Rigid Analytic Geometry

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1 Introduction

For an introduction to Rigid Analytic Geometry it would be better to simply read the introduction by Bosch [Bos].

2 Tate Algebras

2.1 Non-Archimedean Absolute Values

We will be working extensively with fields equipped with a topology induced by a non-Archimedean absolute value.

2.1.1 Definition: Let K be a field. A map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is called a *non-Archimedean absolute value* if it satisfies the following properties.

For all $a, b \in K$ we have:

(a) $|a| = 0 \iff a = 0$;

(b) $|ab| = |a|\,|b|$;

(c) $|a+b| \leq \max\{|a|, |b|\}$.

We may pass back and forth from non-Archimedean absolute values and valuations by setting $\nu(a) = -\log|a|$, and given a valuation we may define $|a| = e^{-\nu(a)}$. An absolute value is trivial if its image only takes on the values 0 and 1, and is discrete if $|K^\times|$ is discrete in $\mathbb{R}_{\geq 0}$. Similarly a valuation is trivial if $\nu(K^\times) = 0$, and discrete if $\nu(K^\times)$ is discrete in $\mathbb{R}$. We will assume that our absolute values / valuations are non-trivial unless otherwise stated.

Let K be a field with a non-Archimedean absolute value. This gives rise to a distance function $d(a, b) = |a - b|$ and this lets us define a topology on K as usual. We will say K is complete if every Cauchy sequence converges. The property of being non-Archimedean has some consequences which may run counter to our intuition in say, $\mathbb{R}$.

2.1.2 Proposition
Let $a, b \in K$. If $|a| \neq |b|$ then $|a+b| = \max\{|a|, |b|\}$.

Proof
Assume that $|b| < |a|$, otherwise there is nothing to prove. Then if $|a+b| < |a|$

$$|a| = |a+b-b| \leq \max\{|a+b|, |b|\} < |a|$$

which is impossible. It follows that $|a+b| = |a| = \max\{|a|, |b|\}$. ■

2.1.3 Lemma
Let $b_n = \sum_{i=0}^{n} a_n$. The sequence $\{b_n\}$ is a Cauchy sequence if and only if the $\{a_n\}$ form a zero sequence, i.e. $\lim_{n\to\infty} |a_n| = 0$. In particular, if K is complete then $\sum_{i=0}^{\infty} a_n$ converges if and only if $\{a_n\}$ is a zero sequence.

Proof
Necessity of the $\{a_n\}$ being a zero sequence is obvious. To show sufficiency, suppose they are a zero sequence and let $\varepsilon > 0$. We must find an N such that for all $i \geq j \geq N$, we have $\left|\sum_{k=i}^{j} a_k\right| < \varepsilon$. As $\lim_{n\to\infty} |a_n| = 0$, we can find N such that $|a_n| < \varepsilon$ for all $n > N$. Now if $j \geq i \geq N$, Applying the non-Archimedean triangle inequality repeatedly tells us that

$$\left|\sum_{k=i}^{j} a_k\right| \leq \max\{|a_k|\}_{i \leq k \leq j} \leq \varepsilon$$

■

When stated in terms of the distance function, the non-Archimedean triangle inequality takes the form

$$d(y,z) \leq \max\{d(x,y), d(x,z)\},$$

where by Proposition 2.1.2 this is an equality if $d(x,y) \neq d(x,z)$. One consequence is that given three points in K, there exists at least one point whose distance to the other two is the same, i.e. every triangle is isosceles. Furthermore, any point in a disk may serve as its center. Let $B = D^{-}(x,r) = \{z \in K | d(x,z) < r\}$. Given $y \in B$, we have for any $z \in B$, $d(y,z) \leq \max\{d(x,y), d(x,z)\} < r$, so that $B \subseteq D^{-}(y,r)$, but as $x \in D_{-}(y,r)$ similarly we have $D^{-i}(y,r) \subseteq B$ (and similar for closed disks). Thus, if two disks intersect, we can take a point of the intersection as a common center to see they are concentric.

By definition of the topology every open disk $D^{-}(y,r)$ is open, but it is also closed. To see this, let z be in the complement, and look at $D^{-}(z,r)$. If this intersects $D^{-}(y,r)$, taking a point x in the intersection we see that $d(y,z) \leq \max\{d(x,y), d(x,z)\} < r$ so that $z \in D^{-}(y,r)$, a contradiction. Thus we see that the complement is open. We also have that the closed disk $D^{+}(y,r) = \{z \in K | d(y,z) \leq r\}$ is both open and closed. It is obviously closed, and to see that it is open we note that it is the union of the open disks of radius r around each point within it. We also have the boundary, $\partial D(y,r) = \{z \in K | d(x,z) = r\}$, it is certainly closed, but it is also open, as Proposition 2.1.2 tells us that for any $x \in \partial D(y,r)$, $D^{-}(x,r) \subseteq \partial D(y,r)$.

2.1.4 Proposition
The topology of K is totally disconnected.

Proof
Let $S \subseteq K$ be any set with at least two points $a \neq b$. Set $\delta = \frac{d(a,b)}{2}$, and let $S_1 = D^-(a, \delta) \cap S$ and $S_2 = S \setminus S_1$. S_1 and S_2 are both closed and open inside of S, and S is the union of S_1 and S_2. Note that $a \in S_1$ and $b \in S_2$, so that S cannot be connected. ∎

We may attempt to try to establish a form of analysis on K. As it turns out, the standard ideas from real or complex analysis do not behave well. For example, there can exist no non-constant paths in K so that the notion of a line integral does not naively work. We could attempt to restrict to analytic functions, those that are modelled on convergent power series. As it turns out, the usual definition from complex analysis only requiring that it locally agree with a convergent power series does not behave well. To demonstrate this, for an open set U, call a function $f: U \to K$ *locally analytic* if there it agrees with a convergent power series locally around each point of U. Taking a closed or open disk $U = D^{\pm}(x, r)$, we have that $U = \bigcup_{y \in U} D^{-i}(y, r)$. As intersecting disks are concentric, we may thus write U as a disjoint union of open disks. Call these disks D_i, and given a collection of convergent power series f_i on D_i, the function $f: U \to K$ defined by letting $f\downharpoonright_{D_i} = f_i$ will be locally analytic. We could let the f_i then be constants for example, and observe that a locally analytic function doesn't enjoy reasonable global properties. For this reason, a basic principle in rigid geometry is that analytic functions on disks must have a global convergent power series expansion.

2.2 Valuation Theory

Here we will state without proof various facts about valuations. The main result we want is that we can uniquely extend an absolute value from K to its algebraic closure. For proofs see Appendix A of [Bos].

Let K be a field with a non-trivial non-Archimedean absolute value $|\cdot|$. Let V be a K-vector space, a *vector space norm* is a function $\|\cdot\| : V \to K$ such that the following holds for $\alpha \in K, x \in V$:

(a) $\|x\| = 0 \iff x = 0$;

(b) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$;

(c) $\|\alpha x\| = |\alpha| \|x\|$.

If confusion is not likely to arise we will denote the vector space norm by $|\cdot|$ as well. Given a finite dimensional vector space V, fix a basis $v_1, \ldots, v_d$. We define the *maximum norm* $|\cdot|_{\max}$ on an element $x \in V$ by writing $x = \sum \alpha_i v_i$, then setting $|x|_{\max} = \max\{|\alpha_i|\}$. If K is complete then V will be complete with respect to this norm.

As usual, a vector space norm will define a topology on V, and the maximum norms obtained by taking different bases for V are all equivalent. If K is complete then a stronger statement is possible:

2.2.1 Theorem
Let V be a finite dimensional vector space over K. If K is complete then all vector space norms on V are equivalent. In particular, V is complete with respect to any such norm.

2.2.2 Corollary
Let L/K be an algebraic extension, and assume that K is complete. Any absolute value on L restricting to the given absolute value on K are equal.

This corollary shows that any extension of our absolute value on K will be unique, but we have to show that an extension exists in general.

2.2.3 Theorem
Let L/K be an algebraic extension, and assume that K is complete. There is a unique extension of the absolute value on K to L. In fact,

$$|\alpha| = \left|N_{K(\alpha)/K}(\alpha)\right|^{1/d}$$

for $\alpha \in L$ where $N_{K(\alpha)/K}$ denotes the norm of $K(\alpha)$ over K and d is the degree of α over K.

This settles how to extend an absolute value to any algebraic extension in the case that K is complete. If K is not complete we are still okay. First, we will construct the completion of K which we denote $\hat{K}$. First consider all sequences, $K^{\mathbb{N}}$ with addition and multiplication defined component-wise. The set of Cauchy sequences $C(K)$ forms a subring of $K^{\mathbb{N}}$, and zero sequences $Z(R)$ an ideal of $C(R)$. Then $\hat{K} = C(R)/Z(R)$ is a field, and the canonical map $K \to \hat{K}$ sending an element α to the residue of the constant α sequence is a homomorphism. We can extend the norm on K to $\hat{K}$, given an element of $a \in \hat{K}$ let $\{a_i\}$ be a Cauchy sequence in K representing it. Then, the sequence $\{|a_i|\}$ is either a zero sequence or becomes constant after a certain index i_0, and so $c = \lim_{i\to\infty} |a_i|$ exists and we set $|a| = c$. This number is well-defined as $\{a_i\}$ is well-defined up to a null sequence. $\hat{K}$ will be complete with respect to this absolute value and contains K as a dense subfield.

Now let L/K be an algebraic extension. Extend the absolute value of K to $\hat{K}$, and then to $\overline{\hat{K}}$ which we know how to do as $\hat{K}$ is complete. We can then pick a K-embedding $L \to \overline{\hat{K}}$ and then pullback the absolute value on $\overline{\hat{K}}$ along this embedding. This does not yield a canonical extension of $|\cdot|$ to L however as it depended on the choice of embedding.

Note that the algebraic closure of a complete field may not be complete, so that $\overline{\hat{K}}$ will not in general be complete. However, if we start with an algebraically closed field its completion will remain algebraically closed.

2.2.4 Theorem
Let K be an algebraically closed field with a non-Archimedean absolute value. Its completion $|K|$ is algebraically closed.

2.3 Restricted Power Series

We continue to let K be a complete field equipped with a non-trivial non-Archimedean absolute value. Note that the absolute value of K extends to a unique absolute value on $\overline{K}$, and that while $\overline{K}$ itself may not be complete, every finite subextension of it will be complete. For integers $n \geq 1$ let

$$\mathbb{B}^n(\overline{K}) = \{(x_1, \dots, x_n) \in \overline{K}^n \,|\, |x_i| \leq 1\}$$

be the unit ball in $\overline{K}^n$.

2.3.1 Lemma
A formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \zeta^\alpha \in K[[\zeta_1, \dots, \zeta_n]]$$

converges on $\mathbb{B}^n(\overline{K})$ if and only if $\lim_{|\alpha| \to \infty} |c_\alpha| = 0$.

Proof
If f converges at $(1, \dots, 1)$ the series $\sum_\alpha c_\alpha$ converges so that we must have $\lim_{|\alpha| \to \infty} |c_\alpha| = 0$ by Lemma 2.1.3. Conversely, given a point $x \in \overline{B}^n(\overline{K})$, there is a finite, and thus complete, subextension K' such that all components of x belong to K'. Then, if $\lim_{|\alpha| \to \infty} |c_\alpha| = 0$, we have $\lim_{|\alpha| \to \infty} |c_\alpha| \, |x^\alpha| = 0$, so that by Lemma 2.1.3 again we have that $f(x)$ is convergent in $K' \subseteq \overline{K}$. ■

2.3.2 Definition: The K-algebra $T_n = K \langle \zeta_1, \dots, \zeta_n \rangle$ of all formal power series in n-variables converging on $\mathbb{B}^n(\overline{K})$ is called the *Tate algebra* of *restricted* or *strictly convergent* power series. We let $T_0 = K$.

We see that the map T_n to the set of maps $\mathbb{B}^n(\overline{K}) \to \overline{K}$ is a K-algebra map, so that we may consider T_n as actual functions on the closed unit ball. We define the Gauss norm on T_n by setting

$$|f| = \max\{|c_\alpha|\} \quad \text{for} \quad f = \sum_\alpha c_\alpha \zeta^\alpha$$

It satisfies all the conditions for a K-algebra norm which are the same as a K-vector space norm with the additional condition that it is multiplicative. Strictly speaking, we only require $|fg| \leq |f| \, |g|$ for a K-algebra norm but the stronger condition that this is an equality is satisfied by the Gauss norm. The only non-obvious condition to verify is that it is multiplicative which we will detail. First, $|fg| \leq |f| \, |g|$ trivially. Now, we will look at the valuation ring $R = \{a \in K \,|\, |a| \leq 1\}$, this is a local subring of K with maximal ideal $\mathfrak{m}$ those of norm strictly less than one. Let $k = R/\mathfrak{m}$ be the residue field of K, and let $R \langle \zeta_1, \dots, \zeta_n \rangle$ be the R-algebra of restricted power series $f \in T_n$ whose coefficients lie in R. Note that this is equivalent to requiring that $|f| \leq 1$. The canonical projection extends to a map $\pi \colon R \langle \zeta_1, \dots, \zeta_n \rangle \to k[\zeta_1, \dots, \zeta_n]$ as all coefficients of f of norm less than one will be killed in this map. As f converges the coefficients tend to 0 so that the result is actually a polynomial. Denote $\pi(f)$ as $\overline{f}$. Consider $f, g \in T_n$ with $|f| = |g| = 1$. Then $f, g, fg \in R \langle \zeta_1, \dots, \zeta_n \rangle$, and we have $\pi(fg) = \overline{f}\overline{g} \neq 0$ as $k[\zeta_1, \dots, \zeta_n]$ is an integral domain. This says that $|fg| = 1$.

In general, let $f, g \in T_n$, then $|fg| = |f|\,|g|$ is trivial if f or g are constant. If both are non-constant, we may write $f = cf'$ and $g = dg'$ with $|f| = |c|, |g| = |d|$ and $|f'| = |g'| = 1$ where c, d are constants. Then

$$|fg| = |cdf'g'| = |cd|\,|f'g'| = |c|\,|d| = |f|\,|g|$$

2.3.3 Proposition
T_n is complete with respect to the Gauss norm. This means it is a Banach K-algebra, i.e. a K-algebra complete under the given K-algebra norm.

Proof
Consider a series $\sum^\infty f_i$ with restricted power series $f_i = \sum_\alpha c_{i\alpha}\zeta^\alpha \in T_n$ satisfying $\lim_{i\to\infty}|f_i| = 0$. Then, as $|c_{i\alpha}| \leq |f_i|$, we have $\lim_{i\to\infty}|c_{i\alpha}| = 0$ for all α; as a result the limits $c_\alpha = \sum^\infty c_{i\alpha}$ exist. We claim that $f = \sum_\alpha c_\alpha \zeta^\alpha$ is strictly convergent and that $f = \sum^\infty f_i$.

Choose $\varepsilon > 0$. As the f_i form a zero sequence there is an integer N such that $|c_{i\alpha}| < \varepsilon$ for all $i \geq N$, and all α. As the coefficients of the series $f_0, \ldots, f_{N-1}$ form a zero sequence, almost all of these coefficients also have absolute value less than ε. This implies that almost all of the values $|c_{i\alpha}|$ with arbitrary i, α are smaller than ε, and hence the elements $c_{i\alpha}$ form a zero sequence in K under any ordering. Now, the non-Archimedean triangle inequality generalizes to convergent series so that

$$\left|\sum^\infty a_n\right| \leq \max\{|a_n|\},$$

and this shows that f belongs to T_n and that $f = \sum^\infty f_i$. ■

2.3.4 Corollary
A series $f \in T_n$ with $|f| = 1$ is a unit if and only if its reduction $\overline{f} \in k[\zeta_1, \ldots, \zeta_n]$ is a unite, i.e. if and only if $|f| \in k^\times$. More generally, an arbitrary series $f \in T_n$ is a unit if and only if $|f - f(0)| < |f(0)|$ i.e. if and only if the absolute value of the constant coefficient of f is strictly bigger than all the other coefficients of f.

Proof
It is only necessary to consider elements $f \in T_n$ with norm one as we may divide by a constant. If f is a unit in T_n, then its inverse also has norm 1 so f is a unit in $R\langle\zeta_1, \ldots, \zeta_n\rangle$. Then, $\overline{f}$ is a unit of $k[\zeta_1, \ldots, \zeta_n]$ so is in $k^\times$. This implies that the constant has norm strictly larger than all other coefficients. Conversely, if $|f| \in k^\times$, the constant term $f(0)$ satisfies $|f(0)| = 1$, and we may in fact assume that $f(0) = 1$. Then $f = 1 - g$ with $|g| < 1$, and $\sum_0^\infty g^i$ is an inverse of f. ■

2.3.5 Proposition
(Maximum Principle). Let $f \in T_n$. Then $|f(x)| \leq |f|$ for all points $x \in \mathbb{B}^n(\overline{K})$, and there exists a point $x \in \mathbb{B}^n(\overline{K})$ such that $|f(x)| = |f|$.

Proof

The first assertion is trivial. For the second claim, assume that $|f| = 1$ and consider the projection $R\langle \zeta_1, \dots, \zeta_n \rangle \to k[\zeta_1, \dots, \zeta_n]$. Then $\widetilde{f} := \pi(f)$ is a non-trivial polynomial in n variables, and there will exist a point $\widetilde{x} \in \overline{k}^n$ with $\widetilde{f}(\widetilde{x}) \neq 0$. $\overline{k}$ can be interpreted as the residue field of $\overline{K}$, so writing $\overline{R}$ for the valuation ring of $\overline{K}$, pick a lifting $x \in \mathbb{B}^n(\overline{K})$ of $\widetilde{x}$. Consider the commutative diagram

$$\begin{array}{ccc} R\langle \zeta_1, \dots, \zeta_n \rangle & \longrightarrow & k[\zeta_1, \dots, \zeta_n] \\ \downarrow & & \downarrow \\ \overline{R} & \longrightarrow & \overline{k} \end{array}$$

Here, the first vertical map is evaluation at x, and the second the evaluation at $\widetilde{x}$. The former map actually lands in $\overline{R}$ by applying the non-archimedean triangle inequality for convergent series along with the fact that the norm of x_i and all coefficients of f are bounded by one. $f(x) \in \overline{R}$ will be mapped to $\widetilde{f}(\widetilde{x})$, and we know the latter is non-zero which tells us that $|f(x)| = 1 = |f|$ which was our goal. ∎

The Tate algebra T_n has much in common with the polynomial ring in n variables over K. In the sequel we will establish various properties of T_n which will illustrate its similarities with polynomial algebras.

2.3.6 Definition: A restricted power series $g = \sum^{\infty} g_k \zeta^k \in T_n$ with coefficients $g_\alpha \in T_{n-1}$ is called ζ_n-*distinguished* of order $s \in \mathbb{N}$ if the following hold:

(a) g_s is a unit in T_{n-1};

(b) $|g_s| = |g|$ and $|g_s| > |g_k|$ for $k > s$.

In particular, if $g = \sum^{\infty} g_k \zeta_n^k$ satisfies $|g| = 1$, then g is ζ_n-distinguished of order s if and only if its reduction $\overline{g}$ can be written

$$\overline{g} = \overline{g_s}\zeta_n^s + \overline{g}_{s-1}\zeta_n^{s-1} + \cdots + \overline{g_0}\zeta_n^0$$

with $\overline{g}_s \in k^\times$ by Corollary 2.3.4. Thus an arbitrary series $g \in T_n$ is distinguished of order 0 if and only if it is a unit. Furthermore, for $n = 1$, every non-zero element $g \in T_1$ is ζ_1-distinguished of some order $s \in \mathbb{N}$.

2.3.7 Lemma

Given finitely many non-zero elements $f_1, \dots, f_r \in T_n$, there is a continuous automorphism

$$\sigma \colon T_n \to T_n, \qquad \zeta_i \mapsto \begin{cases} \zeta_i - \zeta_n^{\alpha_i} & \text{for } i < n \\ \zeta_n & \text{for } i = n \end{cases}$$

for suitable exponents $\alpha_i \in \mathbb{N}$ such that the elements $\sigma(f_1), \ldots, \sigma(f_r)$ are ζ_n-distinguished. Furthermore, $|\sigma(f)| = |f|$ for all $f \in T_n$.

Proof
[Bos, 2.2 Lemma 7] ■

The reason for considering distinguished elements in T_n is that we can perform a form of a division algorithm by such elements.

2.3.8 Theorem
(Weierstrass Division). Let $g \in T_n$ be ζ_n-distinguished of some order s. Then, for any $f \in T_n$, there is a unique series $q \in T_n$ and a unique polynomial $r \in T_{n-1}[\zeta_n]$ of degree $r < s$ satisfying

$$f = qg + r.$$

Furthermore, $|f| = \max\{|q|\,|g|, |r|\}$.

Proof
[Bos, 2.2 Theorem 8] A very cool proof. ■

2.3.9 Corollary
(Weierstrass Preparation Theorem). Let $g \in T_n$ be ζ_n-distinguished of order n. Then there exists a unique monic polynomial $\omega \in T_{n-1}[\zeta_n]$ of degree s such that $g = e\omega$ for a unit $e \in T_n$. Furthermore, $|\omega| = 1$ so that ω is ζ_n-distinguished of order s.

Proof
Apply the Weierstrass division formula to obtain the equation

$$\zeta_n^s = qg + r$$

with $q \in T_n$, and $r \in T_{n-1}[\zeta_n]$ of degree $< s$ satisfying $|r| \leq 1$. Write $\omega = \zeta_n^s - r$, and then $\omega = qg$, satisfies $\omega = 1$, and is ζ_n-distinguished of order s. We have to show that q is a unit in T_n, then we can let $e = q^{-1}$ to get the equality stated in the corollary. Assuming $|g| = |q| = 1$, we look at $\widetilde{\omega} = \widetilde{q}\widetilde{g}$. As $\widetilde{\omega}$ and $\widetilde{g}$ are polynomials of degree s in ζ_n, it follows that $\widetilde{q}$ is a unit in k as $\widetilde{\omega}$ is monic. It follows that ω is a unit in T_n by Corollary 2.3.4.

To show uniqueness, suppose $g = e\omega$. Let $r = \zeta_n^s - \omega$, and then

$$\zeta_n^s = e^{-1}g + r$$

and by uniqueness of Weierstrass division we know that e^{-1} and r are unique, which implies uniqueness of e and of ω. ■

2.3.10 Corollary
The Tate algebra $T_1 = K\langle\zeta_1\rangle$ of restricted power series in a single variable is a Euclidean domain.

Proof

Every non-zero element $g \in T_1$ is ζ_1-distinguished of a well-defined order $s \in \mathbb{N}$. Using Weierstrass division, we see that the map $T_1 \setminus \{0\} \to \mathbb{N}$ sending g to its order s of being distinguished is a Euclidean norm. ■

Monic polynomials $\omega \in T_{n-1}[\zeta_n]$ with $|\omega| = 1$ such as those appearing in the Weierstrass Preparation Theorem are called Weierstrass polynomials in ζ_n. Thus, each ζ_n-distinguished element $f \in T_n$ is associated to a Weierstrass polynomial. Furthermore, if f is any non-zero element of T_n, by Lemma 2.3.7 we can assume that the indeterminates $\zeta_1, \ldots, \zeta_n$ are chosen in a way that f is ζ_n-distinguished of some order s. In fact, we can do this for any finite set of f_i simultaneously.

2.3.11 Corollary

(Noether Normalization). For any proper ideal $I \subsetneq T_n$, there is an injective map of K-algebras $T_d \to T_n$ for some $d \in \mathbb{N}$ such that the composition

$$T_d \to T_n \to T_n/I$$

is a finite injection. The integer d is equal to the Krull dimension of T_n/I.

Proof

As we will see later, T_d is dimension d, and the dimension cannot change in a finite extension so the last part of the corollary is obvious.

Assuming $I \neq 0$ we pick a non-zero $g \in I$. Applying an automorphism to T_n, we assume that g is ζ_n-distinguished of some order $s \geq 0$. By Weierstrass division we know that any $f \in T_n$ is congruent mod g to a polynomial $r \in T_{n-1}[\zeta_n]$ of degree $< s$. In other words, $\{1, \ldots, \zeta_n^{s-1}\}$ forms a spanning set for $T_n/(g)$ over T_{n-1} so that

$$T_{n-1} \to T_n \to T_n/(g)$$

is a finite map. In fact, by uniqueness of Weierstrass division $T_n/(g)$ is a free T_{n-1} module on those elements.

Consider the composition $T_{n-1} \to T_n/(g) \to T_n/I$ and write I_1 for its kernel. If $I_1 = 0$ we are done. Otherwise, we can do the same with I_1 and T_{n-1}, then by composing our finite maps we eventually will get a finite injection $T_d \to T_n/I$ after finitely many steps. ■

One should note that the map $T_d \to T_n/I$ does **not** correspond to the obvious map sending $\zeta_i \in T_d$ to the class of $\zeta_i \in T_n/I$ as we had to twist by an automorphism of T_m at each step. The obvious map will in general be neither finite nor injective.

2.3.12 Corollary

Let $\mathfrak{m} \subseteq T_n$ be a maximal ideal. Then the field $T_n/\mathfrak{m}$ is finite over K.

Proof

Using Noether normalization, we obtain a finite injective map $T_d \to T_n/\mathfrak{m}$ for some $d \in \mathbb{N}$. This is an integral extension of integral domains, so since $T_n/\mathfrak{m}$ is a field so is T_d. This implies that $d = 0$, and that $T_d = K$. ■

As a consequence we obtain the Nullstellensatz

2.3.13 Corollary
(Nullstellensatz) The map

$$\mathbb{B}^n(\overline{K}) \to \mathrm{MSpec} T_n, \quad x \mapsto \mathfrak{m}_x = \{f \in T_n | f(x) = 0\}$$

is surjective.

Proof
[Bos, 2.2 Corollary 13]. ■

We now derive a few other properties of T_n which show that it is a lot like a polynomial ring.

2.3.14 Proposition
T_n is Noetherian.

Proof
Proceeding by induction, we can assume that T_{n-1} is Noetherian. Let I be a non-trivial ideal of T_n, and pick $g \in I$ which using 2.3.7 we an assume is ζ_n-distinguished. By Weierstrass division, $T_n/(g)$ is a finite T_{n-1}-module, and is thus Noetherian. $I/(g)$ is then finitely generated over T_{n-1}, and so I is finitely generated over T_n. ■

2.3.15 Proposition
T_n is a UFD, and thus normal.

Proof
We again proceed by induction, so we assume T_{n-1} is a UFD, so that $T_{n-1}[\zeta_n]$ is a UFD. Let $f \in T_n$ be a non-zero non-unital element. By 2.3.7 and by Weierstrass preparation we may assume that f is a Weierstrass polynomial, so $|f| = 1$, and $f \in T_{n-1}[\zeta_n]$ is monic of degree s, and thus ζ_n-distinguished of order s. Factor $f = \omega_1 \cdots \omega_r$ into prime elements $\omega_i \in T_{n-1}[\zeta_n]$. As f is monic in ζ_n, we assume the same is true for ω_i. Then as $|\omega_i| \geq 1$, we have $|\omega_i| = 1$ as $|f| = 1$. Thus, the ω_i are also Weierstrass polynomials.

It remains to show that the ω_i being prime in $T_{n-1}[\zeta_n]$ implies they are prime in T_n as well. We will show that it is prime, we show that $T_{n-1}[\zeta_n]/(\omega) \to T_n/(\omega)$ is an isomorphism. This is clear however as both are free T_{n-1}-modules generated by the classes of $\zeta_n^0, \ldots, \zeta_n^{s-1}$. Thus T_n is a UFD. ■

2.3.16 Proposition
T_n is Jacobson, i.e. $\sqrt{I}$ is the intersection of all maximal ideal containing I for all ideals I.

Proof
It suffices to only consider prime ideals $\mathfrak{p} \subseteq T_n$. Consider the case of $\mathfrak{p} = 0$ first, we have to show that the Jacobson radical of T_n is zero. If $f \in \mathrm{rad}(T_n)$, then by the Nullstellensatz f vanishes at all points $x \in \mathbb{B}^n(\overline{K})$, and so $f = 0$ by the maximum principle.

Note that being Jacobson is equivalent to having $T_n/\mathfrak{p}$ satisfy the nilradical equaling the Jacobson radical, i.e. the Jacobson radical is trivial. Assume that $\mathfrak{p}$ is an arbitrary non-zero prime ideal. Using Noether normalization we get a finite embedding $T_d \to T_n/\mathfrak{p}$ for some $d < n$. By the going up theorem, for any maximal $\mathfrak{m} \subseteq T_d$, there exists a maximal $\mathfrak{m}' \subseteq T_n/\mathfrak{p}$ lying over $\mathfrak{m}$. As a result, if $\mathfrak{q} \subseteq T_n/\mathfrak{p}$ is the intersection of all maximal ideals in $T_n/\mathfrak{p}$, we know that $\mathfrak{q} \cap T_d = 0$ by induction on n teling us that T_d is Jacobson. Now if $\mathfrak{q}$ is non-zero, pick an element $f \in \mathfrak{q}$, and let

$$f^r + a_1 f^{r-1} + \cdots + a_r = 0$$

be an equation of minimal degree for f over T_d. Necessarily $a_r \neq 0$. We however see that $a_r \in \mathfrak{q} \cap T_d$ and thus, is zero. We therefore must have $\mathfrak{q} = 0$ ■

Proof
Every maximal ideal $\mathfrak{m} \subseteq T_n$ is of height n, and can be generated by n elements. Thus dim $T_n = n$, and furthermore it is a regular local ring.
Proof
[Bos, 2.2, Proposition 17]. ■

2.4 Ideals in Tate Algebras

Consider an ideal $I = (a_1, \dots, a_r)$, with generators a_i satisfying $|a_i| = 1$. If we could show that $f \in I$ admits a presentation $f = \sum f_i a_i, f_i \in T_n$ satisfying $|f_i| \leq |f|$, then we can deduce that I is complete under the Gauss norm; so in addition I is closed in T_n.

2.4.1 Definition: Let R be a ring. A ring norm on R is a map $|\cdot| : R \to \mathbb{R}_{\geq 0}$ satisfying

(a) $|a| = 0 \equiv a = 0$;

(b) $|ab| \leq |a|\,|b|$;

(c) $|a + b| \leq \max\{|a|, |b|\}$;

(d) $|1| \leq 1$.

The norm is called multiplicative if instead of (b) we have

$$\text{(b) } |ab| = |a|\,|b|\,.$$

One sees that these conditions imply that $|1| = 1$ as long as R is non-zero. Indeed, $|1| \leq |1|^2$ by (b), and so $|1| = |1|^2$ since $|1| \leq 1$ by (d). Thus, $|1| = 1$ or 0, but it cannot be 0 by (a) unless R is zero.

2.4.2 Definition: Let R be a ring with a multiplicative ring norm $|\cdot|$ such that $|a| \leq 1$ for all $a \in R$.

(a) R is called a *B-ring* if

$$\{a \in R| \, |a| = 1\} \subseteq R^{\times}.$$

(b) R is called *bald* if

$$\sup\{|a| \, |a \in R \text{ with } |a| < 1\} < 1.$$

Note that a B-ring is local with maximal ideal $\mathfrak{m} = \{a \in R| \, |a| < 1\}$ as the units are precisely those of value 1.

2.4.3 Proposition
Let K be a field with a valuation and R its valuation ring. Then the smallest subring $R' \subseteq R$ containing a given zero sequence $a_0, a_1, \dots \in R$ is bald.

Proof
The smallest subring $S \subseteq R$ equals either $\mathbb{Z}/p\mathbb{Z}$ for some prime p, or $\mathbb{Z}$. It is bald as any valuation on $\mathbb{Z}/p\mathbb{Z}$ is trivial, and since the ideal $\{a \in \mathbb{Z}| \, |a| < 1\}$ is principal, so that $|a|$ for any a in the ideal is less than the absolute value of the generator. If there is an $\in \mathbb{R}$ such that $|a_n| \leq< 1$ for all $n \in \mathbb{N}$, we see trivially that $S[a_0, a_1, \dots,]$ is bald. Thus, it is enough to show that for a bald subring $S \subseteq R$, and an element $a \in R$ of value $|a| = 1$, the ring $S[a]$ is bald. We may first localize S at all elements of value 1, and thus assume that S is a B-ring, so it is local with maximal ideal $\mathfrak{m}$, so that $\widetilde{S} = S/\mathfrak{m}$ is a field. If the reduction $\widetilde{a} \in k$ is transcendental over $\widetilde{S}$, then we can show that $S[a]$ is bald. Indeed, for any polynomial $p = \sum c_i \zeta^i \in S[\zeta]$, we have $|p(a)| < 1$ if and only if $\sum \widetilde{c_i} \widetilde{a}^i = 0$, which implies $\widetilde{c_i} = 0$ for all i as $\widetilde{a}$ is transcendental. Thus, $|p(a)| < 1$ implies

$$|p(a)| \leq \sup\{|c| \, |c \in S, |c| < 1\} < 1$$

and we are done.

If $\widetilde{a}$ is algebraic over $\widetilde{S}$, pick a polynomial

$$g = \zeta^n + c_1 \zeta^{n-1} + \dots + c_n \in S[\zeta]$$

of minimal degree such that $\widetilde{g}$ annihilates $\widetilde{a}$, or equivalently $|g(a)| < 1$. Let $\varepsilon < 1$ be the supremum of $|g(a)|$ and all $|c|$ for $c \in S$ of value less than one. This is less than one since it's the supremum of a single element less than one, and a set whose supremum is less than one. Consider a polynomial $f \in S[\zeta]$ with $|f(a)| < 1$; we wish to show $|f(a)| \leq \varepsilon$ and then we will be done as every element in $S[a]$ is of the form $f(a)$, and the supremum will be bounded by ε. We write $f = qg + r$ with $q, r \in S[\zeta]$ and $\deg r < n = \deg g$. Note that $|g(a)| \leq \varepsilon$, and $|q(a)| \leq 1$ as $|a| = 1$, and q's coefficients have value ≤ 1. As a result,

$|(f - qg)(a)| \leq \max\{|f(a)|, |qg(a)|\} < 1$. Furthermore, if $|(f - qg)(a)| \leq \varepsilon$ then $|f(a)| \leq \varepsilon$. To see this, suppose otherwise, then

$$|(f - qg)(a)| \leq \max\{|f(a)|, |qg(a)|\}$$

but as $|f(a)| > \varepsilon \geq |qg(a)|$ we know that we have equality, implying that $|(f - qg)(a)| > \varepsilon$ contrary to assumption. As a result, we can replace f with $f - qg$ and assume that $f = r$.

If all coefficients of r have value < 1, then they have value $\leq \varepsilon$, and then $|r(a)| \leq \varepsilon$ as desired. If on the other hand one of the coefficients of r has value 1, then $\widetilde{r}$ is non-trivial, but we have $\widetilde{r}(\widetilde{a}) = 0$ since $|r(a)| = |f(a)| = 0$, and this contradicts the minimality of $\widetilde{g}$ with respect to annihilates $\widetilde{a}$ as $\deg \widetilde{r} < n = \deg \widetilde{g}$. As a result, $|f(a)| < 1$ implies $|f(a)| \leq \varepsilon$ and we are done. ■

Given a bald subring $R' \subseteq R$, we can localize R' at all elements of value 1 and obtain a B-ring $R'' \subseteq R$ containing R' and which is bald. Then if R is complete, then the completion of R'' is bald, and is again a B-ring as the completion of a B-ring is a B-ring. To see this, let $x \in \hat{A}$ be an element of the completion of a B-ring A. Choose $a \in A$ with $|x - a| < 1$, then $|a| = 1$ as otherwise

$$|x| = |(x - a) + a| \leq \max\{|x - a|, |a|\} < 1.$$

Now write $x = a(1 - z)$ where $z = a^{-1}(a - x)$, we see then that $|z| < 1$. As a result, $(1 - z)^{-1} = \sum |z|^n$ which converges, and then $x^{-1} = a^{-1}(1 - z)^{-1}$. This shows that the smallest complete B-ring in R containing any given bald subring of R will be bald again.

Now we turn our eyes at vector spaces and norms on them. Let K be a field with a complete non-Archimedean valuation.

2.4.4 Definition: Let V be a K-vector space. A *norm* on V is a map $|\cdot| : V \to \mathbb{R}_{\geq 0}$ such that

(a) $|x| = 0 \equiv x = 0$;

(b) $|x + y| \leq \max\{|x|, |y|\}$;

(c) $|cx| = |c| |x|$ for $c \in K, x \in V$.

2.4.5 Definition: Let V be a complete normed K-vector space. A system $(x_\nu)_{\nu \in N}$ where N is finite or countably infinite is called a *(topological) orthonormal basis* of V if the following hold:

(a) $|x_\nu| = 1$ for all $\nu \in N$;

(b) Each $x \in V$ can be written as a convergent series $x = \sum_{\nu \in N} c_\nu x_\nu$ with coefficients $c_\nu \in K$.

(c) For each equation $x = \sum_{\nu \in N} c_\nu x_\nu$ as in (b) we have $|x| = \max\{|c_\nu|\}_{\nu \in N}$. In particular, the coefficients c_ν in (b) are unique (by subtracting to presentations from each other and factoring).

As an example, the monomials $\zeta^\nu \in T_n$ form an orthonormal basis if we consider T_n as a K-vector space. For any normed K-vector space V, we use the notation

$$V^\circ = \{x \in V \,|\, |x| \leq 1\}$$

for its "unit ball" and

$$\widetilde{V} = V^\circ / \{x \in V \,|\, |x| < 1\}$$

for its reduction.

2.4.6 Theorem
Let K be a field with a complete valuation and V a complete normed K-vector space with an orthonormal basis $(x_\nu)_{\nu \in N}$. Write R for the valuation ring of K, and consider a system of elements

$$y_\mu = \sum_{\nu \in N} c_{\mu\nu} x_\nu \in V^\circ, \quad \mu \in M,$$

where the smallest subring of R containing all coefficients $c_{\mu\nu}$ is bald. Then, if the residue classes $\widetilde{y}_\mu \in \widetilde{V}$ form a k-basis of $\widetilde{V}$, the elements y_μ form an orthonormal basis of V.

PROOF
[Bos, 2.3, Theorem 6]. ■

2.4.7 Corollary
Let I be an ideal in T_n. Then there are generators $a_1, \ldots, a_r$ of I satisfying the following:

(a) $|a_i| = 1$ for all i;

(b) For each $f \in I$, there are elements $f_1, \ldots, f_r \in T_n$ such that

$$f = \sum_1^r f_i a_i, \qquad |f_i| \leq |f|.$$

PROOF
[Bos, 2.3, Corollary 7]. ■

The proof of the above corollary as given in Bosch actually shows that elements $a_1, \ldots, a_r \in I$ with $|a_i| \leq 1$ satisfy the assertion as soon as $\widetilde{a}_i$ generate the ideal $\widetilde{I} \subseteq k[\zeta]$.

2.4.8 Corollary
Each ideal $I \subseteq T_n$ is complete, and thus closed in T_n.

PROOF
Choose generators $a_1, \ldots, a_r \in I$ as in the previous corollary. If $f = \sum^{\infty} f_\lambda$ is convergent in T_n with elements $f_\lambda \in I$, there exist equations

$$f_\lambda = \sum_1^r f_{\lambda i} a_i$$

with coefficients $f_{\lambda i} \in T_n$ satisfying $|f_{\lambda i}| \leq |f_\lambda|$. Then,

$$f = \sum_1^r \left(\sum^{\infty} f_{\lambda i} \right) a_i \in I$$

and we are done. ■

2.4.9 Corollary
Each ideal $I \subseteq T_n$ is *strictly closed*, i.e. for each $f \in T_n$ there is an element $a_0 \in I$ such that

$$|f - a_0| = \inf_{a \in I} |f - a|$$

PROOF
[Bos, 2.3, Corollary 9]. ■

2.4.10 Corollary
Let $N \subseteq T_n^s$ be a T_n-submodule of a finite direct sum of T_n, and consider on T_n^s the maximum norm derived from the Gauss norms on T_n. Then there are generators $x_1, \ldots, x_r$ of N as a T_n-module satsifying:

(a) $|x_i| = 1$ for all i.

(b) For each $x \in N$, there are elements $f_1, \ldots, f_r \in T_n$ such that

$$x = \sum^r f_i x_i, \qquad |f_i| \leq |x|\,.$$

PROOF
[Bos, 2.3, Corollary 10]. ■

References

[Bos] S. Bosch: *Lectures on formal and rigid geometry*, Lecture Notes in Mathematics, vol. 2105, Springer, Cham. MR 3309387