BRAIDING VIA GEOMETRIC LIE ALGEBRA ACTIONS

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ABSTRACT. We introduce the idea of a geometric categorical Lie algebra action on derived categories of coherent sheaves. The main result is that such an action induces an action of the braid group associated to the Lie algebra. The same proof shows that strong categorical actions in the sense of Khovanov-Lauda and Rouquier also lead to braid group actions. As an example, we construct an action of Artin’s braid group on derived categories of coherent sheaves on cotangent bundles to partial flag varieties.

CONTENTS

1. Introduction 2
Acknowledgements 3
2. Definitions and Main Results 3
2.1. Notation 3
2.2. Geometric categorical g actions 3
2.3. Role of the deformation 5
2.4. Example from resolutions of Kleinian singularities 6
2.5. $C^\times$-equivariance 6
2.6. Equivalences via geometric categorical sl2 actions 7
2.7. Braid group action via geometric categorical g actions 7
3. Example: cotangent bundles to flag varieties 8
3.1. The categorical g action 8
3.2. The braid group action 9
3.3. Proof of sl2 conditions (i) - (vii) 9
3.4. Proof of Serre relation (viii) 10
3.5. Existence of deformations: conditions (x) and (xi) 12
4. Preliminaries 14
4.1. Idempotent completeness 14
4.2. Some basic sl2 relations 15
4.3. Spaces of maps 16
4.4. Some basic sl3 relations 17
4.5. Induced maps 18
5. Proof of Main Theorem 2.6 26
5.1. Step 1: Calculation of $\mathcal{E}_{ij} * F_i^{(\lambda,\alpha_i) + s}$ 26
5.2. Step 2: Calculation of $\mathcal{E}_{ij} * F_i^{(\lambda,\alpha_i) + s} * \mathcal{E}_i^{(s)}$ 27
5.3. Step 3: Calculation of $\mathcal{E}_{ij} * T_i$ 28
5.4. Step 4: Proof that $T_{ij} * T_i \cong T_i * T_j$ 31
5.5. Step 5: Proof of braiding relation 31
6. Braiding via strong categorical g-actions 32
References 34

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1. Introduction

Let $X$ be a smooth complex variety. Often the derived category of coherent sheaves on $X$, denoted $D(X)$, possesses interesting autoequivalences, not coming from automorphisms of $X$ itself. For example, if $D(X)$ contains a spherical object $E$, then Seidel-Thomas [ST] defined a spherical twist $T_E : D(X) \rightarrow D(X)$ which is a non-trivial autoequivalence.

The notion of twists in spherical objects has been generalized by various authors (Horja [Ho], Anno [A], and Rouquier [Ro1]) to twists in spherical functors (a relative version). In [CKL2], [CKL3], we (jointly with Anthony Licata and following ideas of Chuang-Rouquier [CR]) defined the notion of geometric categorical $\mathfrak{sl}_2$ actions as a generalization of the notion of spherical functors. We showed that geometric categorical $\mathfrak{sl}_2$ actions give rise to equivalences of derived categories of coherent sheaves.

Often autoequivalences of $D(X)$ can be organized into an action of a braid group. Seidel-Thomas [ST] showed that given a collection of spherical objects which form a type $\Gamma$ arrangement, the spherical twists generate an action of the braid group $B_\Gamma$. An important example from [ST] of this situation concerned the case where $X$ is the resolution of a surface quotient singularity $\mathbb{C}^2/H$ and the spherical objects come from the exceptional $\mathbb{P}^1$s.

Another example of a braid group action was given by Khovanov-Thomason in [KT]. They showed that $B_n$ acts on $D(T^*\text{Fl}((\mathbb{C}^n)))$, the derived category of the cotangent bundle to the full flag variety, with the generators acting by spherical twists. Our purpose in this paper is to introduce a new method of constructing braid group actions (called geometric categorical $\mathfrak{g}$ actions), where the generators act by the equivalences coming from geometric categorical $\mathfrak{sl}_2$ actions. Roughly speaking, spherical objects are a special case of geometric categorical $\mathfrak{sl}_2$ actions and type $\Gamma$ arrangements of spherical objects are a special case of geometric categorical $\mathfrak{g}$ actions.

To explain our motivation for this notion, let us recall that our proof that a geometric categorical $\mathfrak{sl}_2$ action gives an equivalence came in two parts. First in [CKL2], we showed that a geometric categorical $\mathfrak{sl}_2$ action gives a strong categorical $\mathfrak{sl}_2$ action, a notion introduced by Chuang-Rouquier [CR]. We then showed in [CKL3] that a strong categorical $\mathfrak{sl}_2$ action gives an equivalence, using an explicit complex introduced by Chuang-Rouquier [CR]. The reason for introducing the notion of geometric categorical $\mathfrak{sl}_2$ action, rather than working with strong categorical $\mathfrak{sl}_2$ actions, is that the axioms of the former are much easier to check in examples.

The notion of strong categorical $\mathfrak{sl}_2$ action has been generalized by Rouquier [Ro2] and Khovanov-Lauda [KL] to the notion of strong categorical $\mathfrak{g}$ action. Hence it is natural to conjecture that a strong categorical $\mathfrak{g}$ action gives an action of the braid group of type $\mathfrak{g}$ (denoted $B_\mathfrak{g}$). Also it is natural to search for a notion of geometric categorical $\mathfrak{g}$ action which implies strong categorical $\mathfrak{g}$ action but which is easier to check in geometric examples.

In this paper we accomplish these goals with one caveat. More specifically we define a notion of geometric categorical $\mathfrak{g}$ action and prove that a geometric categorical $\mathfrak{g}$ action gives rise to an action of $B_\mathfrak{g}$ (Theorem 2.6). We also show that a strong categorical $\mathfrak{g}$ action gives an action of $B_\mathfrak{g}$ (Theorem 6.1). This essentially answers a conjecture of Rouquier [Ro2] (Rouquier has also recently proven his conjecture via a different method). However, we do not prove that a geometric categorical $\mathfrak{g}$ action gives a strong categorical $\mathfrak{g}$ action, though we expect this to be the case (the proof should follow along the same lines as [CKL2], where we established this result for $\mathfrak{g} = \mathfrak{sl}_2$).

We give a quick example showing how resolutions of $\mathbb{C}^2/H$ give geometric categorical $\mathfrak{g}$ actions. In greater detail in section 3, we discuss the more complicated example of a geometrical categorical $\mathfrak{sl}_n$ action on cotangent bundles to $n$-step partial varieties. This generalizes the work of Khovanov-Thomason [KT] for $T^*\text{Fl}((\mathbb{C}^n))$ and also our previous work [CKL2, CKL3] on cotangent bundles to Grassmannians.

In a forthcoming paper with Anthony Licata [CKL4], we will construct geometric categorical $\mathfrak{g}$ actions on Nakajima quiver varieties, generalizing the two examples in this paper. Using the main result of this paper, this will provide many more examples of braid group actions.
There are also many interesting examples of strong categorical $g$ actions, not involving coherent sheaves. For examples, Khovanov-Lauda [KL] have considered strong categorical $\mathfrak{sl}_n$ actions on categories of modules over cohomology rings of partial flag varieties and Rouquier has defined strong categorical $\mathfrak{sl}_g$ actions on categories of representations of the symmetric group over $\mathbb{F}_q$. Our Theorem 6.1 can be applied to these situations to produce braid group actions.

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2. Definitions and Main Results

In this section we define the concept of a geometric categorical $g$ action, review the construction of equivalences from strong categorical $\mathfrak{sl}_2$ actions and state our main result (Theorem 2.6).

2.1. Notation. Consider a simply-laced Dynkin diagram $\Gamma$ with finite vertex set $I$. Denote by $g$ the corresponding Kac-Moody Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. Let $\{\alpha_i\}_{i \in I}$ denote the simple roots and $\{\omega_i\}_{i \in I}$ the fundamental weights. Denote by $X$ the weight lattice and by $\langle \cdot, \cdot \rangle$ the standard invariant form on $X$ normalized so that $\langle \alpha_i, \alpha_i \rangle = 2$ and $\langle \alpha_i, \alpha_j \rangle = -1$ if $i \neq j \in I$ are joined by an edge.

Let $B_g$ denote the braid group of type $g$. It has generators $\sigma_i$ for $i \in I$ and relations

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if} \quad i \text{ and } j \text{ are connected in } \Gamma$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad i \text{ and } j \text{ are not connected in } \Gamma$$

Recall that $B_g$ maps to the Weyl group $W_g$ of type $g$ which has the same generators and relations, except that the generators square to the identity.

All our varieties will be be defined over $k$. When we write $H^*(X)$ we will mean the cohomology of $X$ shifted so that it lies between degrees $-\dim(X)$ and $\dim(X)$. For example, $H^*(\mathbb{P}^1) = k[-1] \oplus k[1]$.

2.2. Geometric categorical $g$ actions. Let $g$ be a Kac-Moody Lie algebra. In [CKL2] we introduced the concept of a geometric categorical $g$ action when $g = \mathfrak{sl}_2$. We now extend this definition to arbitrary simply-laced $g$. The key example to keep in mind is $g = \mathfrak{sl}_3$.

A geometric categorical $g$ action consists of the following data.

(i) A collection of smooth varieties $Y(\lambda)$ for $\lambda \in X$.

(ii) Fourier-Mukai kernels

$$E_i^{(r)}(\lambda) \in D(Y(\lambda) \times Y(\lambda + r\alpha_i)) \quad \text{and} \quad F_i^{(r)}(\lambda) \in D(Y(\lambda + r\alpha_i) \times Y(\lambda))$$

We will usually write just $E_i^{(r)}$ and $F_i^{(r)}$ to simplify notation whenever possible.

(iii) For each $Y(\lambda)$ a flat deformation $\tilde{Y}(\lambda) \to \mathfrak{h}^*$ (where the fibre over 0 $\in \mathfrak{h}^*$ is identified with $Y(\lambda)$).

Denote by $\tilde{Y}_i(\lambda) \to \text{span}(\omega_i) \subset \mathfrak{h}^*$ the restriction of $\tilde{Y}(\lambda)$ to $\text{span}(\omega_i)$ (this is a one parameter deformation of $Y(\lambda)$).

Remark 2.1. In practice we only need the first order deformation of $\tilde{Y}(\lambda)$, but in geometric examples there exists a natural deformation over $\mathfrak{h}^*$. Replacing this deformation by the corresponding first order deformation does not change the results and arguments in the rest of the paper.

Similarly, one can replace $\mathfrak{h}^*$ by some abstract smooth base of the same dimension, $\text{span}(\omega_i) \subset \mathfrak{h}^*$ by one-dimensional subvarieties etc. But we use $\mathfrak{h}^*$ to keep notation simpler and because in many examples the base is naturally isomorphic to $\mathfrak{h}^*$.
On this data we impose the following conditions.

(i) Each (graded piece of the) Hom space between two objects in \( D(Y(\lambda)) \) is finite dimensional.

In particular, this means that \( \text{End}(O_{Y(\lambda)}) = k \cdot I \).

(ii) All \( \mathcal{E}^{(r)} \)'s and \( \mathcal{F}^{(r)} \)'s are sheaves (i.e. complexes supported in cohomological degree zero).

(iii) \( \mathcal{E}_i^{(r)}(\lambda) \) and \( \mathcal{F}_i^{(r)}(\lambda) \) are left and right adjoints of each other up to shift. More precisely

(a) \( \mathcal{E}_i^{(r)}(\lambda)_R = \mathcal{F}_i^{(r)}(\lambda)[r(\langle \lambda, \alpha_i \rangle + r)] \)

(b) \( \mathcal{E}_i^{(r)}(\lambda)_L = \mathcal{F}_i^{(r)}(\lambda)[-r(\langle \lambda, \alpha_i \rangle + r)] \).

(iv) For each \( i \in I \),

\[
\mathcal{H}^*(\mathcal{E}_i \ast \mathcal{E}_i^{(r)}) \cong \mathcal{E}_i^{(r+1)} \otimes_k H^*(\mathbb{P}^r).
\]

(v) If \( \langle \lambda, \alpha_i \rangle \leq 0 \) then

\[
\mathcal{F}_i(\lambda) \ast \mathcal{E}_i(\lambda) \cong \mathcal{E}_i(\lambda - \alpha_i) \ast \mathcal{F}_i(\lambda - \alpha_i) \oplus \mathcal{P}
\]

where \( \mathcal{H}^*(\mathcal{P}) \cong O_\Delta \otimes_k H^*(\mathbb{P}^{-(\lambda, \alpha_i)\cdot i}). \)

Similarly, if \( \langle \lambda, \alpha_i \rangle \geq 0 \) then

\[
\mathcal{E}_i(\lambda - \alpha_i) \ast \mathcal{F}_i(\lambda - \alpha_i) \cong \mathcal{F}_i(\lambda) \ast \mathcal{E}_i(\lambda) \oplus \mathcal{P}'
\]

where \( \mathcal{H}^*(\mathcal{P}') \cong O_\Delta \otimes_k H^*(\mathbb{P}^{(\lambda, \alpha_i)\cdot i}). \)

(vi) We have

\[
\mathcal{H}^*(i_{123*} \mathcal{E}_i \ast i_{123*} \mathcal{E}_i) \cong \mathcal{E}_i^{(2)}[-1] \oplus \mathcal{E}_i^{(2)}[2]
\]

where \( i_{12} \) and \( i_{23} \) are the closed immersions

\[
i_{12} : Y(\lambda) \times Y(\lambda + \alpha_i) \rightarrow Y(\lambda) \times \tilde{Y}(\lambda + \alpha_i)
\]

\[
i_{23} : Y(\lambda + \alpha_i) \times Y(\lambda + 2\alpha_i) \rightarrow \tilde{Y}(\lambda + \alpha_i) \times Y(\lambda + 2\alpha_i).
\]

(vii) If \( \langle \lambda, \alpha_i \rangle \leq 0 \) and \( k \geq 1 \) then the image of \( \text{supp}(\mathcal{E}^{(r)}(\lambda - r\alpha_i)) \) under the projection to \( Y(\lambda) \) is not contained in the image of \( \text{supp}(\mathcal{E}^{(r+k)}(\lambda - (r+k)\alpha_i)) \) also under the projection to \( Y(\lambda) \). Similarly, if \( \langle \lambda, \alpha_i \rangle \geq 0 \) and \( k \geq 1 \) then the image of \( \text{supp}(\mathcal{E}^{(r)}(\lambda)) \) in \( Y(\lambda) \) is not contained in the image of \( \text{supp}(\mathcal{E}^{(r+k)}(\lambda)) \).

(viii) If \( i \neq j \in I \) are joined by an edge in \( \Gamma \) then

\[
\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i \cong \mathcal{E}_i^{(2)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(2)}
\]

while if they are not joined then \( \mathcal{E}_i \ast \mathcal{E}_j \cong \mathcal{E}_j \ast \mathcal{E}_i \).

(ix) If \( i \neq j \in I \) then \( \mathcal{F}_j \ast \mathcal{E}_i \cong \mathcal{E}_i \ast \mathcal{F}_j \).

(x) For \( i \in I \) the sheaf \( \mathcal{E}_i \) deforms over \( \mathcal{A}_i \) to some

\[
\mathcal{E}_i \in D(\tilde{Y}(\lambda)_{|\mathcal{A}_i} \times \tilde{Y}(\mathcal{A}_i)_{|\mathcal{A}_i}).
\]

(xi) If \( i \neq j \in I \) are joined by an edge, by Lemma 4.5, there exists a unique non-zero map (up to multiple) \( T_{ij} : \mathcal{E}_i \ast \mathcal{E}_j[-1] \rightarrow \mathcal{E}_j \ast \mathcal{E}_i \) whose cone we denote

\[
\mathcal{E}_{ij} := \text{Cone} \left( \mathcal{E}_i \ast \mathcal{E}_j[-1] \xrightarrow{T_{ij}} \mathcal{E}_j \ast \mathcal{E}_i \right) \in D(Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_j))
\]

Then \( \mathcal{E}_{ij} \) deforms over \( B := (\alpha_i + \alpha_j)^{1} \subset \mathfrak{h}^{\ast} \) to some

\[
\mathcal{E}_{ij} \in D(\tilde{Y}(\lambda)_{|B} \times_B \tilde{Y}(\lambda + \alpha_i + \alpha_j)_{|B}).
\]
Remark 2.2. The conditions (i), (ii), (iii), (vii) are technical conditions. The conditions (iv), (v), (viii), (ix) are categorical versions of the relations in the usual presentation of the Kac-Moody Lie algebra $\mathfrak{g}$ (except as in [CKL2], we only impose parts of (iv), (v) at the level of cohomology which is much easier to check). The conditions (vi), (x) and (xi) relate to the deformation.

Notice that conditions (i) - (vii) are precisely equivalent to saying that $\{Y(\lambda + n\alpha_i)\}_{n \in \mathbb{Z}}$, together with $\mathcal{E}_i$ and $\mathcal{F}_i$ and deformations $\tilde{Y}_i(\lambda + n\alpha_i)$ generate a geometric categorical $\mathfrak{sl}_2$ action. Relations (viii) - (xi) then describe how these various $\mathfrak{sl}_2$ actions are related.

One can compare the geometric definition above to the notion of a 2-representation of $\mathfrak{g}$ in the sense of Rouquier [Ro2], which in turn is very similar to the notion of an action of Khovanov-Lauda’s 2-category [KL]. In these definitions, there are functors $\mathcal{E}_i, \mathcal{F}_i$ as well as some natural transformations $X, T$ between these functors. The additional data of our deformations is perhaps equivalent to the additional deformation of these natural transformations. In the case of $\mathfrak{g} = \mathfrak{sl}_2$, we were able to make this connection precise (see [CKL2]). For general $\mathfrak{g}$, it remains an open problem to show that a geometric categorical $\mathfrak{g}$ action gives an action of Khovanov-Lauda or Rouquier’s 2-category. In any case, we work here with the above definition since these axioms can be checked in examples (as in section 3).

Remark 2.3. Since $\mathcal{E}_i, \mathcal{F}_i$ are biadjoint (up to shift), the conditions (iv), (vi) and (viii) immediately imply the same conditions where all $\mathcal{E}_i$ are replaced by $\mathcal{F}_i$.

Finally, we define two more maps we will use repeatedly. The first map is

$$\mathcal{E}_i^{(r+1)} \to \mathcal{E}_i \ast \mathcal{E}_i^{(r)}[-r] \cong \mathcal{E}_i^{(r)} \ast \mathcal{E}_i[-r]$$

and $\mathcal{F}_i^{(r+1)} \to \mathcal{F}_i \ast \mathcal{F}_i^{(r)}[-r] \cong \mathcal{F}_i^{(r)} \ast \mathcal{F}_i[-r]$ which includes into the lowest degree summand of the right hand side. Notice that there is a unique such map (up to multiple) because by Lemma 4.5 we have that $\text{End}^k(\mathcal{E}_i^{(r+1)})$ zero if $k < 0$ and one-dimensional if $k = 0$ (similarly with $\text{End}^k(\mathcal{F}_i^{(r+1)})$).

The second map is

$$\mathcal{F}_i^{(r)} \ast \mathcal{E}_i^{(r)}(\lambda) \to \mathcal{O}_\Delta[\langle (\lambda, \alpha_i + r) \rangle]$$

and $\mathcal{E}_i^{(r)} \ast \mathcal{F}_i^{(r)}(\lambda) \to \mathcal{O}_\Delta[-\langle (\lambda, \alpha_i + r) \rangle]$ given by adjunction (by definition $\mathcal{E}^{(r)}$ and $\mathcal{F}^{(r)}$ are adjoint to each other up to shifts). This map is also uniquely defined (up to multiple).

We say that a geometric categorical $\mathfrak{g}$-action is integrable if for every weight $\lambda$ and $i \in I$ we have $Y(\lambda + n\alpha_i) = 0$ for $n \gg 0$ or $n \ll 0$. From hereon we assume all actions are integrable.

2.3. Role of the deformation. Let us briefly discuss the role of the deformation. In general if $Y$ is a variety, $Y \to \mathbb{A}^1$ is a 1-parameter deformation, and $\mathcal{A} \in D(Y)$, then the obstruction to deforming $\mathcal{A}$ is given by a map $c(\mathcal{A}) : \mathcal{A} \to \mathcal{A}[2]$. By definition, $c(\mathcal{A})$ is the connecting map coming from the exact triangle $\mathcal{A}[1] \to i^* i_* \mathcal{A} \to \mathcal{A}$, where $i$ is the inclusion $i : Y \to \tilde{Y}$ (see [HT, appendix]).

In a geometric categorical $\mathfrak{g}$ action, we have a deformation $\tilde{Y}(\lambda) \to \mathfrak{h}^*$. Hence given $v \in \mathfrak{h}^*$, we can consider the deformation $\tilde{Y}(\lambda) \times_{\text{span}(v)} Y(\lambda + \alpha_i)$. This gives us a linear map $c(\mathcal{E}_i) : \mathfrak{h}^* \to \text{Hom}(\mathcal{E}_i, \mathcal{E}_i[2])$ (linearity follows from results of [HT]). Condition (x) implies that $c(v, \mathcal{E}_i) = 0$ for $v \in \alpha_i \perp$. Similarly, condition (xi) implies that $c(v, \mathcal{E}_{ij}) = 0$ for $v \in (\alpha_i + \alpha_j) \perp$.

On the other hand, condition (vi) implies that $c(\omega_i, \mathcal{E}_i)$ is non-zero. To see why consider the connecting map $\text{Ic}(\mathcal{E}_i) : \mathcal{E}_i \ast \mathcal{E}_i[-1] \to \mathcal{E}_i \ast \mathcal{E}_i[1]$ where $\text{Ic}(\mathcal{E}_i)$ is the connecting map for the deformation $Y(\lambda) \times Y(\lambda + \alpha_i) \to Y(\lambda) \times \tilde{Y}(\lambda + \alpha_i)$. We will examine this map on the level of cohomology.

Let $i_{12}, i_{23}$ be as in (vi). By formal properties of composition of kernels we have $\mathcal{E}_i \ast i_{12}^* \mathcal{E}_i \cong i_{23}^* \mathcal{E}_i \ast i_{12} \mathcal{E}_i$ (see [CKL2, Lemma 4.1]) and hence (vi) implies that

$$\mathcal{H}^* (\text{Cone}(\text{Ic}(\mathcal{E}_i))) \cong \mathcal{E}_i^{(2)}[-1] \oplus \mathcal{E}_i^{(2)}$$.
This means that on cohomology $Ic(\mathcal{E}_i)$ induces an isomorphism
$$\mathcal{E}_i^{(2)} \cong \mathcal{E}_i(\mathcal{E}_i \ast \mathcal{E}_i[-1]) \cong \mathcal{E}_i(\mathcal{E}_i \ast [1]) \cong \mathcal{E}_i^{(2)}$$
which shows that $Ic(\mathcal{E}_i)$ is non-zero.

Although the deformation $Y(\lambda) \times Y(\lambda + \alpha_i) \to Y(\lambda) \times Y(\lambda + \alpha_i)$ is different than the deformation
$Y(\lambda) \times \text{span}\omega_i Y_i(\lambda + \alpha_i)$, the induced map on the cohomology of $\mathcal{E}_i \ast \mathcal{E}_i[-1]$ is the same. This is because the deformation $Y(\lambda) \times Y(\lambda + \alpha_i) \to Y(\lambda) \times Y(\lambda + \alpha_i)$ induces zero at the level of cohomology and by
linearity the map induced by $c(\omega_i, \mathcal{E}_i)$ is the sum of this map and the map induced by $Ic(\mathcal{E}_i)$. Hence $c(\omega_i, \mathcal{E}_i)$ is also non-zero and induces the above isomorphism on cohomology.

More generally, if $v \in h^*$ is any vector, then $c(v, \mathcal{E}_i) : \mathcal{E}_i[-1] \to \mathcal{E}_i[1]$ will be non-zero if and only if $\langle v, \alpha_i \rangle$ is non-zero and in this case $Ic(v, \mathcal{E}_i) : \mathcal{E}_i \ast \mathcal{E}_i[1] \to \mathcal{E}_i \ast \mathcal{E}_i[1]$ will induce the above isomorphism on the cohomology.

Remark 2.4. For all practical purposes we do not actually need that $\mathcal{E}_i$ deforms over all of $\alpha_i^\perp$ and $\mathcal{E}_{ij}$ deforms over all of $(\alpha_i + \alpha_j)^\perp$. We just need them to deform over some one-dimensional subspaces.

We use the definition above since it is satisfied in all examples we know and, in our opinion, is more aesthetically pleasing.

2.4. Example from resolutions of Kleinian singularities. An instructive example of geometric categorical $g$ action comes from the minimal resolution of a Kleinian singularity. Let $H$ denote a finite subgroup of $SL_2(\mathbb{C})$ and let $\pi : Y \to \mathbb{C}^n/H$ be a minimal resolution. Recall that $H$ determines a finite type simply-laced Dynkin diagram $\Gamma$ whose vertex set $I$ is in bijection with the components of the exceptional fibre $\pi^{-1}(0)$.

From the work of Seidel-Thomas [ST], we know that each component $E_i$ of $\pi^{-1}(0)$ determines a spherical object $S_i = O_{E_i}(-1)$. These $S_i$ form a type $\Gamma$ arrangement of spherical object and thus by the work of Seidel-Thomas give an action of the braid group $B_\gamma$ on $D(Y)$ (as usual, here $\gamma$ is the Lie algebra associated to $\Gamma$).

Let us use the same data to construct a geometric categorical $g$ action. Let $Y(\lambda)$ be defined as follows. Let $Y(0) = Y$, $Y(\lambda) = \text{pt}$ for $\lambda$ a root of $g$, and $Y(\lambda) = \emptyset$ for all other $\lambda$. Then we define $E_i(0) : D(Y) \to D(\text{pt})$ using the kernel $S_i \subset D(Y \times \text{pt})$ and similarly with $E_i(-\alpha_i)$, $\mathcal{F}_i(0)$ and $\mathcal{F}_i(-\alpha_i)$ (all other $E_i, \mathcal{F}_i$ we need to define are functors $D(\text{pt}) \to D(\text{pt})$ which we take to be the identity). The deformation $\tilde{Y}$ of $Y$ is the standard deformation (which may be constructed by thinking of $\mathbb{C}^n/H$ as a Slodowy slice or by deforming the polynomial defining the singularity $\mathbb{C}^n/H$).

Let us check condition (viii) of the geometric categorical $\mathfrak{sl}_n$ action (all other conditions are immediate or follow along the same lines). Let $i, j$ be connected by an edge in $\Gamma$ (so $E_i, E_j$ intersect in a point). Then condition (viii) states that
$$E_i(\alpha_j) \ast E_j(0) \ast E_i(-\alpha_i) \cong E_j(\alpha_i) \ast E_i^{(2)}(-\alpha_i) \ast E_i^{(2)}(-\alpha_i + \alpha_j) \ast E_j(-\alpha_i).$$

Now $Y(-\alpha_i + \alpha_j) = \emptyset$ while $E_i^{(2)}(-\alpha_i) = O_{\Delta_{\alpha_i}} = E_i(\alpha_j)$. So we see that this is equivalent to the fact that the composition
$$D(\text{pt}) \xrightarrow{E_i(-\alpha_i)} D(Y) \xrightarrow{E_i^{(2)}} D(\text{pt})$$
is the identity. But this is precisely the condition relating two adjacent spherical objects in an $\Gamma$ arrangement, namely that $\text{Ext}^1(S_i, S_{i+1}) = \mathbb{C}$.

2.5. $\mathbb{C}^\times$-equivariance. There is a slightly more general $\mathbb{C}^\times$-equivariant version of a geometric categorical $g$ action. In this setting every variety $Y(\lambda)$ and deformation $\tilde{Y}(\lambda)$ is equipped with a $\mathbb{C}^\times$-action. The action on $\tilde{Y}(\lambda)$ is equivariant with respect to a weight 2 action on the base $h^*$. Subsequently, each $D(Y(\lambda))$ becomes the derived category of $\mathbb{C}^\times$-equivariant coherent sheaves. This extra $\mathbb{C}^\times$-structure gives us another grading on our categories which we denote by $\{\}$.
The conditions on the data are essentially the same. The adjoint conditions become

\((i)\) \(E^{(r)}_i(\lambda)R = F^{(r)}_i(\lambda)[r((\lambda, \alpha_i) + r)] \{-r((\lambda, \alpha_i) + r)\}\)

\((\text{ii})\) \(E^{(r)}_i(\lambda)L = F^{(r)}_i(\lambda)[-r((\lambda, \alpha_i) + r)] \{r((\lambda, \alpha_i) + r)\}\).

Condition \((vi)\) on composition of deformed \(E_s\) becomes

\[\mathcal{H}^{*}(i_{23}, \mathcal{E}_i \ast i_{12}, \mathcal{E}_i) \cong \mathcal{E}^{(2)}_i[-1]\{1\} \oplus \mathcal{E}^{(2)}_i[2\{-3\}\right]

while \(\mathcal{E}_{ij}\) is defined as the cone of \(\mathcal{E}_i \ast \mathcal{E}_j[-1]\{1\} \rightarrow \mathcal{E}_j \ast \mathcal{E}_i\). All the other conditions are the same once we replace \(H^{*}(\mathbb{P}^n)\) by the doubly graded version

\[H^{*}(\mathbb{P}^n) := \mathbb{C}\{-2\}\{n\} \oplus \mathbb{C}\{-2\}\{n\} \oplus \cdots \oplus \mathbb{C}\{n\}\{-n\}\right].\]

All the results in this paper have natural \(\mathbb{C}^\times\)-equivariant analogues. However, we will not work \(\mathbb{C}^\times\)-equivariantly because keeping track of the extra \(\{\cdot\}\) shifts would make the notation hard to read. One of the reasons to even consider this \(\mathbb{C}^\times\)-equivariant setup is that it shows up naturally in various examples. The cotangent bundles of partial flag varieties considered in section 3 is one such example.

2.6. Equivalences via geometric categorical \(\mathfrak{sl}_2\) actions. In [CKL2] we proved that a geometric categorical \(\mathfrak{sl}_2\) action induces a strong categorical \(\mathfrak{sl}_2\) action. In [CKL3] we showed that a strong categorical \(\mathfrak{sl}_2\) action can be used to construct equivalences (using the ideas of Chuang-Rouquier [CR]). We briefly review this construction starting from a categorical \(\mathfrak{g}\) action.

Given a geometric categorical \(\mathfrak{g}\) action one can construct for each vertex \(i \in I\) a geometric categorical \(\mathfrak{sl}_2\) action. More precisely, we use as kernels \(E_i^{(r)}\) and \(F_i^{(r)}\) and use the one parameter deformation of \(\mathcal{Y}_i(\lambda)\). Consequently by the main result of [CKL2], we obtains a strong \(\mathfrak{sl}_2\) action generated by the functors induced by the kernels \(E_i\) and \(F_i\).

Consider for each \(s\) (such that \(s \geq 0\) and \(s + \langle \lambda, \alpha_i \rangle \geq 0\)) the kernel

\[T_i^s(\lambda) := F_i^{((\lambda, \alpha_i) + s)} \ast E_i^{(s)}(\lambda)[-s] \in D(Y(\lambda) \times Y(\lambda - \langle \lambda, \alpha_i \rangle \alpha_i))\].

There exists a natural map \(d_i^s : T_i^s(\lambda) \rightarrow T_i^{s-1}(\lambda)\) given as the composition

\[d_i^s = F_i^{((\lambda, \alpha_i) + s)} \ast E_i^{(s)}[s] \ast E_i^{((\lambda, \alpha_i) + s)} \ast E_i^{(s-1)}[-(s+1)]\rightarrow T_i^{s-1}(\lambda)\].

Note that \(T_i^0(\lambda) = 0\) for \(s \gg 0\) since we only deal with integrable representations. The main result of [CKL3] is the following.

Theorem 2.5.

\[\cdots \rightarrow T_i^s(\lambda) \xrightarrow{d_i^s} T_i^{s-1}(\lambda) \xrightarrow{d_i^{s-1}} \cdots \xrightarrow{d_i^1} T_i^0(\lambda)\]

is a complex of kernels which has a unique (left) convolution denoted \(T_i(\lambda)\). Moreover, the kernel \(T_i(\lambda)\) induces an equivalence of triangulated categories \(D(Y(\lambda)) \xrightarrow{\sim} D(Y(\lambda - \langle \lambda, \alpha_i \rangle \alpha_i))\).

2.7. Braid group action via geometric categorical \(\mathfrak{g}\) actions. As noted in the previous section, each vertex \(i \in I\) induces a geometric categorical \(\mathfrak{sl}_2\) action and subsequently an equivalence \(T_i\) (or, more precisely, a series of equivalences \(T_i(\lambda)\), one for each weight \(\lambda\)). The main result of this paper is to prove that these equivalences braid.

Theorem 2.6. If \(i, j \in I\) are joined by an edge then the corresponding equivalences \(T_i\) and \(T_j\) satisfy the braiding relation \(T_i \ast T_j \ast T_i \cong T_j \ast T_i \ast T_j\). If \(i, j \in I\) are not joined by an edge then \(T_i \ast T_j \cong T_j \ast T_i\).

Hence there is an action of the braid group of type \(\mathfrak{g}\) on \(D(\mathcal{Y}(\lambda))\) compatible with the action of the Weyl group on the weight lattice.
Example 2.7. As a simple application of this theorem, we can consider the minimal resolution $Y$ of the Kleinian singularity $\mathbb{C}^2/H$. In section 2.4, we explained that $Y = Y(0)$ was the 0 weight space of a geometric categorical $g$ action. Hence by Theorem 2.6, we obtain an action of the braid group $B_g$ on $D(Y)$. As mentioned earlier, such an action was previously studied by Seidel-Thomas [ST]. A more substantial application will be given in the next section.

3. Example: Cotangent bundles to flag varieties

Before we prove Theorem 2.6 we would like to illustrate a geometric categorical $\mathfrak{sl}_n$ action on the $\mathbb{C}^\times$-equivariant derived category of coherent sheaves on the cotangent bundle to partial flag varieties (Theorem 3.1). We work $\mathbb{C}^\times$ equivariantly in order to have condition (i) hold (if not, $\text{End}(\mathcal{O}_{Y(\lambda)}) \cong H^0(\mathcal{O}_{Y(\lambda)})$ will be infinite dimensional).

3.1. The categorical $g$ action. Fix integers $n \leq N$. We consider the variety $Fl_n(\mathbb{C}^N)$ of $n$-step flags in $\mathbb{C}^N$. This variety has many connected components, which are indexed by the possible dimensions of the spaces in the flags. In particular, let

$$C(n, N) := \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^N : \lambda_1 + \cdots + \lambda_n = N \}$$

For $\lambda \in C(n, N)$, we can consider the variety of $n$-steps flags where the jumps are given by $\lambda$:

$$Fl_\lambda(\mathbb{C}^N) := \{ 0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^N : \dim V_i/V_{i-1} = \lambda_i \}$$

Let $Y(\lambda) = T^*Fl_\lambda(\mathbb{C}^N)$. These will be our varieties for the geometric categorical $\mathfrak{sl}_n$ action. We regard each $\lambda$ as a weight for $\mathfrak{sl}_n$, via the identification of the weight lattice of $\mathfrak{sl}_n$ with the quotient $\mathbb{Z}^n/(1, \ldots, 1)$. For compatibility with [CKL4], we choose the convention that the simple roots $\alpha_i$ are equal to $(0, \ldots, 0, -1, 1, 0, \ldots, 0)$ where the $-1$ is in position $i$.

We will make use of the following description of the cotangent bundle to the partial flag varieties.

$$Y(\lambda) = \{(X, V) : X \in \text{End}(\mathbb{C}^N), V \in Fl_\lambda(\mathbb{C}^N), XV_i \subset V_{i-1} \}$$

This description immediately leads to the following deformations of $Y(\lambda)$ over $\mathbb{C}^n$

$$\hat{Y}(\lambda) := \{(X, V, x) : X \in \text{End}(\mathbb{C}^N), V \in Fl_\lambda(\mathbb{C}^N), x \in \mathbb{C}^n, XV_i \subset V_i, X|_{V_i/V_{i-1}} = x_i \cdot \text{id} \}.$$

In more Lie-theoretic terms, $Fl_\lambda(\mathbb{C}^N)$ is the variety of parabolic subalgebras $p$ of $\mathfrak{gl}_N$ of type $\lambda$. $Y(\lambda)$ is the variety of pairs $(X, p)$ where $X \in \mathfrak{gl}_N$, $p$ is a parabolic subalgebra of type $\lambda$ and $X$ is in the nilradical of $p$. Finally $\hat{Y}(\lambda)$ is the variety of triples $(X, p, x)$ where $X$ is in $p$ and its image in the Levi of $p$ is the central element $x$.

We will restrict our deformation over the locus $\{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_n = 0 \}$ which we identify with $\mathfrak{h}^*$ the dual of the Cartan for $\mathfrak{sl}_n$.

We define an action of $\mathbb{C}^\times$ on $\hat{Y}(\lambda)$ by $t \cdot (X, V, x) = (t^2X, V, t^2x)$. Restricting to $Y(\lambda) = T^*Fl_\lambda(\mathbb{C}^N)$ this corresponds to a trivial action on the base and a scaling of the fibres.

To construct the kernels $\mathcal{E}_i^{(r)}$, we consider correspondences $W_i^r$. More specifically, let $\lambda, i, r$ be such that $\lambda \in C(n, N)$ and $\lambda + r\alpha_i \in C(n, N)$ (ie $\lambda_i \geq r$). Then we define $W_i^r(\lambda) := \{(X, V, V') : (X, V) \in Y(\lambda), (X, V') \in Y(\lambda + r\alpha_i), V_j = V'_j \text{ for } j \neq i, \text{ and } V' \subset V_i \}$

From this correspondence we define the kernel

$$\mathcal{E}_i^{(r)}(\lambda) = \mathcal{O}_{W_i^r(\lambda)} \otimes \text{det}(V_{i+1}/V_i)^{-r} \otimes \text{det}(V'_i/V_{i-1})^{r(\lambda_i+1)} \in D(Y(\lambda) \times Y(\lambda + r\alpha_i))$$

where $\{\cdot\}$ denotes an equivariant shift and, abusing notation, $V_i$ denotes the vector bundle on $Y(\lambda)$ whose fibre over $(X, V) \in Y(\lambda)$ is naturally identified with $V_i$. Similarly, we define the kernel

$$\mathcal{F}_i^{(r)}(\lambda) = \mathcal{O}_{W_i^r(\lambda)} \otimes \text{det}(V'/V_i)^{\lambda_i+1 - \lambda_i - r} \{r\lambda_i - r^2\} \in D(Y(\lambda + r\alpha_i) \times Y(\lambda))$$

where we now regard $W_i^r(\lambda)$ as a subvariety of $Y(\lambda + r\alpha_i) \times Y(\lambda)$. 


Lemma 3.5. Let

Proof.

3.2. The braid group action. As a corollary of this theorem and the main result of this paper (Theorem 2.6), we obtain the following.

Theorem 3.2. There is an action of the braid group $B_n$ on the derived category of coherent sheaves on $T^*Fl_n(\mathbb{C}^N)$. This action is compatible with the action of $S_n$ on the set of connected components $C(n, N)$.

In particular, if $N = dn$ for some integer $d$ and we choose $\lambda = (d, \ldots, d)$, then we obtain an action of the braid group of the derived category of coherent sheaves on the connected variety $T^*Fl_\lambda(\mathbb{C}^N)$.

Example 3.3. Consider the case $n = N$. Let $T^*(Fl(\mathbb{C}^n))$ denote the cotangent bundle to the full flag variety. We have constructed an action of the braid group $B_n$ on $D(T^*(Fl(\mathbb{C}^n)))$. Such an action was previously constructed by Khovanov-Thomas [KT] and by Riche [Ric], Bezrukavnikov-Mirkovic-Rumynin [BMR]. Their work served as motivation for this paper. In this case, the generators of the braid group act by spherical twists (see [CKL3, section 2.5]). This is the simplest case of our result, since in general the equivalences which generate the braid group action are given by more complicated complexes than spherical twists.

Even though the construction of each equivalence

via a categorical $\mathfrak{sl}_2$ action is indirect the kernels one obtains are fairly concrete. More precisely, the kernel $T_i$, which induces the functor $T_i$, is always a sheaf supported on the variety

$$Z_i(\lambda) := \{(X, V, V') : X \in \text{End}(\mathbb{C}^N), V \in Fl_\lambda(\mathbb{C}^N), V' \in Fl_{\lambda-(\lambda, \alpha_i)\alpha_i}(\mathbb{C}^N)$$

$$XV_j \subset V_{j-1}, XV_j' \subset V_{j-1}' \text{ and } V_j = V_j' \text{ if } j \neq i\}.$$ 

In general $T_i$ is not the structure sheaf of $Z_i(\lambda)$ but rather some rank one Cohen-Macaulay sheaf on $Z_i(\lambda)$. In [C] we give a concrete description of this sheaf in the Grassmannian case ($n = 2$ case). A similar description of $T_i$ is possible in general.

In the rest of this section we prove Theorem 3.1.

3.3. Proof of $\mathfrak{sl}_2$ conditions (i) - (vii). In [CKL2] (based on the computations in [CKL1]) we proved the $\mathfrak{sl}_2$ relations (i) - (vii) hold for cotangent bundles to Grassmannians (i.e. the case $n = 2$). The same proof with virtually no changes necessary applies to prove these relations for any $n$.

As an example, we will check the adjunction relations (iii). We begin by computing some canonical bundles.

Lemma 3.4. $\omega_{T^*Fl(\mathbb{C}^N)} \cong \mathcal{O}_{T^*Fl(\mathbb{C}^N)} \{-2\sum_{i \neq j} \lambda_i \lambda_j\}$

Proof. The variety $T^*Fl_\lambda(\mathbb{C}^N)$ is symplectic (since it is a cotangent bundle) and the symplectic form has weight 2 for the $\mathbb{C}^\times$ action. Hence the $d$th wedge power of the symplectic form gives a non-vanishing section of the canonical bundle, where $d$ is the dimension of $Fl_\lambda(\mathbb{C}^N)$. Since $d = \sum_{i \neq j} \lambda_i \lambda_j$, the result follows.

Lemma 3.5. $\omega_{W_i^{(r)}}(\lambda) \cong \text{det}(V_{i+1}/V_i)^{-r} \text{det}(V_i/V_i')^{-\lambda_i + \lambda_{i+1} + r} \text{det}(V_i'/V_{i-1})^r \{-2\sum_{i \neq j} \lambda_i \lambda_j + 2\lambda_{i+1}r\}$

Proof. Let $\mu := (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - r, r, \lambda_{i+1}, \ldots, \lambda_n)$ and let $T$ denote the variety

$$T := \{(X, U) : (X, U) \in Y(\mu), Xu_{i+1} \subset u_{i-1}\}$$

We can view $W_i^{(r)}$ as a subvariety of $T$, by setting $U_j = V_j$ for $j < i$, $U_i = V_i'$, $u_{i+1} = V_i$ and $u_j = V_{i-1}$ for $j > i + 1$. Moreover we can see that $W_i^{(r)}$ is carved out of $T$ as the vanishing locus
of $X : U_{i+2}/U_{i+1} \to U_{i+1}/U_i \{2\}$. Thus $W^{(r)}_i$ is cut out of $T$ be a section of the vector bundle $(U_{i+2}/U_{i+1})^\vee \otimes U_{i+1}/U_i \{2\}$.

Similarly, $T$ is cut out of $T^* F L^\mu(\mathbb{C}^N)$ by a section of $(U_{i+1}/U_i)^\vee \otimes U_i/\{2\}$. Thus

$$\omega^{(r)}_i \equiv \det((U_{i+2}/U_{i+1})^\vee \otimes U_{i+1}/U_i \{2\})^\vee \otimes \det((U_{i+1}/U_i)^\vee \otimes U_i/\{2\})^\vee \otimes \omega_{T^* F L^\mu(\mathbb{C}^N)}.$$  Combining all this with our previous calculation of $\omega_{T^* F L^\mu(\mathbb{C}^N)}$ we obtain the desired result. \hfill \Box

We will now give the proof of the first adjunction statement. The other proofs are similar.

**Corollary 3.6.** $\mathcal{E}^{(r)}_i(\lambda)_R \cong \mathcal{F}^{(r)}_i(\lambda)[r(\lambda_{i+1} - \lambda_i) + r^2] \{-r(\lambda_{i+1} - \lambda_i) + r^2\}$

**Proof.** We have

$$\mathcal{E}^{(r)}_i(\lambda)_R \cong \mathcal{E}^{(r)}_i \otimes \pi_2^* \omega_Y(\lambda)[\dim Y(\lambda)]$$

$$\cong \mathcal{O}^{(r)} W^{(r)}_i \otimes \det(V_{i+1}/V_i)^\vee \det(V_i/V_{i-1})^{-r} \{-r\lambda_{i+1}\} \otimes \pi_2^* \omega_Y(\lambda)[\dim Y(\lambda)]$$

$$\cong \omega^{(r)}_i \otimes \det(V_{i+1}/V_i)^\vee \det(V_i/V_{i-1})^{-r} \{-2 \sum_{i \neq j} \lambda_i \lambda_j + 2 \lambda_{i+1} r\} \otimes \pi_1^* \omega_Y(\lambda + \{\lambda_{i+1}\}) [\dim Y(\lambda) - \dim W^{(r)}_i(\lambda)]$$

$$\cong \mathcal{O}^{(r)} W^{(r)}_i \otimes \det(V_{i+1}/V_i)^\vee \det(V_i/V_{i-1})^{-r} \{-r\lambda_{i+1}\}$$

$$\cong \mathcal{F}^{(r)}_i(\lambda)[r(\lambda_{i+1} - \lambda_i) + r^2] \{-r(\lambda_{i+1} - \lambda_i) + r^2\}$$

as desired. \hfill \Box

**3.4. Proof of Serre relation (viii).** Since we are in the Lie algebra $\mathfrak{sl}_n$, having $i, j \in I$ joined by an edge is equivalent to $j = i \pm 1$. So let us consider $j = i + 1$ (the case $j = i - 1$ is the same) We will show that

$$\mathcal{E}_i \ast \mathcal{E}_{i+1} \ast \mathcal{E}_i = \mathcal{E}^{(2)}_i \ast \mathcal{E}_{i+1} \ast \mathcal{E}_i \ast \mathcal{E}^{(2)}_i.$$  

Here is the outline of the proof. On the left hand side computing $\mathcal{E}_i \ast \mathcal{E}_{i+1}$ is straight-forward, meaning that intersections are of the expected dimension and the pushforward is one-to-one. The intersection when computing $\mathcal{E}_i \ast \mathcal{E}_{i+1} \ast \mathcal{E}_i$ is also of the expected dimension but contains two components $A$ and $B$. Pushing forward by $\pi_{12}$ then gives us two terms (one for each component) which are equal to $\mathcal{E}^{(2)}_i \ast \mathcal{E}_{i+1}$ and $\mathcal{E}_{i+1} \ast \mathcal{E}^{(2)}_i$ (these are also easy to compute).

**Lemma 3.7.** We have

$$\mathcal{E}^{(2)}_i \ast \mathcal{E}_{i+1} \cong \mathcal{O}^{(2)} W^{(2)}_{i+1} \otimes \det(V_{i+2}/V_i)^{-1} \det(V_{i+1}/V_i)^{-1} \det(V_i/V_{i-1})^2 \{\lambda_{i+2} + 2\lambda_{i+1} - 2\}$$

$$\mathcal{E}_{i+1} \ast \mathcal{E}^{(2)}_i \cong \mathcal{O}^{(2)} W^{(2)}_{i+1} \otimes \det(V_{i+2}/V_i)^{-1} \det(V_{i+1}/V_i)^{-2} \det(V_i/V_{i-1}) \det(V_i/V_{i-1})^2 \{\lambda_{i+2} + 2\lambda_{i+1} - 2\}$$

where

$$W^{(2)}_{i+1} := \{(X, V, V') : V' \subset V_i \subset V_{i+1} \subset V_{i+1}, \text{ and } V_j = V'_j \text{ for } j \neq i, i + 1\}$$

$$W^{(2)}_{i+1} := \{(X, V, V') : V' \subset V_i, V_{i+1} \subset V_{i+1}, XV_{i+1} \subset V'_i, \text{ and } V_j = V'_j \text{ for } j \neq i, i + 1\}$$

inside $Y(\lambda) \times Y(\lambda + 2\alpha_i + \alpha_{i+1})$. 
Proof. Computing $\mathcal{E}_i^{(2)} \ast \mathcal{E}_{i+1}$ is easy since the intersection $\pi_{i2}^{-1}(W_{i+1}) \cap \pi_{23}^{-1}(W_i^{(2)})$ is of the expected dimension and the map $\pi_{13}$ maps this intersection one-to-one onto its image

$$\pi_{13}(\pi_{i2}^{-1}(W_{i+1}) \cap \pi_{23}^{-1}(W_i^{(2)})) \cong W_{i+1}.$$ 

So neither the tensor product nor the pushforward $\pi_{13}$ have lower or higher terms. Keeping track of the line bundles gives the result.

Computing $\mathcal{E}_{i+1} \ast \mathcal{E}_i^{(2)}$ is very similar.

**Lemma 3.8.** We have

$$\mathcal{E}_i \ast \mathcal{E}_{i+1} \cong \mathcal{O}_{W_{i+1}} \otimes \det(V_{i+2}/V_{i+1}) \gamma \det(V_i'/V_{i-1})\{-\lambda_i+2 - \lambda_{i+1} - 1\}$$

$$\mathcal{E}_{i+1} \ast \mathcal{E}_i \cong \mathcal{O}_{W_{i+1}} \otimes \det(V_{i+2}/V_i) \gamma \det(V_{i+1}/V_{i-1})\{\lambda_i+1 + \lambda_{i+2}\}$$

where

$$W_{i+1} := \{(X, V, V') : V_i' \subset V_i \subset V_{i+1} \subset V_{i+1}, \text{ and } V_j = V_j' \text{ for } j \neq i, i+1\}$$

$$W_{i+1} := \{(X, V, V') : V_{i+1}' \subset V_i \subset V_{i+1} \subset V_{i+1}, XV_{i+1} \subset V_i', \text{ and } V_j = V_j' \text{ for } j \neq i, i+1\}$$

inside $Y(\lambda + \alpha_i) \times Y(\lambda + 2\alpha_i + \alpha_{i+1})$ and $Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_{i+1})$ respectively.

Proof. At the level of sets

$$W_{i+1} \cong \pi_{13}(\pi_{i2}^{-1}(W_{i+1}) \cap \pi_{23}^{-1}(W_i))$$

where the intersection is transverse and the push forward is one-to-one. Hence in computing $\mathcal{E}_i \ast \mathcal{E}_{i+1}$ neither the tensor product nor the pushforward $\pi_{13}$ have lower or higher terms. Keeping track of the line bundles gives the result. Computing $\mathcal{E}_{i+1} \ast \mathcal{E}_i$ is very similar.

Notice that the varieties $W_{i+1}^{(2)}$, $W_{i+1}$, $W_{i+1}$ are all smooth. This is because each of them is a vector bundle over a iterated Grassmannian bundle. The fibre of these vector bundles is given by the $\mathcal{X}$ data and the base is given by the $V_i$, $V_i'$ data.

Now we can compute $(\mathcal{E}_i \ast \mathcal{E}_{i+1}) \ast \mathcal{E}_i$. We find that

$$\pi_{i2}^{-1}(W_i) \cap \pi_{23}^{-1}(W_{i+1}) = \{(X, V, V', V'') : V''_i \subset V_i \subset V_{i+1} \subset V_{i+1} = V_{i+1}, V_j = V_j' \text{ if } j \neq i, i+1\}.$$ 

This intersection is of the expected dimension but the push-forward under $\pi_{13}$ is only generically one-to-one. This variety has two components which we denote by $A$ and $B$ which are defined by

$$A := \{(X, V, V', V'') : V''_i \subset V_i \subset V_{i+1} \subset V_{i+1} = V_{i+1}, \text{ and } V_j = V_j' \text{ if } j \neq i, i+1\}$$

$$B := \{(X, V, V', V'') : V''_i \subset V_i \subset V_i' \subset V_{i+1} \subset V_{i+1} = V_{i+1}, XV_{i+1} \subset V_i'' \text{ and } V_j = V_j' \text{ if } j \neq i, i+1\}.$$ 

The varieties $A, B$ are smooth for the same reasons as explained above for $W_{i+1}^{(2)}$, $W_{i+1}^{(2)}$, $W_{i+1}$.

Keeping track of the line bundles shows that $(\mathcal{E}_i \ast \mathcal{E}_{i+1}) \ast \mathcal{E}_i \cong \pi_{13} \ast \mathcal{E}_i (\mathcal{O}_{A \cup B} \otimes \mathcal{L})$ where

$$\mathcal{L} := \det(V_{i+2}/V_i)^{-1} \det(V''_i/V_i') \det(V''_i/V_{i-1})^2 \{2\lambda_i + \lambda_{i+2}\}.$$ 

Let $E := A \cap B$. It is a divisor inside of each of $A, B$. Consider the standard short exact sequence

$$0 \to \mathcal{O}_A(-E) \oplus \mathcal{O}_B(-E) \to \mathcal{O}_{A \cup B} \to \mathcal{O}_E \to 0.$$ 

Now, $E$ is cut out of $A$ by a section of $\text{Hom}(V_{i+1}/V_{i+1}', V''_i/V''_i\{2\})$, namely the map $X : V_{i+1}/V_{i+1}' \to V''_i/V''_i\{2\}$. Hence

$$\mathcal{O}_A(-E) = \mathcal{O}_A \otimes (V_{i+1}/V_{i+1}') \otimes (V''_i/V''_i)^\gamma\{-2\}.$$ 

Similarly, $E$ is cut out of $B$ by a section of $\text{Hom}(V_i/V_i', V_i'/V_{i+1})$, namely the natural inclusion map $V_i/V_i' \to V_{i+1}/V_{i+1}'$. Hence we see that

$$\mathcal{O}_B(-E) = \mathcal{O}_B \otimes (V_i/V_i') \otimes (V_{i+1}/V_{i+1}')^\gamma.$$ 

Putting all this together, we obtain a distinguished triangle

$$(2) \quad \pi_{13} \ast (\mathcal{O}_A \otimes \mathcal{L}_A) \oplus \pi_{13} \ast (\mathcal{O}_B \otimes \mathcal{L}_B) \to \mathcal{E}_i \ast \mathcal{E}_{i+1} \ast \mathcal{E}_i \to \pi_{13} \ast (\mathcal{O}_E \otimes \mathcal{L})$$
where
\[
\mathcal{L}_A := \det(V_{i+2}/V_i)^{-1} \det(V_{i+1}/V_{i+2}) \det(V_i'/V_{i-1}) \{2\lambda_i + \lambda_{i+2} - 2\},
\]
\[
\mathcal{L}_B := \det(V_i/V_i'') \det(V_{i+1}/V_i') \det(V_{i+2}/V_i)^{-1} \det(V_{i+1}/V_{i+2})^{-1} \{2\lambda_i + \lambda_{i+2}\}.
\]
Now, we note that \(\pi_{13}(A) = W_{i'(2)i+1}\). The map \(\pi_{13}|A : A \to W_{i'(2)i+1}\) is generically one-to-one and \(\mathcal{L}_A\) is pulled back from \(\pi_{13}|A \) and hence
\[
\pi_{13*}(\mathcal{L}_A) \cong \mathcal{O}_{W_{i'(2)i+1}} \otimes \det(V_{i+2}/V_i)^{-1} \det(V_{i+1}/V_{i+2})^{-1} \{2\lambda_i + \lambda_{i+2}\}
\]
\[
\cong \mathcal{E}_i(2) \ast \mathcal{E}_{i+1}\]
A very similar argument shows that \(\pi_{13*}(\mathcal{O}_B \otimes \mathcal{L}_B) \cong \mathcal{E}_{i+1} \ast \mathcal{E}_{i}(2)\).
Finally, we see that \(\pi_{13}|E\) is a \(\mathbb{P}^1\) bundle. Moreover \(\mathcal{L}\) restricts to \(\mathcal{O}_{\mathbb{P}^1}(-1)\) on these fibres. Hence we conclude that \(\pi_{13*}(\mathcal{O}_E \otimes \mathcal{L}) = 0\). So distinguished triangle \((2)\) gives us an isomorphism
\[
\mathcal{E}_{i+1}(2) \ast \mathcal{E}_{i+1} \ast \mathcal{E}_{i}(2) \cong \mathcal{E}_i \ast \mathcal{E}_{i+1} \ast \mathcal{E}_i
\]
as desired.

**Remark 3.9.** There is an interesting similarity between the proof of the braid relation in [KT] and the proof of the Serre relation above. In particular, the proof of Proposition 4.6 of [KT] inspired our proof above. The geometry occurring in that proof is similar to the geometry we consider here.

Finally, the identity \(\mathcal{E}_i \ast \mathcal{E}_j \cong \mathcal{E}_j \ast \mathcal{E}_i\) when \(i, j \in I\) are not joined by an edge (i.e. when \(|i - j| > 1\)) follows from a direct calculation of both sides (all intersections are of the expected dimension and push-forwards are one-to-one so this calculation is straight-forward). The same argument works to show that \(\mathcal{F}_j \ast \mathcal{E}_i \cong \mathcal{E}_i \ast \mathcal{F}_j\) for any \(i, j \in I\) (condition (xi)).

### 3.5. Existence of deformations: conditions (x) and (xi)

We now explain why condition (xi) holds. Since we are in the Lie algebra \(\mathfrak{sl}_n\), having \(i, j \in I\) joined by an edge is equivalent to \(j = i \pm 1\). So let us consider \(j = i+1\) (the case \(j = i-1\) is the same). We must show that the sheaf \(\mathcal{E}_{i+1} \cong \text{Cone}(T_{i+1}^i)\) deforms along over the subspace \((\alpha_i + \alpha_j)^2\).

Here is an outline of the argument. Recall that \(\mathcal{E}_{i+1} \cong \text{Cone}(\mathcal{E}_i \ast \mathcal{E}_{i+1}[1] \xrightarrow{T_{i+1}^i} \mathcal{E}_{i+1} \ast \mathcal{E}_j)\). Now \(\mathcal{E}_i \ast \mathcal{E}_{i+1}\) is a line bundle supported on \(W_{ii+1}\) and \(\mathcal{E}_{i+1} \ast \mathcal{E}_i\) a line bundle supported on \(W_{i+1i}\). We show that the connecting map \(T_{i+1}^i\) must be the connecting map in the standard triangle
\[
\mathcal{O}_{W_{ii+1}}(−D) \to \mathcal{O}_{U_{ii+1}} \to \mathcal{O}_{W_{i+1i}},
\]
where \(U_{ii+1} = W_{ii+1} \cup W_{i+1i}\) and \(D := W_{i+1i} \cap W_{ii+1}\). Thus we identify \(\mathcal{E}_{ii+1}\) with a line bundle supported on \(U_{ii+1}\) and then write down an explicit deformation of it.

**Lemma 3.10.** We have
\[
\mathcal{E}_{ii+1} \cong \mathcal{O}_{U_{ii+1}} \otimes \det(V_{i+2}/V_i) \det(V_{i+1}/V_i) \{\lambda_{i+1} + \lambda_{i+2}\}
\]
where
\[
U_{ii+1} := \{(X, V, V') : V'_i \subset V_i, V'_{i+1} \subset V_{i+1}, V_j = V'_j \text{ if } j \neq i, i + 1 \} \subset Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_j).
\]

**Proof.** Recall that by Lemma 3.8 the objects \(\mathcal{E}_{ii+1}\) and \(\mathcal{E}_{i+1i}\) in \(\text{D}(Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_j))\) are line bundles supported on smooth varieties \(W_{ii+1}\) and \(W_{i+1i}\). Notice that the difference between the varieties \(W_{ii+1}\) and \(W_{i+1i}\) is that in the former we demand that \(V'_i \subset V_{i+1}\), while in the latter we demand that \(XV_{i+1} \subset V_{i+1}'\).

If we relax both conditions we end up with the variety \(U_{ii+1}\). Note that \(V_i/V'_i\) is dimension 1 and \(V_{i+1}/V'_{i+1}\) is codimension 1 (in \(V_{i+1}/V'_i\)) so any point in \(U_{ii+1}\) is either in \(W_{ii+1}\) or in \(W_{i+1i}\). In
particular $U_{ii+1} = W_{ii} \cup W_{i+1}$, and these are the two irreducible components of $U_{ii+1}$. These two components intersect in a divisor $D$ and gives us a short exact sequence of sheaves

$$0 \to \mathcal{O}_{W_{ii+1}}(-D) \to \mathcal{O}_{U_{ii+1}} \to \mathcal{O}_{W_{i+1}} \to 0$$

Using the fact that $D$ is cut out by $W_{i+1}$ by a section of $\text{Hom}(V_i/V'_i, V_{i+1}/V'_{i+1})$, we see that

$$\mathcal{O}_{W_{ii+1}}(-D) \cong \mathcal{O}_{W_{ii+1}} \otimes (V_{i+1}/V'_{i+1}) \otimes (V_i/V')^\vee.$$ 

Substituting this into the previous exact sequence and rotating, we obtain a distinguished triangle

$$\mathcal{O}_{W_{ii+1}}[-1] \to \mathcal{O}_{W_{ii+1}} \otimes (V_{i+1}/V'_{i+1})(V_{i}/V')^\vee \to \mathcal{O}_{W_{i+1}}.$$ 

Tensoring with the line bundle $\det(V_{i+2}/V'_{i+1})$ $\det(V_i'/V_{i-1})\{\lambda_{i+1} + \lambda_{i+2}\}$, we obtain the distinguished triangle

$$\mathcal{E}_i \ast \mathcal{E}_{i+1}[-1]{1} \to \mathcal{E}_{i+1} \ast \mathcal{E}_i \to \mathcal{O}_{U_{ii+1}} \otimes \det(V_{i+2}/V'_{i+1})^\vee \det(V_i'/V_{i-1})\{\lambda_{i+1} + \lambda_{i+2}\}$$

Moreover the first map in this distinguished triangle is non-zero. Since $T_{ii+1}$ is the unique such map (up to multiple) the first map must equal $T_{ii+1}$ up to multiple. The result follows. 

Now that we have identified $\mathcal{E}_{ii+1}$ more explicitly we can write down a deformation $\tilde{\mathcal{E}}_{ii+1}$ over

$$B := (\alpha_i + \alpha_{i+1})^{-1} = (0, \ldots, 0, -1, 0, 1, 0, \ldots, 0)^{-1}.$$ 

Define the variety

$$\tilde{U}_{ii+1} := \{(X, V, V', x) : (X, V, x) \in \hat{Y}(\lambda), (X, V', x) \in \hat{Y}(\lambda + \alpha_i + \alpha_{i+1}), x \in (\alpha_i + \alpha_{i+1})^{-1},$$

$$V'_i \subset V_i, V'_{i+1} \subset V_{i+1}, V_j = V'_j \text{ for } j \neq i, i+1\}$$

and consider

$$\tilde{\mathcal{E}}_{ii+1} := \mathcal{O}_{\tilde{U}_{ii+1}} \otimes \det(V_{i+2}/V'_{i+1})^\vee \det(V_i'/V_{i-1})\{\lambda_{i+1} + \lambda_{i+2}\} \in D(\hat{Y}(\lambda)|_B \times_B \hat{Y}(\lambda + \alpha_i + \alpha_{i+1})|_B).$$

**Proposition 3.11.** We have $j^* \tilde{\mathcal{E}}_{ii+1} = \mathcal{E}_{ii+1}$ where $j$ is the inclusion of the central fibre $Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_{i+1})$ into $Y(\lambda)|_B \times_B Y(\lambda + \alpha_i + \alpha_{i+1})|_B$.

**Proof.** Since the line bundles on both sides agree, it suffices to show that $j^* \mathcal{O}_{\tilde{U}_{ii+1}} = \mathcal{O}_{U_{ii+1}}$. To do this it suffices to show that $\tilde{U}_{ii+1}$ is an irreducible variety of dimension $\dim \, \mathbb{D}_{ii+1} + \dim \, B$ and that the scheme theoretic central fibre of $\tilde{U}_{ii+1} \to B$ is reduced. To do this, we will pass to local coordinates. To simplify our task of finding local coordinates we will use an idea Richo [Ric] and pass to a subvariety.

Note that $\tilde{U}_{ii+1}$ has an action of the group $SL_N$. Let $Z$ denote the subvariety of $\tilde{U}_{ii+1}$ consisting of those points $(X, V, V', x)$ where

$$0 \subset V_1 \subset \cdots \subset V_{i-1} \subset V'_i \subset V'_{i+1} \subset V_{i+1} \subset V'_{i+2} \subset \cdots \subset V_{n-1} \subset \mathbb{C}^N$$

is the standard partial flag. This means that $V_1 = \text{span}(e_1, \ldots, e_{\lambda_1})$, etc. The variety $Z$ has an action of the Borel subgroup $B_N \subset SL_N$. Note that as before, we have a map $Z \to B$.

The variety $\tilde{U}_{ii+1}$ is obtained from $Z$ by induction, $\tilde{U}_{ii+1} \cong Z \times_{B_N} SL_N$. Moreover, this is actually an isomorphism as varieties over $B$. Hence it suffices to prove that $Z$ is irreducible of expected dimension and the central fibre over $B$ is reduced.

We will show that $Z$ can be covered by open affine varieties $A$ such that $A$ can be embedded into $\mathbb{C}^{r+n-1}$ with the map to $B = \mathbb{C}^{n-1}$ given by projection onto the first $n-1$ coordinates. Let us write the coordinates on $\mathbb{C}^{r+n-1}$ as $x_1, \ldots, x_{n-1}, z_1, \ldots, z_r$. We will show that under this embedding $A$ is given by the single equation $x_1 - x_2 = z_1 z_2$. This proves the desired facts concerning the central fibre of $A$ and hence also for $\tilde{U}_{ii+1}$. 


To find this open affine variety $A$, note that $Z$ has a smooth surjective affine map $Z \to \mathbb{P}(V_{i+1}/V_i')$, taking $(X, V, V', x)$ to $V_i$. Pick $k, l$ such that

$$V_{i+1}' = V_i' \oplus \text{span}(e_k, \ldots, e_{l-1})$$

and

$$V_{i+1} = V_i' \oplus \text{span}(e_k, \ldots, e_l).$$

In $\mathbb{P}(V_{i+1}/V_i')$ there is an open affine subspace consisting of those $V_i$ which are of the form

$$V_i = V_i' \oplus \langle e_k + c_{k+1}e_{k+1} + \cdots + c_le_l \rangle.$$

We let $A$ denote the preimage of this affine subspace in $A$. So a point in $A$ is described by $(c_{k+1}, \ldots, c_l)$ and $(x_1, \ldots, x_{n-1})$ and the matrix $X$. The condition that $(X - x_j)V_j' \subset V_j'$ for all $j$ forces the diagonal to be $x_1, \ldots, x_{n-1}$ (with repetition) and many entries of $X$ to be 0. We focus on a small portion of the matrix which contains the interesting action — elsewhere the entries are free or forced to be 0 (this is not quite true, there are also some entries which are forced to be linear combinations of other entries with coefficients $c_{k+1}, \ldots, c_l$). Consider the square submatrix containing matrix coefficients for the basis elements $e_k, \ldots, e_l$.

$$
\begin{bmatrix}
    x_{i+1} & u \\
    \vdots & \ddots \\
    x_{i+1} & u c_{l-1} \\
    \vdots & \vdots \\
    x_i & u c_l \\
\end{bmatrix}
$$

The condition $(X - x_{i+1})e_l \in V_i$ gives the equation $x_i - x_{i+1} = uc_l$. Thus, we see that $A$ embedded in the affine space given by $e_{k+1}, \ldots, c_l, x_1, \ldots, x_{n-1}$, and the free entries of the matrix, subject to the one equation $x_i - x_{i+1} = uc_l$. This gives us the desired local description.

**Remark 3.12.** It is interesting to notice that neither $W_{ii+1}$ nor $W_{i+1i}$ deform over $B$ but that $U_{ii+1} = W_{ii+1} \cup W_{i+1i}$ does deform. In fact $W_{ii+1}$ and $W_{i+1i}$ only deform over $\alpha_i^\perp \cap \alpha_{i+1}^\perp$.

The proof that $E_i$ deforms over $\alpha_i^\perp$ (condition (x)) is the same but easier since we already have an explicit description of $E_i$ as a line bundle supported on $W_i$.

## 4. Preliminaries

In this section we fix some further notation and prove various technical results about compositions of functors $E$ and $F$ and about spaces of maps (natural transformations) between them. The reader can choose to skim this section on a first reading, using it as a reference.

### 4.1. Idempotent completeness.

Let $C$ be a graded additive category over $k$ which is idempotent complete (meaning that every idempotent splits). Notice that the (derived) category of coherent sheaves on any variety is idempotent complete (so all the categories we work with are idempotent complete).

Suppose that (each graded piece of) the space of homs between two objects is finite dimensional (by condition (i) this is true in our setup). Then every object in $C$ has a unique, up to isomorphism, direct sum decomposition into indecomposables (see section 2.2 of [Rin]). In particular, this means that if $A, B, C \in C$ then we have the following cancellation laws:

$$A \oplus B \cong A \oplus C \Rightarrow B \cong C$$

(3)

$$A \otimes_k k^n \cong B \otimes_k k^n \Rightarrow A \cong B.$$  

(4)

Now suppose $Q \in C$ is an object with $\text{End}(Q) = \mathbb{C} \cdot \text{id}$ and $\text{Hom}^i(Q, Q) = 0$ if $i < 0$. We call these basic objects. Given some $M \in C$ we can then talk of the number of direct summands $Q$ contained in $M$. More precisely, this will be the largest possible rank of a graded vector space $V$ such that there exist maps

$$Q \otimes_k V \to M \to Q \otimes_k V.$$
whose composition is the identity.

More generally, given a map \( f : M \to N \) where \( M, N \in \mathcal{C} \) we can speak of the \( Q \)-rank of \( f \) as the largest rank of a graded space \( V \) such that there exist maps

\[
Q \otimes_k V \to M \to N \to Q \otimes_k V
\]

whose composition is the identity.

If we have a commutative diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{f_1} & N_1 \\
\sim & & \sim \\
M_2 & \xrightarrow{f_2} & N_2
\end{array}
\]

where the vertical maps are isomorphisms, then the \( Q \)-ranks of \( f_1 \) and \( f_2 \) are equal.

Finally, we will repeatedly use the following cancellation Lemma which Bar-Natan [BN] calls “Gaussian elimination”.

**Lemma 4.1.** Let \( X, Y, Z, W \) be four objects in a triangulated category. Let \( f = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \oplus Y \to Z \oplus W \) be a morphism. If \( D \) is an isomorphism, then \( \text{Cone}(f) \cong \text{Cone}(A - BD^{-1}C : X \to Z) \).

**Proof.** This is essentially Lemma 4.2 from [BN] (or Lemma 5.25 from [CK2]). \( \square \)

### 4.2. Some basic \( \mathfrak{sl}_2 \) relations.

We begin by reviewing some of the relations which follow from the definition of a geometric categorical \( \mathfrak{sl}_2 \) action. These results are strictly about \( \mathfrak{sl}_2 \) actions and so they all follow from [CKL2].

**Proposition 4.2.** We have the direct sum decomposition

\[
E_i \ast E_i^{(r)} \cong E_i^{(r+1)} \otimes_k H^*(\mathbb{P}^r) \cong E_i^{(r)} \ast E_i.
\]

More generally, we have

\[
E_i^{(r_1)} \ast E_i^{(r_2)} \cong E_i^{(r_1+r_2)} \otimes_k H^*(G(r_1, r_1 + r_2)).
\]

**Proof.** This follows by Proposition 4.2 and Corollary 4.4 of [CKL2]. \( \square \)

**Proposition 4.3.** If \( \langle \lambda, \alpha_i \rangle \leq 0 \) we have the direct sum decomposition

\[
\mathcal{F}_i(\lambda) \ast E_i(\lambda) \cong E_i(\lambda - \alpha_i) \ast \mathcal{F}_i(\lambda - \alpha_i) \oplus O_\Delta \otimes_k H^*((\mathbb{P}^{-\langle \lambda, \alpha_i \rangle} - 1))
\]

and similarly if \( \langle \lambda, \alpha_i \rangle \geq 0 \).

**Proof.** This follows by Proposition 4.6 of [CKL2]. \( \square \)

**Corollary 4.4.** If \( \langle \lambda, \alpha_i \rangle + a + b \geq 0 \) then we have

\[
E_i^{(b)} \ast \mathcal{F}_i^{(a)}(\lambda) \cong \bigoplus_{j=0}^b \mathcal{F}_i^{(a-j)} \ast E_i^{(b-j)} \otimes_k H^*(G(j, \langle \lambda, \alpha_i \rangle + a + b)).
\]

**Proof.** This is a formal consequence of Proposition 4.3 and the cancellation relations. See [CKL3] Lemma 4.2 for a sketch of the proof. \( \square \)
4.3. Spaces of maps. Next we have some results about maps between various combinations of $\mathcal{E}$s.

Lemma 4.5. If $i, j \in I$ are joined by an edge then

\[
\text{Ext}^k(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a)}, \mathcal{E}_j^{(a)} \ast \mathcal{E}_i^{(b)}) \cong \begin{cases} 0 & \text{if } k < ab \\ k & \text{if } k = ab \end{cases}
\]

while

\[
\text{Ext}^k(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a)}, \mathcal{E}_i^{(a)} \ast \mathcal{E}_j^{(b)}) \cong \begin{cases} 0 & \text{if } k < 0 \\ k \cdot \text{id} & \text{if } k = 0 \end{cases}
\]

for any $a, b \geq 0$. The same results hold if we replace all $\mathcal{E}$s by $\mathcal{F}$s.

Remark 4.6. We will denote the unique map (up to non-zero multiple) in (5) when $k = ab$ by

\[T_{ij}^{(b)(a)}: \mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a)} \to \mathcal{E}_j^{(a)} \ast \mathcal{E}_i^{(b)}[ab].\]

When $a = b = 1$ we omit the superscripts.

Proof. Let us assume $(\lambda, \alpha) \leq 0$ and $(\lambda, \alpha_i) \leq 0$ (the opposite inequalities follow similarly). The proof is by (decreasing) induction on $(\lambda, \alpha_j)$ and also $a, b$ (the base case being $a = b = 0$). We have

\[
\text{Ext}^k(\mathcal{E}_i^{(b)}(\lambda + a\alpha_j) \ast \mathcal{E}_j^{(a)}(\lambda), \mathcal{E}_j^{(a)}(\lambda + b\alpha_i) \ast \mathcal{E}_i^{(b)}(\lambda)) \\
\cong \text{Ext}^k(\mathcal{E}_j^{(a)}(\lambda + a\alpha_j) \ast \mathcal{E}_i^{(b)}(\lambda + b\alpha_i) \ast \mathcal{E}_j^{(a)}(\lambda), \mathcal{E}_i^{(b)}(\lambda)) \\
\cong \text{Ext}^k(\mathcal{F}_j^{(a)}(\lambda + b\alpha_i)[-a(\lambda + b\alpha_i, \alpha_j) + a]) \ast \mathcal{E}_i^{(b)}(\lambda + a\alpha_j) \ast \mathcal{E}_j^{(a)}(\lambda), \mathcal{E}_i^{(b)}(\lambda)) \\
\cong \text{Ext}^k(\mathcal{E}_i^{(b)}(\lambda) \ast \mathcal{F}_j^{(a)}(\lambda) \ast \mathcal{E}_j^{(a)}(\lambda)[-a(\lambda, \alpha_j) + a - b], \mathcal{E}_i^{(b)}(\lambda))
\]

Now

\[\mathcal{F}_j^{(a)}(\lambda) \ast \mathcal{E}_j^{(a)}(\lambda) \cong \bigoplus_{s=0}^{s=a} \mathcal{E}_j^{(a-s)} \ast \mathcal{F}_j^{(a-s)} \otimes_k H^*(\mathbb{G}(s, -(\lambda, \alpha_j)))\]

so we need to understand

\[
\text{Ext}^k(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a-s)} \ast \mathcal{F}_j^{(a-s)}(\lambda - (a-s)\alpha_j) \otimes_k H^*(\mathbb{G}(s, -(\lambda, \alpha_j))), \mathcal{E}_i^{(b)}[a((\lambda, \alpha_j) + a - b)]).
\]

But $\mathcal{F}_j^{(a-s)}(\lambda - (a-s)\alpha_j)_L = \mathcal{E}_j^{(a-s)}(\lambda - (a-s)\alpha_j)_L[(a-s)((\lambda - (a-s)\alpha_j) + a - s)]$ so this equals

\[
\text{Ext}^k(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a-s)} \otimes_k H^*(\mathbb{G}(s, -(\lambda, \alpha_j))), \mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a-s)}((2a-s)((\lambda, \alpha_j) + s) - ab)).
\]

Now $H^*(\mathbb{G}(s, -(\lambda, \alpha_j)))$ is supported in degrees $* \leq -s((\lambda, \alpha_j) + s)$. So we get summands of the form

\[
\text{Ext}^k(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a-s)}, \mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a-s)})(2(a-s)((\lambda, \alpha_j) + s) - ab - *)
\]

where $* \geq 0$. If $k < ab$ then $2(a-s)((\lambda, \alpha_j) + s) - ab - + k < 0$ so we get zero (here we use that

\[
\text{Ext}^{<0}(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a-s)}, \mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a-s)}) = 0
\]

by the induction hypothesis). If $k = ab$ then $2(a-s)((\lambda, \alpha_j) + s) - i$ is non-negative precisely when $* = 0$ and $s = 0$ and we get only one such summand since $H^*(\mathbb{G}(s, -(\lambda, \alpha_j)))$ is one-dimensional in top degree.

This completes half the induction argument (i.e. relation (6) implies (5)). To prove the other half we repeat the analogous argument with

\[
\text{Ext}^k(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a)}, \mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a)})
\]

to show that relation (5) assuming (6).

Notice that to ensure the induction terminates we need the assumption that the action is integrable. The corresponding result for $\mathcal{F}$s follows by taking adjoints. \qed
The following result shows that $\mathcal{E}^{(b)} \ast \mathcal{E}^{(a)}$ are basic objects.

**Corollary 4.7.** If $i, j \in I$ are not joined by an edge then

$$\Ext^k(\mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a)}, \mathcal{E}_i^{(b)} \ast \mathcal{E}_j^{(a)}) \cong \begin{cases} 0 & \text{if } k < 0 \\ k \cdot \id & \text{if } k = 0 \end{cases}$$

and similarly if we replace all the $\mathcal{E}s$ by $\mathcal{F}s$.

**Proof.** The proof is precisely the induction from Lemma 4.5. The main difference is that in the computation we replace $\langle \alpha_i, \alpha_j \rangle = -1$ by $\langle \alpha_i, \alpha_j \rangle = 0$. Also, the induction has only one part since now $\mathcal{E}_i^{(b)}$ and $\mathcal{E}_j^{(a)}$ commute. \hfill \Box

### 4.4. Some basic $\mathfrak{g}_3$ relations

We first generalize the relation $\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_k \cong \mathcal{E}_i^{(2)} \ast \mathcal{E}_j \ast \mathcal{E}_k^{(2)}$ when $i, j, k \in I$ are joined by an edge.

**Proposition 4.8.** If $i, j \in I$ are joined by an edge then

$$\mathcal{E}_i^{(a)} \ast \mathcal{E}_j \ast \mathcal{E}_i \cong \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \otimes_k H^*(\mathbb{P}^{a-1}) \ast \mathcal{E}_i \ast \mathcal{E}_j^{(a+1)}$$

and similarly

$$\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i^{(a)} \cong \mathcal{E}_j \ast \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_i \otimes_k H^*(\mathbb{P}^{a-1}) \ast \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j.$$

**Proof.** We prove the first relation by induction on $a$ (the second relation follows similarly). The base case is $a = 1$ which is precisely one of the conditions of having a geometric categorical $\mathfrak{g}$ action.

Applying $\mathcal{E}_i^{(a)}$ to the left of the relation for $a = 1$ we get

$$\mathcal{E}_i^{(a)} \ast \mathcal{E}_j \ast \mathcal{E}_i \cong \mathcal{E}_i^{(a)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_j.$$  

Tensoring both sides with $H^*(\mathbb{P}^1)$ and using that $\mathcal{E}_i^{(2)} = \mathcal{E}_i^{(2)} \otimes_k H^*(\mathbb{P}^1)$ we get

$$\mathcal{E}_i^{(a)} \ast \mathcal{E}_j \ast \mathcal{E}_i \otimes_k H^*(\mathbb{P}^1) \cong \mathcal{E}_i^{(a)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_j \cong \left( \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i \otimes_k H^*(\mathbb{P}^{a-1}) \ast \mathcal{E}_j \ast \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_i \right) \ast \mathcal{E}_i^{(a)} \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_j$$

where the second isomorphism follows by induction. Thus we get

$$\mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i \otimes_k H^*(\mathbb{P}^1 \times \mathbb{P}^a) \cong \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i \otimes_k H^*(\mathbb{P}^{a-1}) \ast \mathcal{E}_j \ast \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_i \otimes_k \mathcal{E}_i^{(a)} \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_j.$$  

Now, $\mathcal{E}_i^{(a)} \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_j$ is a direct sum of summands $\mathcal{E}_i^{(a+2)} \ast \mathcal{E}_j$ which are basic objects by Lemma 4.5. So the right side of the isomorphism above breaks up into a direct sum of basic objects from which it follows by cancellation law (3) that we can cancel to obtain

$$\mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i \otimes_k H^*(\mathbb{P}^{a+1}) \cong \mathcal{E}_j \ast \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_i \otimes_k \mathcal{E}_i^{(a)} \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_j \cong \mathcal{E}_j \ast \mathcal{E}_i^{(a+2)} \otimes_k H^*(\mathbb{P}^{a+1}) \ast \mathcal{E}_j \otimes_k H^*(\mathbb{P}^a).$$

Using cancellation law (4) we get

$$\mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i \cong \mathcal{E}_j \ast \mathcal{E}_i^{(a+2)} \ast \mathcal{E}_i^{(a+2)} \ast \mathcal{E}_j \otimes_k H^*(\mathbb{P}^a)$$

and the induction step is complete. \hfill \Box

**Corollary 4.9.** If $i, j \in I$ are joined by an edge then

$$\mathcal{E}_i^{(a)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(b)} \cong \mathcal{E}_i^{(a+b)} \ast \mathcal{E}_j \otimes_k H^*(\mathbb{G}(b, a + b - 1)) \ast \mathcal{E}_j \ast \mathcal{E}_i^{(a+b)} \otimes_k H^*(\mathbb{G}(a, a + b - 1)).$$
Proof. The proof is by induction on \( a \). To compute \( \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(b)} \) one looks at
\[
\mathcal{E}_i^{(a)} \ast \mathcal{E}_i^{(a)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(b)} \cong \mathcal{E}_i^{(a+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(b)} \otimes \kappa H^*(\mathbb{P}^a).
\]
By induction the left hand side is
\[
\mathcal{E}_i \ast \left( \mathcal{E}_i^{(a+b)} \ast \mathcal{E}_j \otimes \kappa H^*(\mathbb{G}(b,a+b-1)) \oplus \mathcal{E}_j \ast \mathcal{E}_i^{(a+b)} \otimes \kappa H^*(\mathbb{G}(a,a+b-1)) \right).
\]
Now we have
\[
\mathcal{E}_i \ast \mathcal{E}_i^{(a+b)} \ast \mathcal{E}_j \cong \mathcal{E}_i^{(a+b+1)} \ast \mathcal{E}_j \otimes \kappa H^*(\mathbb{P}^{a+b})
\]
and by Proposition 4.8
\[
\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i^{(a+b)} \cong \mathcal{E}_i^{(a+b+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i^{(a+b+1)} \otimes \kappa H^*(\mathbb{P}^{a+b-1})
\]
where the right hand side is a direct sum of basic objects. So by the cancellation law the induction step comes down to proving that
\[
H^*(\mathbb{G}(b,a+b-1) \times \mathbb{P}^{a+b}) \oplus H^*(\mathbb{G}(a,a+b-1)) \cong H^*(\mathbb{G}(b,a+b) \times \mathbb{P}^a)
\]
and
\[
H^*(\mathbb{G}(a,a+b-1) \times \mathbb{P}^{a+b-1}) \cong H^*(\mathbb{G}(a+1,a+b) \times \mathbb{P}^a)
\]
which one can prove by standard techniques.

\[\square\]

4.5. Induced maps.

**Lemma 4.10.** If \( i, j \in I \) are connected by an edge then
\[
\mathcal{E}_{ij} \ast \mathcal{E}_i \cong \text{Cone}(\mathcal{E}_i^{(2)} \ast \mathcal{E}_j \ast [-1]) \xrightarrow{T_{ij}^{(2)(1)}} \mathcal{E}_j \ast \mathcal{E}_i^{(2)} [1]).
\]

In particular,
\[
\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i^{[-1]} \xrightarrow{T_{ij}^{(2)}} \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i
\]
induces an isomorphism on the \( \mathcal{E}_j \ast \mathcal{E}_i^{(2)} [-1] \) summand. Similarly
\[
\mathcal{E}_{ij} \ast \mathcal{E}_j \cong \text{Cone}(\mathcal{E}_i \ast \mathcal{E}_j^{(2)} \ast [-2]) \xrightarrow{T_{ij}^{(1)(2)}} \mathcal{E}_j^{(2)} \ast \mathcal{E}_i.
\]

We also have the analogous results for \( \mathcal{E}_i \ast \mathcal{E}_{ij} \) and \( \mathcal{E}_j \ast \mathcal{E}_{ij} \).

**Proof.** We deal with the case of \( \mathcal{E}_{ij} \ast \mathcal{E}_i \) since the other cases follow similarly.

Now \( (\mathcal{E}_i \ast \mathcal{E}_j \ast [-1]) \xrightarrow{T_{ij}} \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i \) induces a map
\[
\left( \begin{array}{cc} \alpha & 0 \\ \beta & \gamma \end{array} \right) : \left( \begin{array}{cc} \mathcal{E}_j \ast \mathcal{E}_i^{(2)} [-1] \\ \mathcal{E}_i \ast \mathcal{E}_j [-1] \end{array} \right) \rightarrow \left( \begin{array}{cc} \mathcal{E}_j \ast \mathcal{E}_i^{(2)} [1] \\ \mathcal{E}_j \ast \mathcal{E}_i^{(2)} [1] \end{array} \right).
\]

We need to show that \( \alpha \neq 0 \neq \gamma \) because then \( \alpha \) is a non-zero multiple of the identity and by the cancellation Lemma 4.1 the cone is isomorphic to \( \text{Cone}(\gamma) \) where \( \gamma \) must be \( T_{ij}^{(2)(1)} \) (up to a multiple) by Lemma 4.5.

Let \( v \in \mathfrak{h}^* \) be a vector with \( \langle v, \alpha_i \rangle = 1 \) and \( \langle v, \alpha_j \rangle = -1 \).

Denote by \( m \) the natural inclusion
\[
m : Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_j) \rightarrow \bar{Y}(\lambda) \times_{\operatorname{span}(v)} \bar{Y}(\lambda + \alpha_i + \alpha_j)).
\]
Given any \( \mathcal{M} \in D(Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_j)) \) one has the standard exact sequence
\[
\mathcal{M}[1] \rightarrow m^* m_* \mathcal{M} \rightarrow \mathcal{M}
\]
where the second map is the standard adjunction map. Now consider the following diagram

\[
\begin{array}{ccc}
m^*m_*(E_i \star E_j[-1]) & \rightarrow & m^*m_*(E_j \star E_i) \\
\downarrow (adj) & & \downarrow (adj) \\
E_i \star E_j[-1] & \rightarrow & E_j \star E_i \\
\downarrow & & \downarrow \\
E_i \star E_j[1] & \rightarrow & E_j \star E_i[2]
\end{array}
\]

(8)

where the top two rows and left two columns are exact triangles.

The top left square of (8) commutes because \(m^*m_* \rightarrow I\) is a natural transformation.

In general, if one has a commutative square such as the upper left square in (8) then one can fill it with some maps \(f, g\) making all the squares commute and so that \(\text{Cone}(f) \cong \text{Cone}(g)\) (see Proposition 1.1.11 of [BBD]).

We first show using Lemma 4.11 that there is only one choice for \(f\). To see that \(\text{Hom}^{-1}(m^*m_*(E_i \star E_j[-1]), E_j \star E_i) = 0\) we use Lemma 4.5 on the long exact sequence obtained by applying \(\text{Hom}(-, E_j \star E_i)\) to the exact triangle

\[E_i \star E_j \rightarrow m^*m_*(E_i \star E_j[-1]) \rightarrow E_i \star E_j[-1].\]

Analyzing the same sequence also shows that \(\text{Hom}(m^*m_*(E_i \star E_j[-1]), E_j \star E_i) \cong k\). This space is spanned by \(T_{ij} \circ (adj)\) or equivalently \((adj) \circ m^*m_* T_{ij}\). On the other hand, applying \(\text{Hom}(-, E_j \star E_i[-1])\) also shows that \(\text{Hom}(m^*m_*(E_i \star E_j[-1]), E_j \star E_i[-1]) \cong k\) spanned by the adjunction map. Thus the natural map

\[\text{Hom}(m^*m_*(E_i \star E_j[-1]), E_j \star E_i[-1]) \rightarrow \text{Hom}(m^*m_*(E_i \star E_j)[-1], E_j \star E_i)\]

is injective and so all conditions of Lemma 4.11 are satisfied. It follows that there is a unique map \(f\) making the top right square commute, namely the adjunction map.

The fact that \(E_{ij}\) deforms to some \(\tilde{E}_{ij}\) means that the connecting map in the exact triangle

\[E_{ij}[1] \rightarrow m^*m_* E_{ij} \rightarrow E_{ij}\]

is zero. This is because

\[m^*m_* E_{ij} \cong m^*m_* \tilde{E}_{ij} \cong m^* \left( \tilde{E}_{ij} \otimes m_* \mathcal{O}_{Y(\lambda) \times Y(\lambda + \alpha_i + \alpha_j)} \right) \cong m^* \tilde{E}_{ij} \oplus m^* \tilde{E}_{ij}[1] \cong E_{ij} \oplus E_{ij}[1].\]

So we end up with the commutative diagram

\[
\begin{array}{ccc}
m^*m_*(E_i \star E_j[-1]) & \rightarrow & m^*m_*(E_j \star E_i) \\
\downarrow (adj) & & \downarrow (adj) \\
E_i \star E_j[-1] & \rightarrow & E_j \star E_i \\
\downarrow & & \downarrow \\
E_i \star E_j[1] & \rightarrow & E_j \star E_i[2]
\end{array}
\]

(9)
Now apply \( \ast \mathcal{E}_i \) to the whole diagram to get

\[
\begin{array}{c}
\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i[-1] \xrightarrow{m \ast m \ast T_{ij}} \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i \xrightarrow{T_{ij}} \mathcal{E}_j \ast \mathcal{E}_i \xrightarrow{h_2} \mathcal{E}_j \ast \mathcal{E}_i[2] \xrightarrow{g'} \mathcal{E}_j \ast \mathcal{E}_i[2].
\end{array}
\]

Now we claim that \( h_2 I \) induces a map

\[
\mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i \cong \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[-1] \oplus \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[1] \xrightarrow{h_2} \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[1] \oplus \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[3] \cong \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i[2]
\]

which is an isomorphism on the summand \( \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[1] \).

To see this, note that \( h_2 I = c(v, \mathcal{E}_j \mathcal{E}_i) I \) is the connecting map for the deformation from \( \text{span}(v, 0, v, 0) \subset (\mathfrak{h}^*)^\times \). However if we consider the deformation from \( \text{span}(v, 0, 0, 0) \) the map

\[
c(v, 0, 0, 0, \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i) : \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i[-1] \to \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i[1]
\]

induces the zero map between the summands \( \mathcal{E}_j \ast \mathcal{E}_i^{(2)} \). On the other hand, since \( \langle v, \alpha_i \rangle = 1 \), we see that \( c((0, 0, v, 0), \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i) \) gives an isomorphism between these summands by condition (vi) (see section 2.3). Now, \( c(v, 0, v, 0) = c(v, 0, 0, 0) + c(0, 0, v, 0) \) and so the result follows.

Finally, looking back at diagram (10), we see that \( (g') I \circ (h_2 I) = 0 \) so that \( h_2 I \) must factor as

\[
\mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i \to \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i[1] \xrightarrow{g} \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_i[2].
\]

Since \( h_2 I \) induces an isomorphism on the summand \( \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[1] \) then so must \( g I \). But then \( g I \neq 0 \) so \( g I \) must equal \( T_{ij} I[2] \) (up to a multiple). Thus \( T_{ij} I \) induces an isomorphism on the summand \( \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[-1] \). This concludes the proof that \( \alpha \neq 0 \).

It remains to show \( \gamma \neq 0 \). To see this we consider the map

\[
\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i[-1] \xrightarrow{T_{ij} I} \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i.
\]

We will examine this map in two ways, by associating in two different ways. In particular, we will obtain a contraction by examining the \( \mathcal{E}_i^{(3)} \ast \mathcal{E}_j \) rank of this map.

On one hand, we consider the last three factors together and obtain a map

\[
\mathcal{E}_i \ast (\mathcal{E}_j \ast \mathcal{E}_i^{(2)}[-1] \oplus \mathcal{E}_i^{(2)} \ast \mathcal{E}_j[-1]) \to \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[-1] \oplus \mathcal{E}_j \ast \mathcal{E}_i^{(2)}[1])
\]

which is \( I \left( \alpha \beta \gamma \right) \). If \( \gamma = 0 \), the \( \mathcal{E}_i^{(3)} \ast \mathcal{E}_j \) rank of this map is \( \leq 1 \), since the only contribution can come from the \( \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i^{(2)} \) summand which contains one copy of \( \mathcal{E}_i^{(3)} \ast \mathcal{E}_j \) by Proposition 4.8.

On the other hand, we consider the first three factors together and obtain a map

\[
(\mathcal{E}_i^{(2)} \ast \mathcal{E}_j \oplus \mathcal{E}_i^{(2)} \ast \mathcal{E}_j[-2]) \to \mathcal{E}_i^{(2)} \ast \mathcal{E}_j \oplus \mathcal{E}_j \ast \mathcal{E}_i^{(2)} \ast \mathcal{E}_i
\]

We can apply the above reasoning to this map as well since we can write it as \( \left( \alpha \beta \gamma \right) I \). Now since \( \alpha \neq 0 \), we see that the \( \mathcal{E}_i^{(3)} \ast \mathcal{E}_j \) rank of this map is \( \geq 2 \) (since \( \mathcal{E}_i^{(2)} \ast \mathcal{E}_j \ast \mathcal{E}_i \) contains two copies of \( \mathcal{E}_i^{(3)} \ast \mathcal{E}_j \)). This gives a contradiction.

This means \( \gamma \neq 0 \) and we are done.
Lemma 4.11. Consider the diagram

\[
\begin{array}{c}
A \rightarrow B \rightarrow C \rightarrow A[1] \\
\downarrow \quad \downarrow f \\
A' \rightarrow B' \rightarrow C' \rightarrow A'[1]
\end{array}
\]

where the two rows are exact triangles and the left square commutes. If \( \text{Hom}^{-1}(A, B') = 0 \) and the natural map \( \text{Hom}(A, A') \rightarrow \text{Hom}(A, B') \) is an inclusion then there is a unique map \( f \) making all the squares commute.

Proof. By a basic result in triangulated categories, there always exists a map \( f \) making all the squares commute. We will show that under the above circumstance there is at most one \( f \) making the middle square commute. Hence there will exist a unique map making all the squares commute.

The proof is a little diagram chase. Applying \( \text{Hom}(\cdot, C') \) to the first row we get the long exact sequence

\[ \cdots \rightarrow \text{Hom}^{-1}(A, C') \rightarrow \text{Hom}(C, C') \rightarrow \text{Hom}(B, C') \rightarrow \cdots \]

So it suffices to show \( \text{Hom}^{-1}(A, C') = 0 \) because then \( \phi \) is an inclusion. If we now apply \( \text{Hom}(A, \cdot) \) to the second row then we get the long exact sequence

\[ \cdots \rightarrow \text{Hom}^{-1}(A, B') \rightarrow \text{Hom}^{-1}(A, C') \rightarrow \text{Hom}(A, A') \rightarrow \text{Hom}(A, B') \rightarrow \cdots \]

So if \( \text{Hom}^{-1}(A, B') = 0 \) and \( \phi' \) is an inclusion it follows \( \text{Hom}^{-1}(A, C') = 0 \).

Corollary 4.12. If \( i, j \in I \) are connected by an edge then for \( s \geq 0 \)

\[ \mathcal{E}_{ij} * \mathcal{E}_i^{(s)} \cong \text{Cone}(\mathcal{E}_i^{(s+1)} * \mathcal{E}_j [-1] \xrightarrow{T_{ij}[s]} \mathcal{E}_j * \mathcal{E}_i^{(s+1)}[s]). \]

In particular the \( \mathcal{E}_j * \mathcal{E}_i^{(s+1)} \) rank of the map

\[ \mathcal{E}_i * \mathcal{E}_j * \mathcal{E}_i^{(s)} [-1] \xrightarrow{T_{ij}} \mathcal{E}_j * \mathcal{E}_i * \mathcal{E}_i^{(s)} \]

is \( s \). Similarly

\[ \mathcal{E}_{ij} * \mathcal{E}_j^{(s)} \cong \text{Cone}(\mathcal{E}_i * \mathcal{E}_j^{(s+1)} [-s-1] \xrightarrow{T_{ij}[s]} \mathcal{E}_j^{(s+1)} * \mathcal{E}_i). \]

We also have the analogous results for \( \mathcal{E}_i^{(s)} * \mathcal{E}_{ij} \) and \( \mathcal{E}_j^{(s)} * \mathcal{E}_{ij} \).

Remark 4.13. The proof only assumes the result when \( s = 1 \) (everything else is a formal consequence of the fact \( \mathcal{E}_i^{(a)} * \mathcal{E}_j^{(b)} \) are basic objects).

Proof. We prove only the first identity as the others follow similarly.

We proceed by induction on \( s \). The base case \( s = 1 \) is covered in Lemma 4.10. Consider

\[ (\mathcal{E}_i * \mathcal{E}_j [-1] \xrightarrow{T_{ij}} \mathcal{E}_j * \mathcal{E}_i) * \mathcal{E}_i^{(s+1)} \]

which we can rewrite as

\[ \mathcal{E}_j * \mathcal{E}_i^{(s+2)} \otimes_k H^*([P^s]-1) \oplus \mathcal{E}_i^{(s+2)} * \mathcal{E}_j [-1] \xrightarrow{f} \mathcal{E}_j * \mathcal{E}_i^{(s+2)} \otimes_k H^*([P^s+1]). \]

Let \( t \) denote the \( \mathcal{E}_j * \mathcal{E}_i^{(s+2)} \) rank of \( f \).

On the other hand we also have the map

\[ (\mathcal{E}_i * \mathcal{E}_j [-1] \xrightarrow{T_{ij}} \mathcal{E}_j * \mathcal{E}_i) * \mathcal{E}_i \]

Since \( \mathcal{E}_i^{(s)} * \mathcal{E}_i \cong \mathcal{E}_i^{(s+1)} \otimes H^*([P^s+1]) \) we see that the \( \mathcal{E}_j * \mathcal{E}_i^{(s+2)} \) rank of (12) is \( t(s+1) \).
Now we can rewrite 12 as
\[
(E_j \ast E_i^{(s+1)} \otimes_k H^\ast(\mathbb{P}^s)[-1] \oplus E_i^{(s+1)} \ast E_j[-1] f_2 \ast E_j \ast E_i^{(s+1)} \otimes_k H^\ast(\mathbb{P}^s)) \ast E_i.
\]
By induction, the $E_j \ast E_i^{(s+1)}$ rank of $f_2$ is $s$. Now we can rewrite both sides as
\[
E_j \ast E_i^{(s+2)} \otimes_k H^\ast(\mathbb{P}^s \times \mathbb{P}^{s+1})[-1] \oplus E_j \ast E_i^{(s+2)}[-1] \oplus E_i^{(s+2)} \ast E_j \otimes_k H^\ast(\mathbb{P}^s)[-1]
\]
\[
f_2 \ast E_j \ast E_i^{(s+2)} \otimes_k H^\ast(\mathbb{P}^s \times \mathbb{P}^{s+1})
\]
The $E_i^{(s+2)} \ast E_j$ rank of $f_2$ is either $s(s+2)$ or $s(s+2)+1$ (we do not know \textit{a priori} if it induces an isomorphism on the middle summand on the left hand side).

Combining with above, we see that $t(s+1) = s(s+2)$ or $t(s+1) = s(s+2)+1$ for some $t$ with $0 \leq t \leq s+1$. Hence by Gaussian elimination,
\[
E_{ij} \ast E_i^{(s+1)} \cong \text{Cone}(E_i^{(s+2)} \ast E_j[-1] g \ast E_j \ast E_i^{(s+2)}[s+1]).
\]
The only thing that remains is to show $g \neq 0$ since then by Lemma 4.5, $g = T_{ij}^{(s+2)(1)}$ (up to a non-zero multiple) – completing the induction. But if $g = 0$ then applying $\ast E_i$ to (13) we get $s+1$ summands
\[
E_{ij} \ast E_i^{(s+2)} \cong \text{Cone}(E_i^{(s+3)} \ast E_j[-1] \rightarrow E_j \ast E_i^{(s+3)}[s+2])
\]
on the left hand side and
\[
E_i^{(s+2)} \ast E_j \ast E_i \ast E_i^{(s+2)} \ast E_i[s+1]
\]
one the right hand side. But then the right side contains $s+4$ summands $E_j \ast E_i^{(s+3)}$ instead of $s+2$ on the left side (contradiction). Notice that we did not use any information about $g$ in the induction so this is not a circular argument.

\[\square\]

**Corollary 4.14.** If $i, j \in I$ are connected by an edge then the composition
\[
E_i^{(s+1)} \ast E_j \xrightarrow{1}\ E_i \ast E_i^{(s)} \cong E_j \xrightarrow{T_{ij}(s)(1)} E_i \ast E_i^{(s)}
\]
is an isomorphism of $E_i^{(s+1)} \ast E_j$ onto the lone summand in
\[
E_i \ast E_j \ast E_i^{(s)} \cong E_i^{(s+1)} \ast E_j \ast E_i^{(s+1)} \otimes_k H^\ast(\mathbb{P}^s).
\]

**Proof.** Since $1$ is an inclusion (into lowest cohomological degree) it suffices to show that

\[
E_i \ast E_i^{(s)}[-s] \ast E_j \xrightarrow{T_{ij}(s)(1)} E_i \ast E_j \ast E_i^{(s)}
\]
is an isomorphism onto every copy of $E_i^{(s+1)} \ast E_j$ on the right hand side (or equivalently, that it has $E_i^{(s+1)} \ast E_j$ rank at least $s$).

First we consider the map
\[
E_i \ast E_j \ast E_i^{(s-1)}[-1] \xrightarrow{T_{ij}L} E_j \ast E_i \ast E_i^{(s-1)}.
\]
By Corollary 4.12 this map induces
\[
E_i^{(s)} \ast E_j[-1] \xrightarrow{T_{ij}L} E_j \ast E_i^{(s)}[s-1].
\]
So if we apply $E_i$ to (15) we get that the $E_j \ast E_i^{(s+1)}$ rank of

\[
E_i \ast E_i \ast E_j \ast E_i^{(s-1)}[-1] \xrightarrow{T_{ij}L} E_i \ast E_j \ast E_i \ast E_i^{(s-1)}
\]
is the same as that of the map in (14).
Remark 4.16. This result is a formal consequence of Lemma 4.10 and Corollary 4.4.

Proof. We begin by considering the map (16)

$$\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{F}_i(\lambda - \alpha_i)[{-1}] \xrightarrow{T_{ij} II} \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i$$

On the one hand, we can consider the first three factors together and obtain a map

$$\mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i(\lambda - \alpha_i)[{-1}] \xrightarrow{T_{ij} I} \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i$$

The map on the first summands is an isomorphism by Lemma 4.10. Using Corollary 4.4 we have

$$\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{F}_i(\lambda - \alpha_i)[{-1}] \cong \mathcal{E}_i \ast \mathcal{E}_j[{-1}] \otimes_k H^*(\mathbb{P}(\lambda, \alpha_i)) \oplus \mathcal{F}_i \ast \mathcal{E}_j \ast \mathcal{E}_i$$

and similarly

$$\mathcal{F}_i \ast \mathcal{E}_j \ast \mathcal{E}_i \cong \mathcal{F}_i \ast \mathcal{E}_i[{-1}] \otimes_k H^*(\mathbb{P}(\lambda, \alpha_i)) \oplus \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{E}_j[{-1}]$$

So the $\mathcal{E}_i \ast \mathcal{E}_j$ rank of the map (16) is at least $\langle \lambda, \alpha_i \rangle + 1$.

On the other hand, using Proposition 4.3 we have

$$\mathcal{E}_i \ast \mathcal{F}_i(\lambda - \alpha_i) \cong O_{\Delta} \otimes H^*(\mathbb{P}(\lambda, \alpha_i)) \oplus \mathcal{F}_i \ast \mathcal{E}_i$$

Hence we can rewrite the map (16) in the following ways

$$\xrightarrow{T_{ij} I} \mathcal{E}_j \ast \mathcal{F}_i \ast \mathcal{E}_i \xrightarrow{T_{ij} II} \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i \ast \mathcal{E}_i$$

Since $\mathcal{F}_i \ast \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i$ contain no summand $\mathcal{E}_j \ast \mathcal{E}_i$, we see that the $\mathcal{E}_j \ast \mathcal{E}_i$ rank of (16) is at most $\langle \lambda + \alpha_j, \alpha_i \rangle + 2 = \langle \lambda, \alpha_i \rangle + 1$.

Combining these two observations, we see that the $\mathcal{E}_j \ast \mathcal{E}_i$ rank of (16) is exactly $\langle \lambda, \alpha_i \rangle + 1$. Since $\mathcal{E}_i \ast \mathcal{E}_j$ does not contain any $\mathcal{E}_j \ast \mathcal{E}_i$ summands, examining (19), this shows that the $\mathcal{E}_j \ast \mathcal{E}_i$ rank of

$$\xrightarrow{T_{ij} I} \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{F}_i \ast \mathcal{E}_i$$

is $\langle \lambda, \alpha_i \rangle + 1$.

Now, let us consider

$$\xrightarrow{T_{ij} I} \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{F}_i(\lambda)$$

Lemma 4.15. If $i, j \in I$ are connected by an edge and $\langle \lambda, \alpha_i \rangle + 1 \geq 0$ then

$$\mathcal{E}_{ij} \ast \mathcal{F}_i(\lambda) \cong \text{Cone}(\mathcal{F}_i \ast \mathcal{E}_i \ast \mathcal{E}_j[{-1}] \xrightarrow{IT \oplus I} \mathcal{F}_i \ast \mathcal{E}_i \ast \mathcal{E}_j \oplus \mathcal{E}_i[\langle \lambda, \alpha_i \rangle + 1]).$$

In particular,

$$\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{F}_i[-1] \xrightarrow{T_{ij} I} \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i$$

induces an isomorphism on all summands $\mathcal{E}_j$ on the left hand side.
which we can rewrite as
\[ E_j \otimes H^*(\mathbb{P}(\lambda, \alpha_i)) \oplus F_i * E_i \to E_j \otimes H^*(\mathbb{P}(\lambda, \alpha_i)) \oplus F_i * E_i \]
Since the \( E_j * E_i \) rank of (20) is \( \langle \lambda, \alpha_i \rangle + 1 \), the \( E_j \) rank of (21) is \( \langle \lambda, \alpha_i \rangle + 1 \). Hence we can apply Gaussian elimination to conclude that
\[ E_{ij} * F_i(\lambda) \cong \text{Cone}(F_i * E_i \otimes E_j[-1] \xrightarrow{f_1 \oplus f_2} F_i * E_i \otimes E_j \oplus E_j(\langle \lambda, \alpha_i \rangle + 1)) \]
for some maps \( f_1, f_2 \). Now
\[ \text{Hom}(F_i(\lambda + \alpha_i + \alpha_j) * E_i \otimes E_j[-1], E_j(\langle \lambda, \alpha_i \rangle + 1)) \cong \text{Hom}(E_i \otimes E_j[-1], E_i \otimes E_j[-1]) \cong k \]
which is spanned by the adjunction map \( \varepsilon I \). Similarly \( \text{Hom}(E_i (E_i \otimes E_j[-1], F_i * E_i * E_j) \cong k \) which must be spanned by \( IT_{ij} \). So it remains to show that \( f_1 \) and \( f_2 \) are non-zero.

To show that \( f_1 \neq 0 \), we look again at the map (16). When we rewrite it as in (17), we know that the map on first summands is an isomorphism. By (18), these first summands contain a copy of \( F_i * E_j * \mathcal{E}_{i}^{(2)}[-1] \). On the other hand, when we rewrite (16) as in (19), we see that this copy of \( F_i * E_j * E_i \) is a direct summand of \( F_i * E_i * E_j * E_i[-1] \). Thus the map on \( F_i * E_i * E_j * E_i[-1] \) must be non-zero. However, this is precisely \( f_1 I \) and hence \( f_1 \neq 0 \).

To show \( f_2 \neq 0 \) we consider the map
\[ E_i \star (E_i \star E_j \xrightarrow{T_{ij}} E_i \star E_j) \star F_i(\lambda) \]
On the one hand we can rewrite this as
\[ E_i^{(2)} * E_j \star F_i[-2] \oplus E_i^{(2)} * E_j \star F_i \to E_j \star E_i^{(2)} \star F_i \oplus E_i^{(2)} \star E_j \star F_i \]
where the map is an isomorphism on the second summands. Since
\[ E_i^{(2)} * E_j \star F_i(\lambda) \cong E_i \otimes E_j \otimes_k H^*(\mathbb{P}(\lambda, \alpha_i)) \oplus F_i * E_i^{(2)} * E_j \]
This means that \( E_i \star E_j \) rank of (22) is at least \( \langle \lambda, \alpha_i \rangle + 2 \). On the other hand, as we showed above, (21) has \( E_j \) rank equal to \( \langle \lambda, \alpha_i \rangle + 1 \). Now, above we showed using Gaussian elimination that the map (21) can be written direct sum of a piece having \( E_j \) rank equal to \( \langle \lambda, \alpha_i \rangle + 1 \) and \( f_1 \oplus f_2 \). But since (22) is obtain from (21) by \( E_i \star \), this shows that
\[ E_i * F_i * E_i * E_j[-1] \xrightarrow{I f_1 \oplus f_2} E_i * F_i * E_j * E_i \oplus E_i * E_j(\langle \lambda, \alpha_i \rangle + 1) \]
must have \( E_i * E_j \) rank equal to 1. This means that \( If_2 \) must induce an isomorphism on the right hand \( E_i * E_j \) summand. In particular, \( f_2 \neq 0 \). \( \square \)

**Corollary 4.17.** If \( i, j \in I \) are connected by an edge and \( \langle \lambda, \alpha_i \rangle + s \geq 0 \) then
\[ E_{ij} * F_i^{(s)}(\lambda) \cong \text{Cone}(F_i^{(s)} * E_i \otimes E_j[-1] \xrightarrow{IT \oplus \varepsilon I \otimes I} F_i^{(s)} * E_j * E_i \oplus F_i^{(s-1)} \otimes E_j(\langle \lambda, \alpha_i \rangle + s)) \]
In particular,
\[ E_i * E_j * F_i^{(s)}[-1] \xrightarrow{T_{ij}} E_j * E_i * F_i^{(s)} \]
induces an isomorphism on all summands \( F_i^{(s-1)} * E_j \) on the left hand side.

**Remark 4.18.** This result is a formal consequence of Lemma 4.15 and Corollary 4.4.

**Proof.** The proof is by induction on \( s \) and is the same as that of Corollary 4.12.

The base case \( s = 1 \) is covered by Lemma 4.10. Now suppose \( \langle \lambda, \alpha_i \rangle + s + 1 \geq 0 \) and consider
\[ (E_i \star E_j[-1] \xrightarrow{T_{ij}} E_j * E_i) * F_i^{(s+1)}(\lambda). \]
We have
\[
\mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{F}_i^{(s+1)}(\lambda)[-1] \cong \mathcal{F}_i^{(s)} \ast \mathcal{E}_j \otimes_k H^*([\mathbb{P}(\lambda+\alpha_i+\alpha_j)+s+1] \ast \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_i \ast \mathcal{E}_j [-1] \\
\mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i^{(s+1)}(\lambda) \cong \mathcal{F}_i^{(s)} \ast \mathcal{E}_j \otimes_k H^*([\mathbb{P}(\lambda+\alpha_i)+s+1] \ast \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i.
\]

We first want to show that the map induced in (23) is an isomorphism on all \(\langle \lambda, \alpha_i \rangle + s + 1\) summands \(\mathcal{F}_i^{(s)} \ast \mathcal{E}_j\) on the left hand side. So in total (24) is what we wanted to show).

By induction this induces an isomorphism on all the \(\mathcal{E}_j \ast \mathcal{F}_i^{(s)}\) on the left hand side and also induces
\[
(\mathcal{E}_i \ast \mathcal{E}_j [-1] \xrightarrow{T_{ij}} \mathcal{E}_j \ast \mathcal{E}_i) \ast \mathcal{F}_i(\lambda) \ast \mathcal{F}_i^{(s)}(\lambda + \alpha_i)
\]
which induces a map
\[
\left( \mathcal{F}_i \ast \mathcal{E}_i \ast \mathcal{E}_j [-1] \xrightarrow{IT_{ij}} \mathcal{F}_i \ast \mathcal{E}_j \ast \mathcal{E}_i \right) \ast \mathcal{F}_i^{(s)}(\lambda + \alpha_i).
\]

By Lemma 4.15 \(f\) induces an isomorphism on all the \(\langle \lambda, \alpha_i \rangle + 1\) summands \(\mathcal{E}_j \ast \mathcal{F}_i^{(s)}\) on the left hand side and also induces
\[
(\mathcal{F}_i \ast \mathcal{E}_i \ast \mathcal{E}_j [-1] \xrightarrow{IT_{ij}} \mathcal{F}_i \ast \mathcal{E}_j \ast \mathcal{E}_i) \ast \mathcal{F}_i^{(s)}(\lambda + \alpha_i).
\]

By induction this induces an isomorphism on all the \(\langle \lambda + \alpha_i + \alpha_j, \alpha_i \rangle + s + 1\) summands \(\mathcal{F}_i \ast \mathcal{F}_i^{(s-1)} \ast \mathcal{E}_j\). This shows that (23) induces an isomorphism on \(\langle \lambda, \alpha_i \rangle + s + 1\) summands (which is what we wanted to show).

Thus, by the cancellation Lemma 4.1,
\[
\mathcal{E}_{ij} \ast \mathcal{F}_i^{(s+1)} \cong \text{Cone}(\mathcal{F}_i^{(s+1)} \ast \mathcal{E}_i \ast \mathcal{E}_j [-1] \xrightarrow{g_1 \oplus g_2} \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j \oplus \mathcal{F}_i^{(s)} \ast \mathcal{E}_j (\langle \lambda, \alpha_i \rangle + s + 1))
\]
for some maps \(g_1\) and \(g_2\). Now, one can check (this is a long but straightforward calculation so we omit it here) that
\[
\text{Hom}(\mathcal{F}_i^{(s+1)} \ast \mathcal{E}_i \ast \mathcal{E}_j [-1], \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i) \cong \mathbb{K} \cong \text{Hom}(\mathcal{F}_i^{(s+1)} \ast \mathcal{E}_i \ast \mathcal{E}_j [-1], \mathcal{F}_i^{(s)} \ast \mathcal{E}_j (\langle \lambda, \alpha_i \rangle + s + 1)).
\]

Thus it remains to show that \(g_1\) and \(g_2\) are non-zero. In that case \(g_1\) must be \(IT_{ij}\) and \(g_2\) must be the composition
\[
\mathcal{F}_i^{(s+1)} \ast \mathcal{E}_i \ast \mathcal{E}_j [-1] \xrightarrow{IJ} \mathcal{F}_i^{(s)} \ast \mathcal{F}_i(\lambda + (s+1)\alpha_i + \alpha_j)[s] \ast \mathcal{E}_i \ast \mathcal{E}_j [-1] \\
\xrightarrow{\iota \iota} \mathcal{F}_i^{(s)} \ast \mathcal{E}_j (\langle \lambda, \alpha_i \rangle + s + 1)
\]
(up to a non-zero multiple). Recall that \(\iota\) denotes the unique inclusion of \(\mathcal{F}_i^{(s+1)}\) into the lowest degree summand of \(\mathcal{F}_i^{(s)} \ast \mathcal{F}_i\).

To show \(g_1 \neq 0\) we look at \(\mathcal{E}_{ij} \ast \mathcal{F}_i^{(s+1)} \ast \mathcal{F}_i\). Now \(\mathcal{F}_i^{(s+1)} \ast \mathcal{F}_i \cong \mathcal{F}_i^{(s+2)} \otimes_k H^*([\mathbb{P}^s + 1])\) and we can show as above (without using that \(g_1 \neq 0\)) that
\[
\left( \mathcal{E}_i \mathcal{E}_j [-1] \xrightarrow{T_{ij}} \mathcal{E}_j \ast \mathcal{E}_i \right) \ast \mathcal{F}_i^{(s+2)}
\]
induces an isomorphism on all \(\langle \lambda, \alpha_i \rangle + s + 2\) summands \(\mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j\) on the left hand side. Thus \(\mathcal{E}_{ij} \ast \mathcal{F}_i^{(s+1)} \ast \mathcal{F}_i\) induces an isomorphism on \((s+2)(\langle \lambda, \alpha_i \rangle + s + 2)\) such summands.
On the other hand, the map \((\mathcal{E}_i \ast \mathcal{E}_j[-1] \xrightarrow{T_{ij}} \mathcal{E}_j \ast \mathcal{E}_i) \ast \mathcal{F}_i^{(s)}(\lambda) \ast \mathcal{F}_i\) induces an isomorphism on \(\langle \lambda, \alpha_i \rangle + s + 1\) summands \(\mathcal{F}_i^{(s)} \ast \mathcal{E}_j \ast \mathcal{F}_i\) or equivalently \((\langle \lambda, \alpha_i \rangle + s + 1)\) summands \(\mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j\) and what is left over is the map from equation (24):

\[
\left( \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_i \ast \mathcal{E}_j[-1] \xrightarrow{g_1 \oplus g_2} \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i \right) \ast \mathcal{F}_i
\]

So \(g_1 \neq 0\) since otherwise we would get no further isomorphisms on summands \(\mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j\).

To show that \(g_2 \neq 0\) we apply \(\mathcal{E}_s\) to (23). On the one hand we get

\[
\mathcal{E}_i \ast \mathcal{E}_i \ast \mathcal{E}_j[-1] \xrightarrow{T_{ij}} \mathcal{E}_i \ast \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i^{(s+1)}.
\]

By Lemma 4.10 this induces an isomorphism \((\mathcal{E}_i^{(2)} \ast \mathcal{E}_j \xrightarrow{y} \mathcal{E}_j^{(2)} \ast \mathcal{E}_j) \ast \mathcal{F}_i^{(s+1)}\) and the map

\[
\left( \mathcal{E}_i^{(2)} \ast \mathcal{E}_j[-2] \xrightarrow{T_{ij}^{(1)(2)}} \mathcal{E}_j \ast \mathcal{E}_j^{(2)} \right) \ast \mathcal{F}_i^{(s+1)}.
\]

Now one can show, along the same lines as above, that the map \(T_{ij}^{(1)(2)}\) induces an isomorphism on all summands \(\mathcal{F}_i^{(s-1)} \ast \mathcal{E}_j\) on the left hand side. Thus the map in (25) induces an isomorphism on all summands \(\mathcal{F}_i^{(s-1)} \ast \mathcal{E}_j\) on the left hand side. On the other hand, if \(g_2 = 0\) then the map

\[
\mathcal{E}_i \ast \left( \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_i \ast \mathcal{E}_j[-1] \xrightarrow{g_1 \oplus g_2} \mathcal{F}_i^{(s+1)} \ast \mathcal{E}_j \ast \mathcal{E}_i \ast \mathcal{F}_i^{(s+1)}, \mathcal{E}_j \right) \ast \mathcal{F}_i^{(s+1)}(\langle \lambda, \alpha_i \rangle + s + 1)\]

cannot induce an isomorphism on all summands \(\mathcal{F}_i^{(s-1)} \ast \mathcal{E}_j\) on the left hand side (contradiction). So we must \(g_2 \neq 0\).

\[
\Box
\]

5. Proof of Main Theorem 2.6

In this section we will assume that \(i, j \in I\) are joined by an edge (unless explicitly stated otherwise).

The main idea of the proof is as follows. We will show that \(T_i \ast T_j = T_{ij} \ast T_i\) where \(T_{ij}\) is an equivalence coming from an \(A_2\) action generated by the kernel \(\mathcal{E}_{ij}\). From a similar argument, we will also show that \(T_i \ast T_j = T_{ij} \ast T_j\). This immediately implies the braid relation. The kernel \(\mathcal{E}_{ij}\) should be thought of as a root vector for the root \(\alpha_i + \alpha_j\).

In order to prove that \(T_i \ast T_j = T_{ij} \ast T_i\), we will compute \(\mathcal{E}_{ij} \ast T_i\). Recall that \(T_i\) is the convolution of a complex where each term in the complex is of the form \(T_i^s = \mathcal{F}_i^{(\langle \lambda, \alpha_i \rangle + s)} \ast \mathcal{E}_i^{(s)}\). So to compute \(\mathcal{E}_{ij} \ast T_i\) we first calculate \(\mathcal{E}_{ij} \ast \mathcal{F}_i^{(\langle \lambda, \alpha_i \rangle + s)}\) (Step 1) which follows directly from Corollary 4.17. Next we calculate \(\mathcal{E}_{ij} \ast \mathcal{F}_i^{(\langle \lambda, \alpha_i \rangle + s)} \ast \mathcal{E}_i^{(s)}\) (Step 2) which basically follows from Corollary 4.12. This gives us a simplified expression for \(\mathcal{E}_{ij} \ast T_i^s\).

Next, in the most difficult step, we put all these terms together and simplify to come up with an expression for \(\mathcal{E}_{ij} \ast T_i\). We compare with a similarly simplified expression for \(T_i \ast \mathcal{E}_j\) (this is much easier to calculate) and conclude that \(\mathcal{E}_{ij} \ast T_i \equiv T_i \ast \mathcal{E}_j\) (Corollary 5.4). It then follows by formal arguments that \(T_{ij} \ast T_i \equiv T_i \ast T_{ij}\).

5.1. Step 1: Calculation of \(\mathcal{E}_{ij} \ast \mathcal{F}_i^{(\langle \lambda, \alpha_i \rangle + s)}\). The first step is to compute

\[
\mathcal{E}_{ij} \ast \mathcal{F}_i^{(\langle \lambda, \alpha_i \rangle + s)} \in \mathcal{D}(Y(\lambda + s \alpha_i) \times Y(\lambda - \langle \lambda, \alpha_i \rangle \alpha_i)).
\]

To simplify things we will now notation a little and write \(d_i^k\) for any map obtained as the composition

\[
\mathcal{F}_i^{(k)} \ast \mathcal{E}_i^{(s)} \xrightarrow{T_i} \mathcal{F}_i^{(k-1)} \ast \mathcal{F}_i \ast \mathcal{E}_i \ast \mathcal{E}_i^{(s-1)} \xrightarrow{T_i} \mathcal{F}_i^{(k-1)} \ast \mathcal{E}_i^{(s-1)}
\]

for any \(k \in \mathbb{N}\).
Proposition 5.1. \( E_{ij} \ast F_i^{(\lambda,\alpha_i)+s} (\lambda - \langle \lambda, \alpha_i \rangle \alpha_i) \) is isomorphic to

\[
\text{Cone} \left( F_i^{(\lambda,\alpha_i)+s} \ast E_i \ast E_j[-1] \xrightarrow{IT_{ij} \oplus d^I} F_i^{(\lambda,\alpha_i)+s} \ast E_j \ast F_i^{(\lambda,\alpha_i)+s-1} \ast E_i \left[ s \right] \right).
\]

Proof. This follows directly from Corollary 4.17 since

\[
\langle \lambda - \langle \lambda, \alpha_i \rangle \alpha_i, \alpha_i \rangle + (\langle \lambda, \alpha_i \rangle + s) = s \geq 0.
\]

\( \square \)

5.2. Step 2: Calculation of \( E_{ij} \ast F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s)} \). The second step is to compute

\[
E_{ij} \ast F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s)}(\lambda) \in D(Y(\lambda) \times Y(\lambda - \langle \lambda, \alpha_i \rangle)).
\]

Proposition 5.2. \( E_{ij} \ast F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s)}(\lambda) \) is isomorphic to

\[
\text{Cone} \left( F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s+1)} \ast E_j[-1] \xrightarrow{IT_{ij}^{(s+1)(1)} \oplus \gamma_s} F_i^{(\lambda,\alpha_i)+s} \ast E_j \ast E_i^{(s+1)} \ast E_j \ast F_i^{(\lambda,\alpha_i)+s-1} \ast E_i^{(s)} \left[ s \right] \right)
\]

where \( \gamma_s \) is the composition

\[
F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s+1)} \ast E_j[-1] \xrightarrow{d_i^{s+1} \ast I} F_i^{(\lambda,\alpha_i)+s-1} \ast E_j \ast F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s)} \ast E_j \\
\xrightarrow{IT_{ij}^{(s+1)(1)}} F_i^{(\lambda,\alpha_i)+s-1} \ast E_j \ast E_i^{(s)} \left[ s \right].
\]

Proof. This is a direct consequence of Proposition 5.1. Applying \( *E_i^{(s)} \) to the main expression in Proposition 5.1 we get the two maps

(26) \[
F_i^{(\lambda,\alpha_i)+s} \ast E_i \ast E_j[-1] \ast E_i^{(s)} \xrightarrow{IT_{ij} \oplus I} F_i^{(\lambda,\alpha_i)+s} \ast E_j \ast E_i \ast E_i^{(s)}
\]

(27) \[
F_i^{(\lambda,\alpha_i)+s} \ast E_i \ast E_j[-1] \ast E_i^{(s)} \xrightarrow{d_i^{s+1}} F_i^{(\lambda,\alpha_i)+s-1} \ast E_j \ast E_i^{(s)} \left[ s \right].
\]

By Corollary 4.12, (26) induces an isomorphism on all summands \( F_i^{(\lambda,\alpha_i)+s} \ast E_j \ast E_i^{(s+1)} \) on the left hand side and cancelling out these terms leaves

\[
F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s+1)} \ast E_j[-1] \xrightarrow{IT_{ij}^{(s+1)(1)}} F_i^{(\lambda,\alpha_i)+s} \ast E_j \ast E_i^{(s+1)} \left[ s \right].
\]

Now the map in (27) when restricted to the summand \( F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s+1)} \ast E_j[-1] \) is by Lemma 4.14 the composition

\[
F_i^{(\lambda,\alpha_i)+s} \ast E_i^{(s+1)} \ast E_j[-1] \xrightarrow{IT_{ij}} F_i^{(\lambda,\alpha_i)+s} \ast E_i \ast E_i^{(s)} \ast E_j[-s-1] \\
\xrightarrow{d_i^{IT_{ij}^{(s+1)(1)}}} F_i^{(\lambda,\alpha_i)+s-1} \ast E_j \ast E_i^{(s)} \left[ s \right].
\]

Up to multiple this is the same as the map \( \gamma_s \) (completing the proof). \( \square \)
5.3. Step 3: Calculation of $E_{ij} \ast T_i$. Recall that $T_i(\lambda)$ is the (left) convolution of the complex
\[
\ldots \to d_i^{s+1}T_i(\lambda) \xrightarrow{d_i^s} T_i^{s-1}(\lambda) \xrightarrow{d_i^{s-1}} \ldots \to d_i^0T_i(\lambda)
\]
where $T_i^s(\lambda) = T_i^{((\lambda,\alpha_i)+s)} \ast E_i(\lambda)[s]$. The complex is finite on the left since the categorical $g$ action is integrable. We denote the partial convolution
\[
\mathcal{T}_i^{\geq s}(\lambda) := \text{Conv} \left( \ldots \to d_i^{s+1}T_i(\lambda) \xrightarrow{d_i^s} T_i^{s+1}(\lambda) \xrightarrow{d_i^{s-1}} \ldots \to d_i^0T_i(\lambda) \right).
\]
Since convolution of a complex is nothing but an iterated cone we have a standard exact triangle
\[
T_i^{\geq s} \to T_i^{s+1} \to T_i^{s+1}.
\]

**Proposition 5.3.** For any $s \geq -1$ we have $E_{ij} \ast T_i^{\geq s}$ is isomorphic to
\[
\text{Cone} \left( T_i^{\geq s+1} \ast E_j \xrightarrow{\gamma'_s} E_j \ast F_i^{((\lambda,\alpha_i)+s-1)} \ast E_i(\lambda) \right)
\]
where the map $\gamma'_s$ is uniquely determined by the fact that the composition
\[
T_i^{s+1}(\lambda + \alpha_j) \ast E_j \to T_i^{s+1}(\lambda + \alpha_j) \ast E_j \xrightarrow{\gamma'_s} E_j \ast F_i^{((\lambda,\alpha_i)+s-1)} \ast E_i(\lambda)
\]
is the map $\gamma_s$ from Proposition 5.2.

**Proof.** First we explain why $\gamma'_s$ is well defined. By Lemma 5.7 below
\[
\text{Ext}^k(T_i^t \ast E_j(\lambda), E_j \ast F_i^{((\lambda,\alpha_i)+s-1)} \ast E_i(\lambda)) = 0
\]
for $k \leq 0$ and $t \leq s$. So using the triangle $T_i^{s+1} \to T_i^{s+1} \to T_i^{s+1}[1]$ it follows by induction that
\[
\text{Ext}^k(T_i^{s} \ast E_j, E_j \ast F_i^{((\lambda,\alpha_i)+s-1)} \ast E_i(\lambda)) = 0
\]
for $k \leq 0$. Thus, from the triangle $T_i^{s} \to T_i^{s+1} \to T_i^{s+1}$ we get that
\[
\text{Hom}(T_i^{s+1} \ast E_j, E_j \ast F_i^{((\lambda,\alpha_i)+s-1)} \ast E_i(\lambda)) \cong \text{Hom}(T_i^{s+1} \ast E_j, E_j \ast F_i^{((\lambda,\alpha_i)+s-1)} \ast E_i(\lambda))
\]
and so $\gamma'_s$ is uniquely determined by $\gamma_s$.

The proof is by a decreasing induction on $s$. The base case is when $s \gg 0$ in which case everything is zero and there is nothing to prove. Now we will prove the result for $s-1$ assuming it holds for $s$.

The key is the following commutative diagram.

\[
\begin{align*}
& T_i^{s+1} \ast E_j \xrightarrow{d_i^{s+1}} T_i^s \ast E_j \xrightarrow{IT_i^{(s+1)} \ast \gamma_{s+1}} E_j \ast F_i^{((\lambda,\alpha_i)+s-1)} \ast E_i(\lambda) \xrightarrow{f} E_{ij} \ast T_i^{\geq s} \\
& T_i^s \ast E_j \xrightarrow{I \ast E_i(\lambda)} E_j \ast F_i^{((\lambda,\alpha_i)+s-2)} \ast E_i(\lambda) \xrightarrow{g} E_{ij} \ast T_i^{\geq s} \\
& T_i^{s} \ast E_j \xrightarrow{g} E_j \ast F_i^{((\lambda,\alpha_i)+s-2)} \ast E_i(\lambda)
\end{align*}
\]

The first two rows and two columns are exact triangles – note that the second column is exact by the cancellation Lemma 4.1. The maps $f$ and $g$ are to be determined as explained below. But first we need to explain why the top left square commutes.
Because of the vanishing from Lemma 5.7 it suffices to show that the following square commutes.

\[
\begin{array}{ccc}
\mathcal{T}_i^{s+1} \ast \mathcal{E}_j & \xrightarrow{\gamma_s} & \mathcal{E}_j \ast \mathcal{F}_i^{(\lambda, \alpha_i) + s - 1} \ast \mathcal{E}_i^{(s)} \\
\downarrow I \ast d_i^{s+1} I & & \downarrow I \ast I \ast d_i^{s+1} I \\
\mathcal{T}_i^s \ast \mathcal{E}_j & \xrightarrow{IT_i^{(s+1)} \ast \gamma_s - 1} & \mathcal{E}_j \ast \mathcal{F}_i^{(\lambda, \alpha_i) + s - 1} \ast \mathcal{E}_i^{(s-1)} \oplus \mathcal{E}_j \ast \mathcal{F}_i^{(\lambda, \alpha_i) + s - 2} \ast \mathcal{E}_i^{(s-1)}
\end{array}
\]

Now by Lemma 5.7 we have \(\text{Hom}(\mathcal{T}_i^{s+1} \ast \mathcal{E}_j, \mathcal{E}_j \ast \mathcal{F}_i^{(\lambda, \alpha_i) + s - 1} \ast \mathcal{E}_i^{(s)}) = 0\) so we just need to show that \(\gamma_s = (IT_i^{(s+1)}) \circ (d_i^{s+1} I)\) and this is the definition of \(\gamma_s\).

Now, as in the proof of Lemma 4.10, if one has a commutative square such as the upper left square in (28) then one can fill it with some maps \(f, g\) making all the squares commute and so that \(\text{Cone}(f) \cong \text{Cone}(g)\). Now by Lemma 5.7 we have that

\[
\text{Hom}(\mathcal{T}_i^{s+1} \ast \mathcal{E}_j, \mathcal{T}_i^s \ast \mathcal{E}_j) \cong k
\]

and hence is spanned by \(d_i^{s+1} I\). Hence to check that the map

\[
\text{Hom}(\mathcal{T}_i^{s+1} \ast \mathcal{E}_j, \mathcal{T}_i^s \ast \mathcal{E}_j) \rightarrow \text{Hom}(\mathcal{T}_i^{s+1} \ast \mathcal{E}_j, \mathcal{E}_j \ast \mathcal{F}_i^{(\lambda, \alpha_i) + s - 1} \ast \mathcal{E}_i^{(s)} \oplus \mathcal{E}_j \ast \mathcal{F}_i^{(\lambda, \alpha_i) + s - 2} \ast \mathcal{E}_i^{(s-1)})
\]

is injective it suffices to check that \(IT_i^{(s+1)} \ast \gamma_s - 1 \circ d_i^{s+1} I \neq 0\). But this composition equals the composition going the other way around the square, so it suffices to check that \(\gamma_s'\) is non-zero. Now \(\gamma_s \neq 0\) (and hence \(\gamma_s' \neq 0\)) because if it were zero then the complex defining \(\mathcal{T}_i\) would break up into two summands and would not be an equivalence.

Hence the conditions of Lemma 4.11 are satisfied. In other words, there are unique maps \(f, g\) (up to multiple) making all the squares commute. It is not hard to see that \(f\) and \(g\) cannot be zero, so we must have \(f = I \ast d_i^s\) and \(g = \gamma_s' \ast \mathcal{T}\) (up to multiple). Consequently

\[
\mathcal{E}_{ij} \ast \mathcal{T}_i^{s-1} \cong \text{Cone}(I \ast d_i^s) \cong \text{Cone}(f) \cong \text{Cone}(g) \cong \text{Cone}(\gamma_s' \ast \mathcal{T})
\]

and the induction is complete.

If we take \(s = -1\) in Proposition 5.3 then we find that

**Corollary 5.4.** \(\mathcal{E}_{ij} \ast \mathcal{T}_i \cong \mathcal{T}_i \ast \mathcal{E}_j\).

The significance of this is that \(\mathcal{E}_{ij}\) is conjugate to \(\mathcal{E}_j\), namely \(\mathcal{E}_{ij} \cong \mathcal{T}_i \ast \mathcal{E}_j \ast \mathcal{T}_i^{-1}\). Thus, for any \(k \geq 0\) we can define

\[
\mathcal{E}_{ij}^{(k)}(\lambda) := \mathcal{T}_i \ast \mathcal{E}_{ij}^{(k)}(\lambda) \ast \mathcal{T}_i^{-1} \quad \text{and} \quad \mathcal{F}_{ij}^{(k)}(\lambda) := \mathcal{T}_i \ast \mathcal{F}_{ij}^{(k)}(\lambda) \ast \mathcal{T}_i^{-1}.
\]

**Corollary 5.5.** \(\mathcal{E}_{ij}^{(r)}\) and \(\mathcal{F}_{ij}^{(r)}\) generate a geometric categorical \(\mathfrak{sl}_2\) action where the one parameter deformation of \(Y(\lambda)\) is the restriction of \(\tilde{Y}(\lambda)\) to the subspace spanned by \(\alpha_i + \alpha_j\). Moreover, we have

\[
\mathcal{E}_{ij}^{(r)} \ast \mathcal{T}_i \cong \mathcal{T}_i \ast \mathcal{E}_{ij}^{(r)} \quad \text{and} \quad \mathcal{F}_{ij}^{(r)} \ast \mathcal{T}_i \cong \mathcal{T}_i \ast \mathcal{F}_{ij}^{(r)}.
\]

**Remark 5.6.** Strictly speaking, this is not true, since we do not know that the \(\mathcal{E}_{ij}\) are sheaves. In examples, however, this fact can be verified directly. On the other hand, we will not use this Corollary in the sequel.

**Lemma 5.7.** If \(k \leq 0\) and \(t \geq s + 1\) we have

\[
\text{Ext}^k(\mathcal{T}_i^t \ast \mathcal{E}_j(\lambda), \mathcal{T}_i^s \ast \mathcal{E}_j(\lambda)) = 0 \quad \text{and} \quad \text{Ext}^k(\mathcal{T}_i^t \ast \mathcal{E}_j(\lambda), \mathcal{E}_j \ast \mathcal{F}_i^{(\lambda, \alpha_i) + s - 1} \ast \mathcal{E}_i^{(s)}(\lambda)) = 0
\]

where \(\mathcal{T}_i^t = \mathcal{F}_i^{(\lambda + \alpha_i + \alpha_j + t)} \ast \mathcal{E}_i^{(s)}\). The only exception is when \(k = 0\) and \(t = s + 1\) in which case they are both one-dimensional.
Proof. In Proposition 5.2 of [CKL3] we showed that \( \text{Ext}^k(T^t_i, T^s_i) = 0 \) if \( t \geq s + 1 \) and \( k \leq 0 \) (unless both equalities hold in which case the space is one-dimensional). However, what we showed there is a little stronger than that. One has

\[
\text{Ext}^k(T^t_i, T^s_i) \cong \text{Ext}^k(F^t_i(\lambda + \alpha_i, \alpha_i) + s, \lambda + \alpha_i, \alpha_i) + t) * \mathcal{E}_i^{(t)}[-t] * \mathcal{E}_i^{(s)}[-s])
\]

and one can repeatedly use Corollary 4.4 to write \( F^t_i(\lambda + \alpha_i, \alpha_i) + s, \lambda + \alpha_i, \alpha_i) + t) * \mathcal{E}_i^{(t)}[-t + s] \) as a direct sum

\[
\bigoplus_{l \geq 0} F^{(l)}_i * \mathcal{E}_i^{(s+l)} \otimes_k V^l_i
\]

for some graded vector spaces \( V^l_i \). Then one can rewrite \( \bigoplus_{l \geq 0} \text{Ext}^k(F^{(l)}_i * \mathcal{E}_i^{(s+l)} \otimes_k V^l_i, \mathcal{E}_i^{(s)}) \) as

\[
\bigoplus_{l \geq 0} \text{Ext}^k(\mathcal{E}_i^{(s+l)} \otimes_k V^l_i, F^{(l)}_i R * \mathcal{E}_i^{(s)}) \cong \bigoplus_{l \geq 0} \text{Ext}^k(\mathcal{E}_i^{(s+l)}, \mathcal{E}_i^{(s+l)} \otimes_k \mathcal{V}^l_i)
\]

for some (other) graded vector spaces \( \mathcal{V}^l_i \). Then the statement proven in Proposition 5.2 in [CKL3] is that each \( \mathcal{V}^l_i \) is supported in degrees

\[
d \leq -(2l^2 + 2l(-2b + t - s) + (t-s)^2 + 2ab - (t-s))
\]

for some integers \( l, a, b \). If we view this as a quadratic in \( l \) then the discriminant simplifies to give \(-4(t-s)^2 + 4\) (as it happens it is independent of \( a, b \)). This is negative (resp. zero) if \( t > s + 1 \) (resp. \( t = s + 1 \)) which shows that \( \mathcal{V}^l_i \) must lie in negative (resp. non-positive) degrees. It follows, by Lemma 4.5, that

\[
\text{Ext}^k(T^t_i * \mathcal{E}_j(\lambda), T^s_i * \mathcal{E}_j(\lambda)) \cong \text{Ext}^k(\mathcal{E}_i^{(s+l)} * \mathcal{E}_j, \mathcal{E}_i^{(s+l)} * \mathcal{E}_j \otimes_k \mathcal{V}^l_i)
\]

must also be zero for \( k \leq 0 \) and \( t \geq s + 1 \). In the case \( k = 0 \) and \( t = s + 1 \) then precisely one \( \mathcal{V}^l_i \) contains a one-dimensional subspace in degree zero (everything else lying in negative degrees) and so the same argument shows that \( \text{Ext}^0(T^t_i * \mathcal{E}_j, T^s_i * \mathcal{E}_j) \cong k \).

In the case of \( \text{Ext}^k(T^t_i * \mathcal{E}_j, \mathcal{F}_i(\lambda, \alpha_i) + s + 1, \mathcal{E}_j * \mathcal{E}_i^{(s)}) \) the same argument works. Suppose we are not in the case \( k = 0 \) and \( t = s + 1 \). As above we first get

\[
\bigoplus_{l \geq 0} \text{Ext}^k(\mathcal{F}_i^{(l)}, \mathcal{E}_i^{(s+l)} * \mathcal{E}_j \otimes_k \mathcal{V}_i, \mathcal{E}_j * \mathcal{E}_i^{(s)}[s]).
\]

By adjunction we can move the term \( \mathcal{F}_i^{(l)} \) from the left side to the right side and try to simplify as before. But now, using Corollary 4.9, we get terms of the form

\[
\mathcal{E}_i^{(l)} * \mathcal{E}_j * \mathcal{E}_i^{(s)} \cong \mathcal{E}_i^{(l+s)} * \mathcal{E}_j \otimes_k H^*(G(s, s + l - 1)) \oplus \mathcal{E}_j * \mathcal{E}_i^{(l+s)} \otimes_k H^*(G(l, s + l - 1))
\]

instead of terms of the form

\[
\mathcal{E}_i^{(l)} * \mathcal{E}_i^{(s)} * \mathcal{E}_j \cong \mathcal{E}_i^{(l+s)} * \mathcal{E}_j \otimes_k H^*(G(s, s + l)).
\]

In the old case we saw that we end up with terms of the form \( \text{Ext}^k(\mathcal{E}_i^{(l+s)} * \mathcal{E}_j, \mathcal{E}_i^{(l+s)} * \mathcal{E}_j[n]) \) where \( n < 0 \). Since \( H^*(G(s, s + l)) \) is supported in degrees \(-ls \leq m \leq ls \) and \( H^*(G(s, s + l - 1)) \) in degrees \(-(l-1)s \leq m \leq (l-1)s \) it must be that now (if \( l > 0 \)) we end up with terms of the form

\[
\text{Ext}^k(\mathcal{E}_i^{(l+s)} * \mathcal{E}_j, \mathcal{E}_i^{(l+s)}[n-s][s]) \text{ and } \text{Ext}^k(\mathcal{E}_i^{(l+s)} * \mathcal{E}_j, \mathcal{E}_j * \mathcal{E}_i^{(l+s)}[n-s][s])
\]

where \( n < 0 \). These vanish by Lemma 4.5. The only possible exception is when \( l = 0 \) but then we end up with

\[
\text{Ext}^k(\mathcal{E}_i^{(s)} * \mathcal{E}_j \otimes_k \mathcal{V}_0, \mathcal{E}_j \otimes \mathcal{E}_i^{(s)}[s])
\]

which also vanishes by Lemma 4.5 since \( \mathcal{V}_0 \) is supported in negative degrees. The argument when \( k = 0 \) and \( t = s + 1 \) is the same except for the last step when \( l = 0 \) (in that case \( \mathcal{V}_0 \) is supported in negative degrees except for a one-dimensional subspace supported in degree zero).
Remark 5.8. Notice that in this section we proved that if $i, j \in I$ are joined by an edge then
$$\mathcal{T}_i \ast \mathcal{E}_j \ast \mathcal{T}_i^{-1} \cong \text{Cone}(\mathcal{E}_i \ast \mathcal{E}_j[-1] \to \mathcal{E}_j \ast \mathcal{E}_i).$$

One can use the same techniques to also prove that
$$\mathcal{T}_i \ast \mathcal{E}_i \ast \mathcal{T}_i^{-1} \cong \mathcal{F}_i \text{ and } \mathcal{T}_i \ast \mathcal{F}_i \ast \mathcal{T}_i^{-1} \cong \mathcal{E}_i.$$

5.4. Step 4: Proof that $\mathcal{T}_{ij} \ast \mathcal{T}_i \cong \mathcal{T}_i \ast \mathcal{T}_j$. In analogy with $\mathcal{T}_i^s(\lambda)$ we define
$$\mathcal{T}_{ij}^s(\lambda) := \mathcal{F}_{ij}^{(\lambda, \alpha_i + \alpha_j + s)} \ast \mathcal{E}_{ij}^s(\lambda)[-s] \in D(Y(\lambda) \times Y(\lambda - (\lambda, \alpha_i + \alpha_j)(\alpha_i + \alpha_j))).$$

Notice that $\mathcal{T}_{ij}^s \cong \mathcal{T}_i \ast \mathcal{T}_j^s \ast \mathcal{T}_i^{-1}$ so that
$$\text{Hom}(\mathcal{T}_{ij}^s, \mathcal{T}_{ij}^{s-1}) \cong \text{Hom}(\mathcal{T}_i \ast \mathcal{T}_j^s \ast \mathcal{T}_i^{-1}, \mathcal{T}_i \ast \mathcal{T}_j^{s-1} \ast \mathcal{T}_i^{-1}) \cong \text{Hom}(\mathcal{T}_j^s, \mathcal{T}_j^{s-1}) \cong \mathbb{k}.$$

We denote this map by $d_{ij}^s$ (it is well defined only up to a non-zero multiple). One can also describe $d_{ij}^s$ as before by the composition

<table>
<thead>
<tr>
<th>$d_{ij}^s(\lambda)$</th>
<th>$\cong$</th>
<th>$\mathcal{F}<em>{ij}^{(\lambda, \alpha_i + \alpha_j + s)} \ast \mathcal{E}</em>{ij}^s(\lambda)[-s]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ast$</td>
<td>$\mapsto$</td>
<td>$\mathcal{F}<em>{ij}^{(\lambda, \alpha_i + \alpha_j + s) + s} \ast \mathcal{F}</em>{ij}[-(\lambda, \alpha_i + \alpha_j) - s + 1] \ast \mathcal{E}<em>{ij} \ast \mathcal{E}</em>{ij}^{s-1}[-(s - 1)][-s]$</td>
</tr>
<tr>
<td>$\triangleright$</td>
<td>$\mapsto$</td>
<td>$\mathcal{F}<em>{ij}^{(\lambda, \alpha_i + \alpha_j) + s} \ast \mathcal{E}</em>{ij}^s[-s + 1] \cong \mathcal{T}_{ij}^{s-1}(\lambda)$.</td>
</tr>
</tbody>
</table>

Proposition 5.9. The complex

$$\cdots \to \mathcal{T}_{ij}^s(\lambda) \xrightarrow{d_{ij}^s} \mathcal{T}_{ij}^{s-1}(\lambda) \xrightarrow{d_{ij}^{s-1}} \cdots \xrightarrow{d_{ij}} \mathcal{T}_{ij}^{0}(\lambda)$$

has a unique convolution which we denote $\mathcal{T}_{ij}(\lambda)$. Moreover,
$$\mathcal{T}_{ij} \ast \mathcal{T}_i \cong \mathcal{T}_i \ast \mathcal{T}_j.$$

Proof. The key is the following commutative diagram

$$
\begin{array}{ccc}
\cdots & \rightarrow & \mathcal{T}_{ij}^{s} \\
\downarrow & \cong & \downarrow & \cong \\
\mathcal{T}_i \ast \mathcal{T}_j^s \ast \mathcal{T}_i^{-1} & \rightarrow & \mathcal{T}_j \ast \mathcal{T}_j^s \ast \mathcal{T}_j^{-1} & \rightarrow & \cdots \\
\end{array}
$$

where one needs to choose appropriate multiples of the vertical isomorphisms. This is a consequence of the fact that $\text{Hom}(\mathcal{T}_{ij}^s, \mathcal{T}_i \ast \mathcal{T}_j^{s-1} \ast \mathcal{T}_i^{-1}) \cong \text{Hom}(\mathcal{T}_j^s, \mathcal{T}_j^{s-1}) \cong \mathbb{k}$ and that $d_{ij}^s \neq 0 \neq d_{ij}^{s-1}$. Now $\mathcal{T}_{ij}^s$ has a unique convolution so $\mathcal{T}_i \ast \mathcal{T}_{ij}^s \ast \mathcal{T}_i^{-1}$ has a unique convolution and hence so does $\mathcal{T}_{ij}^s$. Finally, the commutativity of the diagram also implies that the convolutions must be isomorphic: i.e. $\mathcal{T}_{ij} \cong \mathcal{T}_i \ast \mathcal{T}_j \ast \mathcal{T}_i$.

5.5. Step 5: Proof of braiding relation. Proposition 5.9 claims that $\mathcal{T}_{ij} \ast \mathcal{T}_i \cong \mathcal{T}_i \ast \mathcal{T}_j$ which is follows from the fact that $\mathcal{E}_{ij} \ast \mathcal{T}_i \cong \mathcal{T}_i \ast \mathcal{E}_j$.

Now the same proof can be used to show that $\mathcal{T}_j \ast \mathcal{T}_{ij} \cong \mathcal{T}_i \ast \mathcal{T}_j$. Namely, Step 1 is a consequence of the analogous version of Corollary 4.17 which computes $\mathcal{F}(s) \ast \mathcal{E}_{ij}$ (this in turn can be traced back to follow formally from Lemma 4.10 and Corollary 4.4 – see Remarks 4.18 and 4.16). Then Step 2 and 3 follow formally (they also use Lemma 4.10 and there is some vanishing one needs to check which is a formal consequence of the Lie algebra relations). This shows that $\mathcal{T}_j \ast \mathcal{E}_{ij} \cong \mathcal{E}_i \ast \mathcal{T}_j$ and then Step 4 follows as before.

Putting these two identities together we get the braid relation
$$\mathcal{T}_i \ast \mathcal{T}_j \ast \mathcal{T}_i \cong \mathcal{T}_j \ast \mathcal{T}_i \ast \mathcal{T}_i \cong \mathcal{T}_j \ast \mathcal{T}_i \ast \mathcal{T}_j.$$
Finally, if \( i, j \in I \) are not joined by an edge then any \( \mathcal{E}_i \) or \( \mathcal{F}_i \) commutes with any \( \mathcal{E}_j \) or \( \mathcal{F}_j \). Since \( \mathcal{T}_i \) is built out of \( \mathcal{E}_i \)'s and \( \mathcal{F}_i \)'s and \( \mathcal{T}_j \) is built out of \( \mathcal{E}_j \)'s and \( \mathcal{F}_j \)'s we get the commutativity relation \( \mathcal{T}_i \ast \mathcal{T}_j \cong \mathcal{T}_j \ast \mathcal{T}_i \). This concludes the proof of Theorem 2.6.

6. BRAIDING VIA STRONG CATETGICAL \( \mathfrak{g} \)-ACTIONS

Strong categorical \( \mathfrak{g} \)-actions have been defined by Khovanov and Lauda in [KL] and independently by Rouquier in [Ro2]. Their definitions are very similar though not identical. One should think of the geometric categorical \( \mathfrak{g} \)-action introduced here as a geometric analogue of their definition which is easier to check in practice.

In [CKL2] we prove that when \( \mathfrak{g} = \mathfrak{sl}_2 \) a geometric \( \mathfrak{g} \)-action implies a strong \( \mathfrak{g} \)-action in the sense of Rouquier. There is good reason to believe the same is true for arbitrary (simply-laced) Kac-Moody Lie algebras \( \mathfrak{g} \). Nevertheless, in this paper we show that the braiding relation follows directly from the geometric \( \mathfrak{g} \)-action.

On the other hand, our proof of Theorem 2.6 works to show that a strong \( \mathfrak{g} \)-action gives a braid group action. In fact it seems that not all the axioms of a strong \( \mathfrak{g} \)-action are needed to obtain the braid group action. We will now explain this, starting with a simplified version of Rouquier’s definition.

A (simplified) strong categorical \( \mathfrak{g} \) action consists of

(i) For each weight \( \lambda \) we have a \( \kappa \)-linear additive \( \mathbb{Z} \)-graded category \( \mathcal{D}(\lambda) \) which is idempotent complete (this category may be empty). Graded means that each category \( \mathcal{D}(\lambda) \) has a shift functor \([\ ]\) which is an equivalence.

(ii) Functors \( E^i(\lambda) : \mathcal{D}(\lambda) \to \mathcal{D}(\lambda + r\alpha_i) \) and \( F^i(\lambda) : \mathcal{D}(\lambda + r\alpha_i) \to \mathcal{D}(\lambda) \) with relations among the functors

(i) For any weight \( \lambda \), \( \text{Hom}(\text{id}_{\mathcal{D}(\lambda)}, \text{id}_{\mathcal{D}(\lambda)}[l]) = 0 \) if \( l < 0 \) while \( \text{End}(\text{id}_{\mathcal{D}(\lambda)}) = \kappa \cdot \text{id} \).

(ii) (a) \( E^i(\lambda) = F^i(\lambda)[r(\langle \lambda, \alpha_i \rangle + r)] \)

(b) \( E^i(\lambda) = F^i(\lambda)[-r(\langle \lambda, \alpha_i \rangle + r)] \).

(iii) \( E_i \circ E^i(\lambda) \cong E^i(\lambda + 1)(\lambda) \otimes_{\kappa} H^{*}(\mathbb{P}^{r}) \cong E^i(\lambda) \circ E_i(\lambda) \)

while \( E_i \circ E_j \cong E_j \circ E_i \) if \( i, j \in I \) are not joined by an edge and

\[ E_i \circ E_j \circ E_i \cong E^{(i)}_i \circ E_j \circ E_i \]

if \( i, j \in I \) are joined by an edge.

(iv) If \( \langle \lambda, \alpha_i \rangle \leq 0 \) then

\( F_i(\lambda) \circ E_i(\lambda) \cong E_i(\lambda - \alpha_i) \circ F_i(\lambda - \alpha_i) \oplus \text{id} \otimes_{\kappa} H^{*}(\mathbb{P}^{-\langle \lambda, \alpha_i \rangle} - 1) \)

while if \( \langle \lambda, \alpha_i \rangle \geq 0 \) then

\( F_i(\lambda - \alpha_i) \circ E_i(\lambda - \alpha_i) \cong E_i(\lambda) \circ F_i(\lambda) \oplus \text{id} \otimes_{\kappa} H^{*}(\mathbb{P}^{\langle \lambda, \alpha_i \rangle} - 1) \).

If \( i \neq j \in I \) then \( F_j \circ E_j \cong E_i \circ F_j \).

along with the following natural transformations (2-morphisms):

(i) \( X_i : E_i(\lambda)[-1] \to E_i(\lambda)[1] \) for each \( i \in I \) and weight \( \lambda \)

(ii) \( T_{ij} : E_i(\lambda)[\langle \alpha_i, \alpha_j \rangle] \to E_j \circ E_i(\lambda) \) for any \( i, j \in I \) and weight \( \lambda \)

with relations

(i) For each \( i \in I \) the \( X_i \)'s and \( T_{ii} \)'s satisfy the nil affine Hecke relations:

(a) \( T_{ii}^{2} = 0 \)

(b) \( (IT_{ii}) \circ (IT_{ii}) = (T_{ii}I) \circ (IT_{ii}) \circ (T_{ii}I) \) as endomorphisms of \( E_i \circ E_i \circ E_i \).

(c) \( (X_iI) \circ T_{ii} - T_{ii} \circ (IX_i) = I = -(IX_i) \circ T_{ii} + T_{ii} \circ (X_iI) \) as endomorphisms of \( E_i \circ E_i \).

(ii) If \( i \neq j \in I \) are joined by an edge then \( T_{ji} \circ T_{ij} = X_iI + IX_j \) as endomorphisms of \( E_i \circ E_j \).
This list of 2-morphisms and the relations appear in both the Khovanov-Lauda and Rouquier definitions and it turns out they suffice to prove the braiding relations. (One small caveat is that the condition $\text{Hom}(\text{id}, \text{id}[l]) = 0$ if $l < 0$ does not appear in either of their definitions.)

**Theorem 6.1.** A categorical strong $\mathfrak{g}$-action as defined above gives rise to equivalences $T_i$ ($i \in I$) satisfying the braiding relations.

**Proof.** The fact that we have the nil affine Hecke relations means that for each $i \in I$ we have a strong categorical $\mathfrak{sl}_2$ action (in the sense of [CKL3]) so we can construct equivalences $T_i$. What remains is to show that they braid, which we do by running again through the proof of Theorem 2.6.

The more complicated relations among compositions of $E$'s and $F$'s (such as Propositions 4.2, 4.3, 4.8 and Corollary 4.9) were all formal arguments which work in any abstract (idempotent complete) category. The same goes for the calculations of Hom-spaces (Lemma 4.5 and Corollary 4.7).

The only place where something more interesting happens is in the proof of Lemma 4.10. Notice that the $T_{ij}$ from that Lemma and our $T_{ij}$ defined above must be equal (up to non-zero scalars) since $\text{Hom}(E_i \circ E_j[-1], E_j \circ E_i) \cong k$ (by Lemma 4.5).

The whole proof of Lemma 4.10 comes down to showing that the map

$$E_j \circ E_i^{(2)}[-1] \oplus E_i^{(2)} \circ E_j[-1] \cong E_i \circ E_j \circ E_i[-1] \xrightarrow{T_{ij}} E_j \circ E_i \circ E_i \cong E_j \circ E_i^{(2)}[-1] \oplus E_j \circ E_i^{(2)}[1]$$

induces an isomorphism on the $E_j \circ E_i^{(2)}[-1]$ summand. In 4.10 we use the fact that $E_{ij}$ deforms to $\tilde{E}_{ij}$ to show this. In the abstract setting we use instead the relation $T_{ji} \circ T_{ij} = X_iI + IX_j$.

More precisely, suppose the map does not induce an isomorphism. This means it must induce zero since $\text{End}(E_j \circ E_i^{(2)}) = k \cdot \text{id}$. But then, the composition

$$E_j \circ E_i^{(2)}[-1] \oplus E_i^{(2)} \circ E_j[-1] \cong E_i \circ E_j \circ E_i[-1] \xrightarrow{T_{ji}} E_i \circ E_i \circ E_i \xrightarrow{IT_{ii}} E_i \circ E_i \circ E_i[-2] \cong E_i \circ E_i^{(2)}[-3] \oplus E_j \circ E_i^{(2)}[-1]$$

must be zero. On the other hand, pre-composing with $(T_{ji}) \circ (IT_{ii})$ we get

$$(IT_{ii}) \circ (T_{ji}) \circ (T_{ji}) \circ (IT_{ii}) = (IT_{ii})(X_jII + IX_iI) \circ (IT_{ii}) = (X_jII) \circ (IT_{ii})^2 + (IT_{ii}) \circ ((IT_{ii}) \circ (IX_iI) + I)$$

where we use $T_{ji}^2 = 0$ twice in the last equality. This is non-zero (contradiction).

This proves Lemma 4.10. Then Corollaries 4.12 and 4.14 follow by formal arguments. Lemma 4.15 is a formal consequence of Lemma 4.10 and Corollary 4.4 and Corollary 4.17 follows from Lemma 4.15 and Corollary 4.4.

This brings us up to section 5. One can easily check that the arguments there are formal consequences of the Lie algebra relations and results from section 4. So the braiding relation follows.

\[\square\]

**Remark 6.2.** In the setting of (non-categorified) quantum groups, there is a braid group action on $U_q(\mathfrak{g})$ (constructed by Lusztig) which is compatible with the braid group action on representations. Hence we would expect there should be a braid group action on the 2-category of Rouquier/Khovanov-Lauda which is compatible with the above action of the braid group on the representations.

From the proof of the main theorem in this paper, we would expect that generators $\sigma_i$ of this braid group action would obey the following two conditions

$$\sigma_i(E_j) = T_i \circ E_j \circ T_i^{-1} = \text{Cone}(E_i \circ E_j[-1] \to E_j \circ E_i) \quad \text{if } i, j \in I \text{ are joined by an edge}$$

$$\sigma_i(E_i) = T_i \circ E_i \circ T_i^{-1} = F_i$$

(see Remark 5.8 for the second equation).
In a forthcoming paper, Khovanov-Lauda will construct a braid group action on the 2-categories from [KL] which satisfies (29) and (30) above. Our braid group action will then be compatible with theirs.

References

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