} \) is a single closed point for almost all \( c \in Q \). \( \square \)

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