WEIL–DELINE REPRESENTATIONS I
LOCAL LANGLANDS SEMINAR

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1. Notation

- \( p := \) a fixed prime
- \( K := \) a \( p \)-adic field, i.e. a finite extension of \( \mathbb{Q}_p \)
- \( \overline{K} := \) an algebraic closure of \( K \)
- \( \mathcal{O}_K := \) the ring of integers of \( K \)
- \( \kappa := \) the residue field of \( \mathcal{O}_K \)
- \( q := \) the cardinality of \( \kappa \)
- \( G_K := \text{Gal}(\overline{K}/K) \)

2. The Weil group

- Let \( \sigma_K \) be the arithmetic Frobenius automorphism \((x \mapsto x^q)\) and \( \phi_K = \sigma_K^{-1} \) the geometric Frobenius.
- Let \( I_K \) be the inertia group of \( K \), i.e. \( I_K := \ker(\pi : G_K \to G_\kappa) \).
- Note that \( G_\kappa = \text{Gal}(\overline{K}/K_{nr}) \cong \hat{\mathbb{Z}} \) where \( K_{nr} \) is the maximal unramified extension.
- The Weil group \( W_K \) is the inverse image of \( \langle \sigma_K \rangle \) under \( \pi \),

\[
0 \to I_K \to W_K \to \langle \sigma_K \rangle \to 0,
\]

endowed with the topology of a locally compact group such that \( W_K \to \langle \sigma_K \rangle \cong \mathbb{Z} \) is continuous where \( \mathbb{Z} \) has the discrete topology and \( I_K \) has

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the profinite topology from $G_K$. This is not the subspace topology. The canonical injective homomorphism $\Phi_K : W_K \hookrightarrow G_K$ is continuous and from the inclusion.

- Alternatively, $W_K \cong \text{Proj lim}_L W_{L/K}$, where $W_{L/K} := W_K/W^c_L$ and

$$
1 \longrightarrow L^\times \longrightarrow W_{L/K} \longrightarrow \text{Gal}(L/K) \longrightarrow 1
$$

so the canonical injective homomorphism $\Phi_K : W_K \hookrightarrow G_K$ with dense image is the projective limit of homomorphisms $W_{L/K} \hookrightarrow \text{Gal}(L/K)$.

3. Representations of the Weil group

**Definition 3.1.** Let $\text{Rep}(G)$ denote the category of representations of $G$.

**Remark 3.2.** Since $\Phi_K$ is injective with dense image, we can identify $\text{Rep}(G_K)$ with a sub-category of $\text{Rep}(W_K)$.

**Definition 3.3.** A representation of $W_K$ that lies in the subcategory corresponding to $\text{Rep}(G_K)$ is called *Galois-type*.

**Example 3.4.** Via $r_K : K^\times \cong W_K^{ab}$, the absolute value $|\cdot|_K$ on $K^\times$ gives the absolute value character $\omega : W_K \rightarrow \mathbb{C}^\times$ sending $x \mapsto |x|_K$. This has infinite image and therefore is not a character of $G_K$.

**Proposition 3.5.** A representation $\rho$ of $W_K$ is of Galois-type if and only if $\rho(W_K)$ is finite.

**Proof.** The open subgroups of $W_K$ of finite index are the $W_L$ for finite $L/K$. Their intersection is $\ker \Phi = 1$. \qed
Definition 3.6. Denote $\omega_s : W_K \to \mathbb{C}^\times$ the quasi-character sending $x \mapsto |x|^s$ for $s \in \mathbb{C}$.

Proposition 3.7 ([Tat67, Lemma 2.3.1.]). Every one-dimensional representation of $W_K$ that is unramified (i.e. trivial on $I_K$) is of the form $\omega_s$ for some $s \in \mathbb{C}$.

Proof. Any unramified quasi-character $\chi$ of $W_K$ will depend only on $\omega$ and, as a function of $\omega$, is itself a quasi-character $\chi'$ of the value group $\{N(p)^n \mid n \in \mathbb{Z}\}$ of $W_K$. This is given by $s = -\log(\log N(p))/\log N(p)$.

Theorem 3.8 ([Del73, Section 4.10]). Every irreducible representation of $W_K$ is of the form $r = r' \otimes \omega_s$ for some $s \in \mathbb{C}$ and representation $r'$ of Galois-type. In fact, this is true for any extension of $\mathbb{Z}$ by a profinite group.

Proof. Every representation of $W_K$ is trivial on a finite-index subgroup $J$ of $I_K$. Since $I_K/J$ is finite, $\phi^n$ acts trivially on $I_K/J$ by conjugation for some $n > 0$ and so is central in $W_K/J$. Each power $\pi^m$ of $\phi^n$ has exactly one eigenvalue $a_m$ if the representation is irreducible. Then each irreducible representation has a type given by

$$(a_m) \in \lim_{\to} (X_m, \phi_{n,m})_m,$$

where $X_m = \mathbb{C}^\times$ and $\phi_{n,m}(x) = x^{m^m}$.

The representations of $W_K$ of type $s$ form an abelian category $M_s(W_K)$, and

$$\text{Rep}(W_K) = \bigoplus_{s \in \mathbb{C}} \text{Rep}_s(W_K)$$

The representations of type $1$ are precisely the Galois-type representations $\text{Rep}(G_K)$. Then we have an isomorphism

$$\cdot \otimes \omega_s : \text{Rep}(G_K) \to \text{Rep}_s(W_K).$$
Proposition 3.9. A Galois-type representation of $W_K$ is irreducible iff it is irreducible as a $G_K$-representation. Furthermore, if $\rho$ is any irreducible $W_K$-representation, it is of Galois-type iff the image of $\det \circ \rho$ is a subgroup of $\mathbb{C}^\times$ of finite order.

Definition 3.10. For any finite extension $L/K$, let $W_L := \phi_K^{-1}(G_L) \subset W_K$ where $G_L := \text{Gal}(\overline{K}/L)$. Note: $W_K/W_L \cong G_K/G_L \cong \text{hom}_K(L,K)$ is finite.

Then we have the restriction functor

$$\text{res}_{L/K} : \text{Rep}(W_K) \to \text{Rep}(W_L)$$

given by $\rho \mapsto \rho|_{W_L}$. The induction functor

$$\text{ind}_{L/K} : \text{Rep}(W_L) \to \text{Rep}(W_K)$$

is given by $(\rho, V) \mapsto (\tau, \{f : W_K \to V \mid f(xw) = \rho(x)f(w) \text{ for all } x \in W_L, w \in W_K\})$. These functors satisfy Frobenius reciprocity.

4. Weil–Deligne Representations

Definition 4.1. A Weil–Deligne representation of $W_K$ is a triple $(\rho, V, N)$ where $(\rho, V)$ is a representation of $W_K$ and $N$ is a nilpotent $\mathbb{C}$-linear endomorphism of $V$ such that

$$\rho(x)N\rho(x)^{-1} = |x| N.$$ 

It is called Frobenius semisimple if $\rho$ is semisimple.

Definition 4.2. Let $(\rho_1, V_1, N_1)$ and $(\rho_2, V_2, N_2)$ be two Weil–Deligne representations.
Define the representation \((\rho, V, N) = (\rho_1, V_1, N_1) \otimes (\rho_2, V_2, N_2)\) by \(V = V_1 \otimes V_2\) and, for \(x \in W_K\) and \(v_i \in V_i\),

\[\rho(x)(v_1 \otimes v_2) := \rho_1(x)v_1 \otimes \rho_2(x)v_2\]

\[N(v_1 \otimes v_2) := N_1v_1 \otimes v_2 + v_1 \otimes N_2v_2.\]

The formula is a result of:

\[\log(\rho_1(x) \otimes \rho_2(x)) = \log(\rho_1(x) \otimes 1 + 1 \otimes \rho_2(x)).\]

Define the representation \((\rho, V, N) = \text{hom}((\rho_1, V_1, N_1), (\rho_2, V_2, N_2))\) by \(V = \text{hom}(V_1, V_2)\) and, for \(\phi \in \text{hom}(V_1, V_2), x \in W_K\) and \(v_i \in V_i\),

\[(\rho(x)\phi)(v_1) := \rho_2(x)(\phi(\rho_1(x)^{-1}v_1))\]

\[(N\phi)(v_1) := N_2(\phi(v_1) - \phi(N_1v_1)).\]

The contragredient \(\rho^\vee\) of a Weil–Deligne representation is \(\text{hom}(\rho, 1)\) where 1 is the trivial one-dimensional representation.

**Remark 4.3.** If \(x \in W_K\) corresponds to the uniformizer \(\pi_K\) via the Artin reciprocity map \(\text{Art}_K : K^\times \to G_K^{ab}\), then \(N\) is conjugate to \(qN\) and hence has no nonzero eigenvalues, i.e. \(N\) is automatically nilpotent.

**Remark 4.4.** The kernel of \(N\) is stable under \(W_K\), so \((\rho, V, N)\) is irreducible iff \((\rho, V)\) is irreducible and \(N = 0\). So the irreducible Weil–Deligne representations of \(W_K\) are the irreducible representations of \(W_K\).

**Remark 4.5.** The category of \(\text{WDRep}_k(W_K)\) does not depend on the topology on \(k\). Thus, we can identify \(\text{WDRep}_C(W_K)\) with \(\text{WDRep}_{\mathbb{Q}_\ell}(W_K)\).

**Example 4.6.** If \(n = 1\), then \(N\) is nilpotent and 1-by-1 and hence zero. Then a Weil–Deligne representation is just a continuous homomorphism \(W_K \to \mathbb{C}^\times\).
**Definition 4.7.** The Weil–Deligne group $W'_K$ is the group scheme $W_K \ltimes \mathbb{G}_a$ over $\mathbb{Q}$ given by the action

$$wxw^{-1} = |w|x,$$

for all $w \in W_K$. Composition is given by

$$(w_1, x_1)(w_2, x_2) = (w_1w_2, |w_2|^{-1}x_1 + x_2).$$

**Remark 4.8.** A Weil–Deligne representation of $W_K$ is the same as a representation of $W'_K$. This arises from the fact that finite-dimensional representations of the additive group $\mathbb{G}_a$ correspond to nilpotent endomorphisms.

5. **L-adic representations**

**Theorem 5.1** (Grothendieck’s l-adic monodromy theorem). Let $F$ be an $\ell$-adic field, where $\ell \neq p$ is prime. Let $(\rho, V)$ be a finite-dimensional representation of $W_K$ over $F$. Then there exists a finite-index open subgroup $H \subset I_K$ such that $\rho(x)$ is unipotent for all $x \in H$.

**Remark 5.2.** A similar theorem is true if we replace $W_K$ by $G_K$ because unipotent subgroups are closed in the image of $G_K$ and $W_K \subset G_K$ is dense.

**Definition 5.3.** Let $t_\ell : I_K \to \mathbb{Q}_\ell$ be a nonzero homomorphism. (This exists and is unique up to a constant multiple because the wild ramification group $P_K$ is a pro-$p$-group and $I_K/P_K \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$).

We have $t_\ell(xy^{-1}) = |x|t_\ell(y)$ for all $x \in W_K$, $y \in I_K$ (because conjugation by $x$ induces raising to the $|x|$ power in $I_K/P_K$).

**Corollary 5.4.** There exists a unique nilpotent operator $N$ of $V$ such that $\rho(x) = \exp(t_\ell(x)N)$ for all $x \in H$ in some open subgroup of $I_K$. (This is $N$ from now on.)
Proof. Nilpotency and uniqueness follow directly from writing $N = t_\ell(x_0)^{-1} \log(\rho(x_0))$ for some $x_0 \in H \cap I_K$ such that $t_\ell(x_0)$ is nontrivial (using the $\ell$-adic monodromy theorem for nilpotency).

Existence follows because $\rho|_{H \cap I_K}$ factors through $t_\ell$ as some continuous representation of $Z_\ell(1)$ which coincides with the continuous representation $Z_\ell(1) \to \text{GL}_F(V), x \mapsto \exp(xN)$ on $t_\ell(x_0)$ and hence on $t_\ell(x_0)Z_\ell(1)$ for all $x_0 \in H \cap I_K$ such that $t_\ell(x_0)$ is nontrivial. Thus, they coincide on $H \cap I_K$. $\square$

Remark 5.5. Corollary 5.4 allows us to attach a Weil–Deligne representation to each representation of $W_K$. But we cannot naively use $(\rho, V) \mapsto (\rho, V, N)$ since $(\rho, V)$ is not smooth in general.

Theorem 5.6 ([Del73, Section 8]). There is an equivalence of categories between finite dimensional continuous representations of $W_K$ and the Weil–Deligne representations of $W_K$

$$(\dashrightarrow)_{WD} : \text{Rep}_k(W_K) \to \text{WDRep}_k(W_K)$$

$$(\rho, V) \mapsto (\rho_\phi, V, N)$$

$$\rho_\phi(\phi^n x) = \rho(\phi^n x) \exp(-t_\ell(x)N).$$

Proof. The condition

$$\rho_\phi(x)N\rho_\phi(x)^{-1} = |x| N,$$

holds because the exponential commutes with $N$. Exercise: show that $(\rho_\phi, V)$ is a continuous representation of $W_K$.

For a map $f : (\rho_1, V_1) \to (\rho_2, V_2)$, the uniqueness of the $N_i$ gives

$$f \circ N_1 = N_2 \circ f.$$ 

So $(\dashrightarrow)_{WD}$ is a faithful functor.
The uniqueness of the monodromy operator implies that $N$ is the monodromy operator associated to $(\rho_\phi, V)$. 

Remark 5.7. The functor depends on our choice of $\phi$ and $t_\ell$, but only up to a natural automorphism of the identity functor.

Remark 5.8. We can view the $\ell$-adic representations of $G_K$ over $\mathbb{Q}_\ell$ as a subcategory of Weil–Deligne representations over $\mathbb{C}$ (or over $\overline{\mathbb{Q}}_\ell$) via Theorem 5.6, Remark 4.5 and Remark 3.2.

REFERENCES


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