AN EXAMPLE OF UNRAMIFIED LIFTINGS OF REPRESENTATIONS

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1. Goal

In this note, let us describe a pair of representations ρ_i over $\mathbb{Z}_p$ of a finite group G such that:

- their reductions modulo p as $\mathbb{F}_p$-representations of G are the same: $\rho_1/p \cong \rho_2/p$; and
- after base changing to $\mathbb{C}_p$, they are different representations of G.

Remark 1.1. (1) Had we allowed ramified base ring (instead of $\mathbb{Z}_p$), it would be easy to find such an example.

(2) When $p = 2$, the two characters of $G = \mathbb{Z}/2$ is such an example. But we regard it as "too special" because of the next remark.

(3) If we set $G = \mathbb{Z}/p$ and $p \geq 3$, then there is no such an example. Consequently, if $p \geq 3$ and exactly divides the order of G, then there is no such an example.

2. Relevant rings

In the following, we set $G = \mathbb{Z}/p^2$. Let us introduce some rings and their properties for later reference. First let us consider the group ring $\mathbb{Z}_p[G] = \mathbb{Z}_p[T]/(T^{p^2} - 1)$.

Lemma 2.1.

(1) *The generic fibre of the group ring is*

$$\mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p \times \mathbb{Q}_p(\zeta_p) \times \mathbb{Q}_p(\zeta_{p^2});$$

(2) *the special fibre of the group ring is*

$$\mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \mathbb{F}_p[T]/((T-1)^{p^2}).$$

Our example is inspired by [Kan98, Definition 1.4-1.5]. Following the notation in loc. cit. , let us denote:

(1) $A_1 = \mathbb{Z}_p[T]/(\Phi_{p^2}(T)) = \mathbb{Z}_p[\zeta_{p^2}]$, where $\Phi_{p^2}(T) = \sum_{i=0}^{p-1} T^{ip}$;
(2) $A_2 = \mathbb{Z}_p[T]/(T^p - 1) = \mathbb{Z}_p[\mathbb{Z}/p]$; and
(3) $A_3 = \mathbb{Z}_p[T]/(\Phi_p(T)) = \mathbb{Z}_p[\zeta_p]$, where $\Phi_p(T) = \sum_{i=0}^{p-1} T^i$.

The following lemma can be easily verified.

Lemma 2.2.

(1) *All of A_i's are p-torsion free $\mathbb{Z}_p[G]$-algebras;*
(2) $A_1[1/p] = \mathbb{Q}_p(\zeta_{p^2})$ *and* $A_1/p = \mathbb{F}_p[T]/((T-1)^{p^2-p})$;
(3) $A_2[1/p] = \mathbb{Q}_p \times \mathbb{Q}_p(\zeta_p)$ *and* $A_2/p = \mathbb{F}_p[T]/((T-1)^p)$; *and*
(4) $A_3[1/p] = \mathbb{Q}_p(\zeta_p)$ *and* $A_3/p = \mathbb{F}_p[T]/((T-1)^{p-1})$.

3. Construction

Now we are ready to give the following:

Construction 3.1. For all $0 \le j \le p$, let Λ_j be the following fibre product of $\mathbb{Z}_p[G]$-algebras:

$$\begin{CD} \Lambda_j @>>> A_1 \\ @VVV @VVV \\ A_2 @>>> \mathbb{F}_p[T]/((T-1)^j) \end{CD}$$

To our interest is the $\mathbb{Z}_p[G]$-algebra structure of Λ_j after inverting p and the reduction modulo p. Inverting p kills $\mathbb{F}_p[T]/((T-1)^j)$, hence we know that $\Lambda_j[1/p] = \mathbb{Q}_p[G]$. So these Λ_j's are just different integral lattices inside the group ring. As for the special fiber, let us look at the two extreme cases. When $j = 0$, we have $\Lambda_0 = A_1 \times A_2$, and so $\Lambda_0/p = \mathbb{F}_p[T]/((T-1)^{p^2-p}) \times \mathbb{F}_p[T]/((T-1)^p)$. On the other hand, when $j = p$, we have $\Lambda_p = \mathbb{Z}_p[G]$. Therefore, by Lemma 2.1(2), we get $\Lambda_p/p = \mathbb{F}_p[T]/((T-1)^{p^2})$.

After observing these huge amount of data[1], we may confidently guess the following:

Lemma 3.2. *For all* $0 \le j \le p$, *we have*

$$\Lambda_j/p = \mathbb{F}_p[T]/((T-1)^{p^2-p+j}) \times \mathbb{F}_p[T]/((T-1)^{p-j}).$$

Now let us make another construction:

Construction 3.3. For all $0 \le j \le p-1$, let Λ_j be the following fibre product of $\mathbb{Z}_p[G]$-algebras:

$$\begin{CD} \Lambda'_j @>>> A_1 \\ @VVV @VVV \\ A_3 @>>> \mathbb{F}_p[T]/((T-1)^j) \end{CD}$$

Similar to the case of Lemma 3.2, we guess the following:

Lemma 3.4. *For all* $0 \le j \le p-1$, *we have*

$$\Lambda'_j/p = \mathbb{F}_p[T]/((T-1)^{p^2-p+j}) \times \mathbb{F}_p[T]/((T-1)^{p-1-j}).$$

With everything stated, we can now state the example meeting our goal:

Example 3.5. Let $\rho_1 = \bigoplus_{i=1}^{p-1} \Lambda_i$ and $\rho_2 = A_3 \oplus \bigoplus_{j=1}^{p-1} \Lambda'_j$.

References

[Kan98] Ming-chang Kang. Integral representations of cyclic groups of order p^2. *J. Algebra*, 207(1):82–126, 1998.

[1] As two examples are really quite a lot, right?