WEIL RESTRICTION FOR SCHEMES AND BEYOND

LENA JI, SHIZHANG LI, PATRICK MCFADDIN, DREW MOORE, & MATTHEW STEVENSON

Abstract. This note was produced as part of the Stacks Project workshop in August 2017, under the guidance of Brian Conrad. Briefly, the process of Weil restriction is the algebro-geometric analogue of viewing a complex-analytic manifold as a real-analytic manifold of twice the dimension. The aim of this note is to discuss the Weil restriction of schemes and algebraic spaces, highlighting pathological phenoma that appear in the theory and are not widely-known.

1. Weil Restriction of Schemes

Definition 1.1. Let $S' \to S$ be a morphism of schemes. Given an S'-scheme X', consider the contravariant functor $\mathrm{R}_{S'/S}(X')\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets})$ given by

$$T \mapsto X'(T \times_S S').$$

If the functor $\mathrm{R}_{S'/S}(X')$ is representable by an S-scheme X, then we say that X is the *Weil restriction of X' along $S' \to S$*, and we denote it by $\mathrm{R}_{S'/S}(X')$.

Remark 1.2. Given any morphism of schemes $f\colon S' \to S$ and any contravariant functor $F\colon (\mathrm{Sch}/S')^{\mathrm{opp}} \to (\mathrm{Sets})$, one can define the pushforward functor $f_*F\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets})$ given on objects by $T \mapsto F(T \times_S S')$. If F is representable by an S'-scheme X', then $f_*F = \mathrm{R}_{S'/S}(X')$. This approach is later used to define the Weil restriction of algebraic spaces.

Theorem 1.3. *Let $S' \to S$ be a finite and locally free morphism. Let X' be an S'-scheme such that for any $s \in S$ and any finite set $P \subseteq X' \times_S \mathrm{Spec}(\kappa(s))$, there exists an affine open subscheme $U' \subseteq X'$ containing P. Then, the functor $\mathrm{R}_{S'/S}(X')$ is representable by an S-scheme. In particular, if X' is a quasi-projective S'-scheme, then the Weil restriction of X' exists.*

Proof. Omitted. See [BLR90, Theorem 7.6/4]. □

In Proposition 2.10, we will show that $\mathrm{R}_{S'/S}(X')$ is quasi-projective over S when X' is quasi-projective over S', where "quasi-projective" is defined as in [EGA, II, Définition 5.3.1]. In [Wei82, §1.3], Weil introduced his version of the Weil restriction of a quasi-projective scheme over a field. His construction is presented below and it is shown to coincide with the modern definition in the case of quasi-projective schemes over a field.

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Theorem 1.4. *Let k'/k be a finite separable extension, let X' be a quasi-projective k'-scheme, and let F/k be a finite Galois extension that splits k'/k. Let*

$$\overline{X} := \prod_{j\colon k' \hookrightarrow F} X' \times_{k',j} F,$$

where the product runs over all embeddings $k' \hookrightarrow F$ over k. Then, for $\sigma \in \mathrm{Gal}(F/k)$, there exists an isomorphism $\varphi_\sigma\colon \overline{X} \simeq \overline{X}$ over $\mathrm{Spec}(\sigma)$, such that $\big(\overline{X}, \{\varphi_\sigma\}_{\sigma\in\mathrm{Gal}(F/k)}\big)$ is an effective descent data, giving the k-scheme $\mathrm{R}_{k'/k}(X')$.

Proof. Let J denote the set of embeddings $j\colon k' \hookrightarrow F$ over k and let $G = \mathrm{Gal}(F/k)$. If $j \in J$, let F_j denote the field F viewed as a k'-algebra via the embedding j, and let $X'_j = X' \times_{k'} F_j$. If $j \in J$ and $\sigma \in G$, then $\sigma \circ j \in J$, and this induces an isomorphism $X'_{\sigma\circ j} \simeq X'_j$ over $\mathrm{Spec}(\sigma)$. Taking the product over all $j \in J$ yields an isomorphism $\varphi_\sigma\colon \overline{X} \simeq \overline{X}$ over $\mathrm{Spec}(\sigma)$. For any $\sigma, \tau \in G$, it is clear that $\varphi_\sigma \circ \varphi_\tau = \varphi_{\sigma\circ\tau}$, so $\big(\overline{X}, \{\varphi_\sigma\}_{\sigma\in G}\big)$ is a descent datum.

If $X := \mathrm{R}_{k'/k}(X')$, the F-scheme $X \times_k F$ has the canonical descent datum $\{\phi_\sigma\}_{\sigma\in G}$, where ϕ_σ is the automorphism of $X \times_k F$ over $\mathrm{Spec}(\sigma)$ given by $1 \times \mathrm{Spec}(\sigma)$. It suffices to construct an F-isomorphism $\psi\colon X \times_k F \simeq \overline{X}$ such that, for any $\sigma \in G$, $\psi \circ \phi_\sigma \circ \psi^{-1} = \varphi_\sigma$ (i.e. under the isomorphism ψ, the descent data $\{\phi_\sigma\}$ is sent to $\{\varphi_\sigma\}$).

To construct the isomorphism ψ, consider

$$\begin{aligned} X \times_k F &\simeq \mathrm{R}_{(k'\otimes_k F)/F}\left(X' \times_{k'} (k' \otimes_k F)\right) \\ &\simeq \mathrm{R}_{(\bigsqcup_{j\in J} F_j)/F}\left(\bigsqcup_{j\in J} X' \times_{k'} F_j\right) \\ &\simeq \prod_{j\in J} X' \times_{k'} F_j \\ &= \overline{X}, \end{aligned}$$

where the first isomorphism follows from Proposition 2.2(2) below, and the third isomorphism follows from Lemma 2.3. One can verify that the canonical descent datum $\{\phi_\sigma\}_{\sigma\in G}$ is carried to the descent datum $\{\varphi_\sigma\}_{\sigma\in G}$ under this isomorphism. □

2. Basic Properties of Weil Restriction

Proposition 2.1. *Let $S' \to S$ be a morphism of schemes, let X be an S-scheme and $X' = X \times_S S'$. Then we have*

$$\mathrm{R}_{S'/S}(X') = \underline{\mathrm{Hom}}_S(S', X).$$

Proof. This is immediate from the universal property. □

Proposition 2.2. *Let $S' \to S$ be a finite, locally free morphism of schemes and let X' be an S'-scheme such that the Weil restriction $\mathrm{R}_{S'/S}(X')$ exists as an S-scheme.*

1. *If X', S', and S are affine with X' of finite type over S', then $\mathrm{R}_{S'/S}(X')$ is affine of finite type over S.*
2. *If T is an S-scheme and $T' = T \times_S S'$, then there is an isomorphism*

$$\mathrm{R}_{T'/T}(X' \times_{S'} T') \simeq \mathrm{R}_{S'/S}(X') \times_S T.$$

of functors on (Sch/T).

(3) *If $X' \to Z'$ and $Y' \to Z'$ are morphisms of S'-schemes, then there is an isomorphism*

$$\mathrm{R}_{S'/S}(X' \times_{Z'} Y') \simeq \mathrm{R}_{S'/S}(X') \times_{\mathrm{R}_{S'/S}(Z')} \mathrm{R}_{S'/S}(Y')$$

of functors on (Sch/S).

Proof. Omitted. See the proof of [CGP15, Proposition A.5.2]. □

Lemma 2.3. *Let S be a scheme. Given a finite collection $\{S_i\}_{i\in I}$ of S-schemes, consider the S-scheme $S' = \bigsqcup_{i\in I} S_i \to S$. Given a collection, for each $i \in I$, of S_i-schemes X_i, consider the S'-scheme*

$$X' = \bigsqcup_{i\in I} X_i.$$

Then, there is an isomorphism

$$\mathrm{R}_{S'/S}(X') \simeq \prod_{i\in I} \mathrm{R}_{S_i/S}(X_i)$$

of functors on (Sch/S).

Proof. For any S-scheme Y, observe that

$$\begin{aligned}
\mathrm{Hom}_S(Y, \mathrm{R}_{S'/S}(X')) &\simeq \mathrm{Hom}_{S'}(Y \times_S S', X') \\
&\simeq \prod_{i\in I} \mathrm{Hom}_{S_i}(Y \times_S S_i, X_i) \\
&\simeq \prod_{i\in I} \mathrm{Hom}_S(Y, \mathrm{R}_{S_i/S}(X_i)) \\
&\simeq \mathrm{Hom}_S\left(Y, \prod_{i\in I} \mathrm{R}_{S_i/S}(X_i)\right).
\end{aligned}$$

□

Proposition 2.4. *Let k be a field and let k' be a nonzero, finite, reduced k-algebra. If G' is a k'-group scheme of finite type, then $\mathrm{R}_{k'/k}(G')$ exists as a k-scheme and it is a k-group scheme of finite type.*

Proof. Omitted. See [CGP15, Proposition A.5.1]. □

Proposition 2.5. *Let $S' \to S$ be a finite, locally free morphism of schemes. Assume that either S is locally Noetherian or $S' \to S$ is étale. Let X' be an S'-scheme such that the Weil restriction $\mathrm{R}_{S'/S}(X')$ exists as an S-scheme. If $X' \to S'$ is quasi-compact, then so is $\mathrm{R}_{S'/S}(X') \to S$.*

Proof. Omitted. See [BLR90, Proposition 7.6/5(a)]. □

Proposition 2.6. *Let $S' \to S$ be a finite, locally free morphism of schemes. Let* P *be one of the following properties of morphisms:*

(1) *monomorphism;*
(2) *open immersion;*
(3) *closed immersion;*
(4) *separated.*

If $f'\colon X' \to Y'$ is a morphism of S'-schemes with the property P*, then the morphism* $\mathrm{R}_{S'/S}(f')\colon\ \mathrm{R}_{S'/S}(X') \to \mathrm{R}_{S'/S}(Y')$ *also has the property* P.

Moreover, if the Weil restrictions $\mathrm{R}_{S'/S}(X')$ *and* $\mathrm{R}_{S'/S}(Y')$ *exist as S-schemes, then* P *may be one of the following properties of morphisms:*

(5) smooth;
(6) étale;
(7) locally of finite type;
(8) locally of finite presentation;
(9) finite presentation.

Proof. The condition (1) follows immediately from the definition of the Weil restriction, and (4) follows immediately from (3). For (2) and (3), see [BLR90, Proposition 7.6/2]. For conditions (5-9), we may assume that S and S' are affine, by working Zariski-locally on S. For (5-6), see the proof of [CGP15, Proposition A.5.2(4)]. For (7-9), see [BLR90, Proposition 7.6/5]. □

Proposition 2.7. *Let $S' \to S$ be a finite, locally free morphism of schemes. Let $f'\colon X' \to Y'$ be a smooth, surjective S'-morphism between schemes locally of finite type over S', and assume that the Weil restrictions of X' and Y' exist as S-schemes. Then,* $\mathrm{R}_{S'/S}(f')\colon\ \mathrm{R}_{S'/S}(X') \to \mathrm{R}_{S'/S}(Y')$ *is smooth and surjective.*

Proof. Omitted. See the proof of [CGP15, Corollary A.5.4(1)]. □

Proposition 2.8. *Let $S' \to S$ be a finite, flat morphism of Noetherian schemes that is surjective and radicial.*

(1) If $X'_1, \ldots, X'_n$ are quasi-projective S'-schemes, then

$$\mathrm{R}_{S'/S}\left(\bigsqcup_{i=1}^{n} X'_i\right) = \bigsqcup_{i=1}^{n} \mathrm{R}_{S'/S}(X'_i).$$

(2) If $\{U'_i\}$ is an open (resp. étale) cover of a quasi-projective S'-scheme X' then $\{\mathrm{R}_{S'/S}(U'_i)\}$ is an open (resp. étale) cover of $\mathrm{R}_{S'/S}(X')$.

Proof. For (1), denote $X' = \bigsqcup_{i=1}^{n} X'_i$, $X = \mathrm{R}_{S'/S}(X')$, and $X_i = \mathrm{R}_{S'/S}(X'_i)$ for each i. To show that the X_i are disjoint and cover X it suffices to check on each fibre of $X \to S$, so by base changing to an algebraic closure of $\kappa(s)$ for each $s \in S$, we may assume that $S = \operatorname{Spec} k$ for an algebraically closed field k. Then $S' = \operatorname{Spec} B'$ where B' is a finite flat k-algebra, and B' is local, by the radicial hypothesis. Each intersection is

$$X_i \cap X_j = X_i \times_X X \cong \mathrm{R}_{S'/S}(X'_i \times_{X'} X'_j) = \mathrm{R}_{S'/S}(\emptyset),$$

which is empty because $\mathrm{Hom}_k(Y, \mathrm{R}_{S'/S}(\emptyset)) = \mathrm{Hom}_{B'}(Y_{B'}, \emptyset)$ for any k-scheme Y (and since $B' \neq 0$ then $Y_{B'}$ is nonempty unless Y is the empty scheme). To show that the X_i cover X it suffices to show that the $X_i(k)$'s cover $X(k)$. As B' is local, any morphism $\mathrm{Spec}(B') \to X' = \bigsqcup X'_i$ factors through some X'_i, so $X(k) = X'(B')$ is covered by the sets $X_i(k) = X'_i(B')$.

For (2), let $\{U'_i\}_{i=1}^{n}$ be a finite subcover and let $U' \coloneqq \bigsqcup_{i=1}^{n} U'_i$. Then, U' is quasi-projective over S' and it admits an étale surjection to X', and hence Proposition 2.7 implies, via (1), that $\mathrm{R}_{S'/S}(U') = \bigsqcup \mathrm{R}_{S'/S}(U'_i) \to \mathrm{R}_{S'/S}(X')$ is a smooth surjection. Moreover, by Proposition 2.6, $\mathrm{R}_{S'/S}(U'_i) \hookrightarrow \mathrm{R}_{S'/S}(X')$ is an open immersion. Thus, $\{\mathrm{R}_{S'/S}(U'_i)\}$ is an open cover of $\mathrm{R}_{S'/S}(X')$.

□

Proposition 2.9. *Let $S' \to S$ be a finite, étale morphism of schemes. If $f'\colon X' \to Y'$ is flat (resp. proper) and the Weil restrictions of X' and Y' exist as S-schemes, then $\mathrm{R}_{S'/S}(f')\colon \mathrm{R}_{S'/S}(X') \to \mathrm{R}_{S'/S}(Y')$ is flat (resp. proper).*

Proof. Omitted. See [BLR90, Proposition 7.6/5(f,g)]. □

Proposition 2.10. *Let $S' \to S$ be a finite flat morphism of Noetherian schemes. If X' is a quasi-projective S'-scheme, then $\mathrm{R}_{S'/S}(X')$ is quasi-projective S-scheme.*

Proof. Let $X = \mathrm{R}_{S'/S}(X')$. Let $g' : X' \to S'$ and $f : X \to S$ denote the structure morphisms of X' and X, respectively. By [Sta17, Tag 01VW], it suffices to construct an f-relatively ample invertible $\mathcal{O}_X$-module. Let $\mathcal{N}'$ be a g'-relatively ample invertible $\mathcal{O}_{X'}$-module, and consider the adjunction morphism $q : X \times_S S' \to X'$ and the projection morphism $\pi : X \times_S S' \to X$. Note that π is finite and flat, and $\mathcal{L}' := q^*(\mathcal{N}')$ is an invertible sheaf on $X \times_S S'$. We claim that its norm $\mathcal{L} = \mathrm{Norm}_\pi(\mathcal{L}')$, as defined in [EGA, II, §6.5], is f-relatively ample on X.

Relative ampleness is local on the base, so in order to verify that $\mathcal{L}$ is f-relatively ample, we may assume that S is affine and $\mathcal{N}'$ is ample on X'. By [Sta17, Tag 01PS], it suffices to show that for every $x \in X$, there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that

$$X_s := \left\{ z \in X \colon s(z) \notin \mathfrak{m}_z (\mathcal{L}^{\otimes n})_z \right\}$$

is an affine neighborhood of x.

Note that if $\{U'_i\}$ is an open cover of X' such that every finite subset of X' of bounded size is contained in some U'_i, then the open sets $U_i := X \setminus \pi(q^{-1}(X' \setminus U'_i))$ form an open cover of X. Here, a subset of X' has bounded size if its image in S' lies in fibres of $S' \to S$ with maximal fibre degree. Indeed, for any $x \in X$, $E := \pi^{-1}(x)$ is such a finite subset of $X \times_S S'$, so $q(E) \subseteq U'_i$ for some index i. It follows that $E \cap q^{-1}(X' \setminus U'_i) = \emptyset$, and hence $x \in U_i$.

For any $n \geq 1$, we have $\mathcal{L}^{\otimes n} = \mathrm{Norm}_\pi(\mathcal{L}'^{\otimes n})$. For any $s' \in \Gamma(X', \mathcal{N}'^{\otimes n})$, define $\mathrm{N}(s') \in \Gamma(X, \mathcal{L}^{\otimes n})$ to be the norm of the section $q^*(s') \in \Gamma(X_{S'}, \mathcal{L}'^{\otimes n})$. Since $\mathcal{N}'$ is ample, [Sta17, Tag 09NV] asserts that there is a collection $\{s'_j\}$ of global sections of various powers $\mathcal{N}'^{\otimes n_j}$ of $\mathcal{N}'$, with $n_j \geq 1$, such that all loci $X'_{s'_j}$ are affine, they cover X', and every finite subset of X' of bounded size is contained in some $X'_{s'_j}$. We claim that the subsets $\{X_{\mathrm{N}(s'_j)}\}$ form an open affine cover of X. By the above observation and Proposition 2.2, it suffices to show that

$$X_{\mathrm{N}(s')} = X \setminus \pi(q^{-1}(X' \setminus X'_{s'})) = \mathrm{R}_{S'/S}(X'_{s'}).$$

The first equality is an application of [EGA, II, Corollaire 6.5.7]. The second equality holds by functorial considerations. □

3. Geometric Connectedness

Proposition 3.1. *Let k be a field, k' a finite, Artinian, local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k. Let A be a smooth k-algebra of pure dimension n. Assume that $\Omega^1_{A/k}$ is globally generated by the differentials of the functions $f_1, \ldots, f_n \in A$. Then, there is a non-canonical isomorphism*

$$\mathrm{R}_{k'/k}(\mathrm{Spec}(A \otimes_k k')) \simeq \mathrm{Spec}(A) \times \mathbb{A}^d_k$$

over $\mathrm{Spec}(A)$*, where $d = (\dim_k(k') - 1) \cdot n$.*

Remark 3.2. Let k, k' be as in Proposition 3.1, let X' be a scheme over k', and denote the special fibre by $X'_0 \coloneqq X' \times_{\mathrm{Spec}(k')} \mathrm{Spec}(k)$. There is a natural morphism $q_{X'} \colon \mathrm{R}_{k'/k}(X') \to X'_0$ defined as follows: to a k-scheme T and a T-point $t \in \mathrm{R}_{k'/k}(X')(T) = X'(T \times_k k')$, we associate the reduction modulo $\mathfrak{m}$, i.e. the morphism

$$T = (T \times_k k') \times_{k'} (k'/\mathfrak{m}) \to X'_0.$$

Thus, $q_{X'}$ is surjecive when X' is k'-smooth, and, in the setting of Proposition 3.1, $\mathrm{R}_{k'/k}(\mathrm{Spec}(A \otimes_k k'))$ is *naturally* a $\mathrm{Spec}(A)$-scheme.

Proof. By Proposition 2.1, it suffices to show that

$$\underline{\mathrm{Hom}}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k'), \mathrm{Spec}(A)) = \mathrm{Spec}(A[x_1, \dots, x_d]).$$

Consider the surjective multiplication map $A \otimes_k A \twoheadrightarrow A$ and denote the kernel by I. By our assumption, we have a non-canonical isomorphism

$$(A \otimes_k A)/I^m \simeq A[x_1, \dots, x_n]/(x_i)^m$$

as A-algebras, for any $m \geq 1$. Let M denote the nilpotence order of the maximal ideal $\mathfrak{m} \subset k'$.

For every k-algebra B, an element

$$\phi \in \underline{\mathrm{Hom}}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k'), \mathrm{Spec}(A))(B)$$

corresponds to a k-algebra homomorphism $\Phi \colon A \to k' \otimes_k B$.

Given Φ as above, we get an A-algebra structure on B via the composition

$$p_1^* \colon A \xrightarrow{\Phi} k' \otimes_k B \twoheadrightarrow B,$$

where the morphism $k' \otimes_k B \twoheadrightarrow B$ is induced from the surjection $k' \twoheadrightarrow k$. Similarly, denote by $p_1'^*$ the composition

$$p_1'^* \colon A \xrightarrow{p_1^*} B \to k' \otimes_k B,$$

where the second morphism is induced by the inclusion from $k \hookrightarrow k'$. Observe that $p_1'^*$ and Φ give rise to a k-algebra homomorphism

$$\Psi \coloneqq (p_1'^* \otimes \Phi) \colon A \otimes_k A \to k' \otimes_k B.$$

This morphism has the property that the "first projection" (i.e. precomposition of Ψ with the inclusion $A \xrightarrow{\mathrm{id} \otimes 1} A \otimes_k A$ into the first factor) agrees with $p_1'^*$, and the ideal $I \subseteq A \otimes_k A$ is mapped to $\mathfrak{m} \otimes B$ under Ψ. Given a pair $(p_1'^*, \Psi)$ as above, we can consider the "second projection", i.e. the composition

$$\Phi \colon A \xrightarrow{1 \otimes \mathrm{id}} A \otimes_k A \xrightarrow{\Psi} k' \otimes_k B.$$

It is easy to check that these procedures invert one another, and they are functorial in B.

The discussion above shows that the datum of Φ is equivalent to the following data:

(1) a k-algebra homomorphism $p_1^* \colon A \to B$;
(2) a k-algebra homomorphism $\Psi \colon A \otimes_k A \to k' \otimes_k B$

satisfying the following two conditions:

(i) the "first projection" $A \xrightarrow{\mathrm{id} \otimes 1} A \otimes_k A \xrightarrow{\Psi} k' \otimes_k B$ agrees with $p_1'^*$;
(ii) the ideal $I \subseteq A \otimes_k A$ is mapped to $\mathfrak{m} \otimes B$ under Ψ.

As $\mathfrak{m}$ has nilpotence order M, it follows that to give a k-algebra homomorphism Ψ is equivalent to giving a k-algebra homomorphism

$$\Psi' \colon (A \otimes_k A)/I^M \simeq A[x_1, \dots, x_n]/(x_i)^M \to k' \otimes_k B$$

with the properties that

(1) the induced morphism $A \to A[x_1, \dots, x_n] \to k' \otimes_k B$ agrees with $p_1'^*$;
(2) the image of the $\overline{x_i}$'s lands in $\mathfrak{m} \otimes B$.

Hence, the datum of Φ is functorially equivalent to a k-algebra homomorphism $A \to B$ along with n elements of $\mathfrak{m} \otimes_k B$. Therefore, by examining the coefficients of the n elements with respect to some chosen k-basis of $\mathfrak{m}$, we obtain an isomorphism

$$\underline{\mathrm{Hom}}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k'), \mathrm{Spec}(A)) \to \mathrm{Spec}(A[x_1, \dots, x_d]).$$

□

Lemma 3.3. *Let k be a field, let k' be a finite, local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k, and let X' be a smooth k'-scheme. Consider the surjective morphism $q_{X'} \colon \mathrm{R}_{k'/k}(X') \to X'_0$ as defined in Remark 3.2. The geometric fibres of $q_{X'}$ are affine spaces; in particular, the geometric fibres of $q_{X'}$ are connected.*

Proof. The problem is local on the base, so we may assume that X' is affine and that $\Omega^1_{X'/k'}$ is free (the latter is equivalent to saying that $\Omega^1_{X'_0/k}$ is free, by Nakayama's Lemma). By [SGAI, Exposé III, Corollaire 6.8], there is a non-canonical k'-isomorphism $X' \simeq X'_0 \times_k k'$. The conclusion follows from Proposition 3.1. □

Lemma 3.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces of finite type over S. Suppose that:*

(1) the geometric fibres of f are non-empty and connected;
(2) étale-locally on Y, f admits a section;
(3) Y is geometrically connected over S.

Then, X is geometrically connected over S.

Remark 3.5. Below, we recall certain definitions on the (geometric) connectedness of algebraic spaces.

(1) Let X be an algebraic space locally of finite type over an algebraically closed field k, and denote by $|X|$ the topological space associated to X (as in [Sta17, Tag 03BY]). We say X is *connected* if the associated topological space $|X|$ is connected. Note that, with our assumptions on X, the topological space $|X|$ is automatically locally connected.
(2) A morphism of algebraic spaces is *geometrically connected* if all geometric fibres are non-empty and connected.

Lemma 3.6. *If X be a non-empty algebraic space locally of finite type over an algebraically closed field k, then X is connected if and only if for any two points $x, x' \in |X|$, there is a finite chain $x_1, \dots, x_n \in |X|$ of points and connected étale scheme neighborhoods U_{x_i} of x_i with $x_1 = x$ and $x_n = x'$ and such that $U_{x_i} \times_X U_{x_{i+1}} \neq \emptyset$ for all $i = 1, \dots, n-1$.*

Proof. It is clear that the existence of such finite chains of points and neighborhoods as in the statement implies that $|X|$ is connected. Conversely, we can define an equivalence relation on $|X|$, where two points $x, x' \in |X|$ are equivalent if there is a finite chain of points and neighborhoods as in the statement. To each equivalence class $\widetilde{x}$, consider $W_{\widetilde{x}} = \bigcup_{x\in\widetilde{x}} W_x$, where W_x denotes the image of U_x in X. As subsets of $|X|$, the $W_{\widetilde{x}}$'s are easily seen to be open and disjoint, hence closed. The connectedness of $|X|$ implies any two points are equivalent, which completes the proof. □

Proof of Lemma 3.4. We may assume that S is the spectrum of an algebraically closed field, due to the definition of geometric connectedness as in Remark 3.5(2). Given a point $x \in |X|$, there is a connected scheme $V_{f(x)}$ étale over Y whose image contains $f(x)$ and such that the pullback morphism $U_x \coloneqq X \times_Y V_{f(x)} \to V_{f(x)}$ admits a section (this guaranteed by condition (2)). Now, condition (1) along with existence of section and connectedness of $V_{f(x)}$ implies that U_x is connected by Lemma 3.6.

Furthermore, for any two points $x, x' \in |X|$, there is a finite collection

$$\{x_1, \ldots, x_n\} \subseteq |X|$$

of points with $x_1 = x$, $x_n = x'$, and such that $V_{f(x_i)} \cap V_{f(x_{i+1})} \neq \emptyset$ for all $i = 1, \ldots, n-1$. By condition (1), we must have that $U_{x_i} \cap U_{x_{i+1} \neq \emptyset}$. Therefore, X is connected. □

Proposition 3.7. *Let k be a field, k' a nonzero finite k-algebra, and $X' \to \operatorname{Spec} k'$ a smooth, surjective, quasi-projective morphism of schemes with geometrically connected fibres. Then, the smooth k-scheme $X = \mathrm{R}_{k'/k}(X')$ is non-empty and geometrically connected.*

An alternate proof is given in [CGP15, Proposition A.5.9]. If X' is not smooth, then the result fails: in Example 4.6, we describe an example due to Gabber of a quasi-projective scheme X' over a field k' that is geometrically connected and smooth away from one point, and whose Weil restriction $\mathrm{R}_{k'/k}(X')$ is not geometrically connected (in fact, it has a nowhere reduced connected component!).

Proof. By Proposition 2.2(2), we may assume that k is algebraically closed. In particular, k' is a finite algebra over an algebraically closed field, and thus it is a semi-local ring over k. Applying Lemma 2.3, we can assume that k' is a local ring over k with nilpotent maximal ideal $\mathfrak{m}$. Furthermore, since k is algebraically closed, the map $k \to k'/\mathfrak{m}'$ is an isomorphism. In the sequel, we identify k with the residue field $k'/\mathfrak{m}'$ via this isomorphism.

Let X'_0 be the fibre of X' over $\operatorname{Spec} k = \operatorname{Spec}(k'/\mathfrak{m}') \to \operatorname{Spec} k'$. As in Lemma 3.3, there is a natural morphism $q_{X'} \colon \mathrm{R}_{k'/k}(X') \to X'_0$ of k-schemes. Let V' be a non-empty, open affine subset of X', and V'_0 the special fibre over $\operatorname{Spec} k$. Since V' is k'-smooth and affine, it is necessarily a trivial deformation of V'_0, i.e. $V \cong V'_0 \times_k k'$. In particular, the adjunction map $V'_0 \to \mathrm{R}_{k'/k}(V')$ gives a section of $\mathrm{R}_{k'/k}(X') \to X'_0$ over V'_0. Thus, $\mathrm{R}_{k'/k}(V')$ is nonempty.

The geometric fibres of $\mathrm{R}_{k'/k}(X') \to X'_0$ are connected by Lemma 3.3. The preceding discussion shows that, Zariski-locally on X'_0, the morphism $q_{X'} \colon \mathrm{R}_{k'/k}(X') \to X'_0$ admits a section. By assumption, X'_0 is connected, and hence $\mathrm{R}_{k'/k}(X')$ is geometrically connected by Lemma 3.4. □

The methods of this section can also be applied to show the following result on the dimension of the Weil restriction.

Corollary 3.8. *Let k'/k be a finite field extension of degree d and let X' be a smooth, quasi-projective k'-scheme of pure dimension d'. Then, the smooth k-scheme $X = \mathrm{R}_{k'/k}(X')$ is of pure dimension $d \cdot d'$.*

Proof. By Proposition 2.2(2) and Lemma 2.3, we may assume that k is algebraically closed and k' is a finite local k-algebra. In particular, if the original field extension k'/k is finite *separable*, then we're done. As dimension is an (étale) local property, we may assume by Proposition 2.8(2) that X' is as in Proposition 3.1, in which case the result immediately follows from Proposition 3.1. □

If k'/k is a finite separable extension, then we may drop the smoothness assumption on X' appearing in Corollary 3.8. On the other hand, if k'/k is non-separable, then the smoothness hypothesis in Corollary 3.8 is necessary: see Example 4.7.

4. Examples

Example 4.1. In general, it is not true that Weil restriction preserves Zariski-open covers. Let k'/k be a finite separable extension of degree $d > 1$ and let $X' = \mathbb{A}^1_{k'}$. Consider the open cover $\{U'_0, U'_1\}$ of X', where $U'_i = \{t \neq i\}$. Then $X = \mathrm{R}_{k'/k}(\mathbb{A}^1_{k'})$ contains the Weil restrictions $U_i = \mathrm{R}_{k'/k}(U'_i)$ as open subschemes by Proposition 2.6(2). We claim that $\{U_0, U_1\}$ does not cover X. Indeed, if k_s denotes a separable closure of k, there is a canonical k_s-algebra isomorphism $k_s \otimes_k k' \simeq k_s^d$ and an identification

$$X_{k_s} \simeq \mathbb{A}^d_{k_s}.$$

If $(t_1, \dots, t_d)$ are the coordinates on $\mathbb{A}^d_{k_s}$ induced by the isomorphism $k_s \otimes_k k' \simeq k_s^d$, then

$$(U_j)_{k_s} = \{(t_1, \dots, t_d) \in \mathbb{A}^d_{k_s} : t_i \neq j \text{ for } i = 1, \dots, d\}$$

for $j = 0, 1$. Hence, $(U_0)_{k_s} \cup (U_1)_{k_s}$ does not contain points such as $(1, 0, ..., 0)$. In particular, $\{U_0, U_1\}$ is does not cover X.

In fact, it is possible that the Weil restriction of all members of a Zariski-open cover is the empty set! Let k be a field and let $k' = k^d$, with $d > 1$. For any quasi-projective k'-scheme X' with fibres X_i, we have $\mathrm{R}_{k'/k}(X') = \prod_{i=1}^d X_i$ by Lemma 2.3. Thus, $\mathrm{R}_{k'/k}(X')$ is non-empty if and only if all of the X_i's are non-empty. In particular, if X'_i is the open subset of X' with fibre X_i over the i-th point of $\operatorname{Spec} k'$ and empty fibre over the other points, then clearly $\{X'_i\}$ is an open cover of X', but $\mathrm{R}_{k'/k}(X'_i) = \emptyset$ for all i!

Note that this phenomenon of failure to carry Zariski-open covers to Zariski-open covers cannot occur when k'/k is purely inseparable by Proposition 2.8(2).

Example 4.2. Let k be a field and let $k[\epsilon] \coloneqq k[x]/(x^2)$. If X is a smooth quasi-projective k-scheme, then the functor of points of $\mathrm{R}_{k[\epsilon]/k}(X_{k[\epsilon]})$ is given by

$$U \mapsto \operatorname{Hom}_{k[\epsilon]}\left(U \times_k k[\epsilon], X_{k[\epsilon]}\right) = \operatorname{Hom}_k\left(U \times_k k[\epsilon], X\right).$$

In particular, $\mathrm{R}_{k[\epsilon]/k}(X_{k[\epsilon]})$ is isomorphic to the tangent bundle T_X of X as k-schemes. Moreover, the structure maps $\mathrm{R}_{k[\epsilon]/k}(X_{k[\epsilon]}) \to X$ and $T_X \to X$ are the adjunction morphisms obtained from the respective universal properties, so $\mathrm{R}_{k[\epsilon]/k}(X_{k[\epsilon]})$ and T_X are in fact isomorphic as X-schemes.

Example 4.3. Let k be a field and $k[\epsilon] \coloneqq k[x]/(x^2)$ be the ring of dual numbers. We consider the Weil restriction of a smooth scheme X' over $k[\epsilon]$ along the structure morphism $\operatorname{Spec}(k[\epsilon]) \to \operatorname{Spec}(k)$. If $X_0 \coloneqq X' \times_{k[\epsilon]} k$ is the special fibre of X', then we claim that $\mathrm{R}_{k[\epsilon]/k}(X')$ is a principal homogeneous space over the tangent bundle of X_0.

Let $X \coloneqq X_0 \times_k k[\epsilon]$ be the trivial deformation of X_0 over $k[\epsilon]$. We claim that there is a canonical action

$$\begin{array}{ccc} \mathrm{R}_{k[\epsilon]/k}(X') \underset{X_0}{\times} \mathrm{R}_{k[\epsilon]/k}(X) & \longrightarrow & \mathrm{R}_{k[\epsilon]/k}(X') \\ \downarrow & \swarrow & \\ X_0 & & \end{array}$$

where $\mathrm{R}_{k[\epsilon]/k}(X) = T_{X_0}$ via Example 4.2 (an X_0-group), and the morphism $\mathrm{R}_{k[\epsilon]/k}(X') \to X_0$ is as in Remark 3.2. To see this, it suffices to construct the *canonical* action for affine schemes X' by Proposition 2.8(2), so we may assume $X' = \mathrm{Spec}(B)$ for some smooth $k[\epsilon]$-algebra B. Then, we have $X_0 = \mathrm{Spec}(B_0)$ and $X = \mathrm{Spec}(B_0[\epsilon])$, where $B_0 = B/(\epsilon)$. For any k-algebra A, we seek an action

$$\begin{array}{ccc} \mathrm{Hom}_{k[\epsilon]}(B, A[\epsilon]) \underset{\mathrm{Hom}_k(B_0, A)}{\times} \mathrm{Hom}_{k[\epsilon]}(B_0[\epsilon], A[\epsilon]) & \longrightarrow & \mathrm{Hom}_{k[\epsilon]}(B, A[\epsilon]) \\ \downarrow & \swarrow & \\ \mathrm{Hom}_k(B_0, A) & & \end{array}$$

that is functorial in A and B.

Let $\phi' \in \mathrm{Hom}_{k[\epsilon]}(B, A[\epsilon])$ and $\widetilde{\phi} \in \mathrm{Hom}_{k[\epsilon]}(B_0[\epsilon], A[\epsilon])$ be such that both have the same image $\phi \in \mathrm{Hom}_k(B_0, A)$. Given the data above, we must produce a $\psi \in \mathrm{Hom}_{k[\epsilon]}(B, A[\epsilon])$ whose reduction is ϕ again. We have

$$\widetilde{\phi}(b_0) = \phi(b_0) + \epsilon \cdot D(b_0), \quad \forall b_0 \in B_0$$

where $D : B_0 \to A$ is k-linear and satisfies the condition

$$D(b_0 \cdot b_0') = \phi(b_0) \cdot D(b_0') + \phi(b_0') \cdot D(b_0). \tag{4.1}$$

Define

$$\psi(b) \coloneqq \phi'(b) + \epsilon \cdot D(\overline{b}), \quad \forall b \in B$$

where $\overline{b} \in B_0$ is the image of b under canonical reduction $B \twoheadrightarrow B_0$. This map ψ is a $k[\epsilon]$-algebra homomorphism precisely because of (4.1) and the fact that $\epsilon^2 = 0$. The set of ψ's is clearly in bijection with the set of $\widetilde{\phi}$'s, and $(\phi', \widetilde{\phi}) \mapsto \psi$ is easily seen to be an action with respect to addition in D's, and it has the desired functoriality in A and B. Thus, $\mathrm{R}_{k[\epsilon]/k}(X')$ is a principal homogeneous space of T_{X_0}, the tangent bundle of X_0.

Example 4.4. This is a continuation of the discussion of Example 4.3. Let k' be a local finite k-algebra with nonzero maximal ideal $\mathfrak{m}'$, where k is an algebraically closed field, and let X' be a smooth, proper k'-scheme of positive dimension. We claim the Weil restriction $\mathrm{R}_{k'/k}(X')$ is never proper over k.

Indeed, let $X_0 \coloneqq X' \times_{k'} k'/\mathfrak{m}'$ be the special fibre of X', which is proper by assumption. Consider the morphism $\mathrm{R}_{k'/k}(X') \to X_0$ as in Remark 3.2. Note that the k-smooth $\mathrm{R}_{k'/k}(X')$ has pure dimension $\dim_k(k') \cdot \dim(X_0')$ by Corollary 3.8. The geometric fibre at any $x_0 \in X_0(k)$ is a principal homogeneous space for the nonzero vector space $T_{x_0}(X_0)$ by Example 4.3. Thus, it cannot be proper.

Example 4.5. If k is an imperfect field, k'/k is a nontrivial finite inseparable extension, and A' is a nonzero abelian variety over k', then we claim $\mathrm{R}_{k'/k}(A')$ violates the conclusion of Chevalley's structure theorem over perfect fields. Recall that Chevalley's structure theorem asserts that for a smooth connected group G over a perfect field k, there is a unique short exact sequence of smooth connected k-groups

$$1 \to H \to G \to A \to 1,$$

where H is affine and A is an abelian variety. See [CGP15, Theorem A.3.7] for a reference for its proof.

Note that $\mathrm{R}_{k'/k}(A')$ exists as a k-group scheme by Proposition 2.4, and it is smooth and connected by Proposition 2.6(5). Suppose there exists a smooth, connected, affine k-group H, an abelian variety A over k, and an exact sequence of k-group homomorphisms

$$1 \to H \to \mathrm{R}_{k'/k}(A') \to A \to 1.$$

By the universal property of the Weil restriction, the k-group homomorphism $H \to \mathrm{R}_{k'/k}(A')$ corresponds to a k'-group homomorphism $H_{k'} \to A'$. If $K \subseteq H_{k'}$ is the kernel of $H_{k'} \to A'$, then $H_{k'}/K$ exists as an affine k'-group scheme of finite type, and the induced k'-group homomorphism $H_{k'}/K \to A'$ is a closed immersion, by [CGP15, Proposition A.2.1]. Thus, the image of $H_{k'}/K \to A'$ is an affine abelian subvariety of A', i.e. it is the inclusion of 0. Thus, $H \to \mathrm{R}_{k'/k}(A')$ must be the constant map given by 0. It follows that $\mathrm{R}_{k'/k}(A') \simeq A$; however, $\mathrm{R}_{k'/k}(A')$ cannot be proper due to Example 4.4 (applied over $\overline{k}$) and Lemma 2.3 (applied to $S' = \mathrm{Spec}(k' \otimes_k \overline{k})$), a contradiction.

Example 4.6. [Gabber] If k is a field, k' is a non-zero finite k-algebra, and X' is a quasi-projective k'-scheme with geometrically connected fibres, then it is not necessarily the case that $\mathrm{R}_{k'/k}(X')$ is geometrically connected (unless we also assume that X' is k'-smooth and the structure morphism $X' \to \mathrm{Spec}(k')$ is surjective).

Let k be an imperfect field of characteristic $p > 0$, and let k' be a nontrivial purely inseparable finite extension of k of degree $d > 1$. Pick $a' \in k' \setminus k$, $q = p^e \in \mathbf{N}$ such that $(k')^q \subseteq k$, and $m \in \mathbf{N}_{>2q}$ such that m is not divisible by p. Let X' be the geometrically integral curve over k' given by $y^{qp} = a'x^q + x^m$. The open subscheme $U' = X' \setminus \{(0,0)\} = X' \cap \{x \neq 0\}$ is k'-smooth. Therefore, $U \coloneqq \mathrm{R}_{k'/k}(U')$ is smooth and geometrically connected due to Proposition 3.7, and it is an open subscheme of $X \coloneqq \mathrm{R}_{k'/k}(X')$ due to Proposition 2.6(2). We claim that X is the disjoint union of U and a non-empty open subscheme V which is nowhere reduced.

Choose a k-basis $\{a'_i\}_{i=0}^{d-1}$ of k' with $a'_0 = 1$ and $a'_1 = a'$, and substitute $\sum_{i=0}^{d-1} a'_i x_i$ for x and $\sum_{i=0}^{d-1} a'_i y_i$ for y in the equation defining X'. If $c_i \coloneqq a_i'^q \in k^\times$, we have that X is cut out, as a closed subscheme of $\mathrm{R}_{k'/k}(\mathbb{A}^2_{k'}) \simeq \mathbb{A}^{2d}_k$, by the equation

$$\left(\sum_{i=0}^{d-1} c_i y_i^q\right)^p = a'_1 \left(\sum_{i=0}^{d-1} c_i x_i^q\right) + \left(\sum_{i=0}^{d-1} a'_i x_i\right)^{m-2q} \left(\sum_{i=0}^{d-1} c_i x_i^q\right)^2.$$

Expanding the $(m-2q)$-th power, we may write

$$\left(\sum_{i=0}^{d-1} a'_i x_i\right)^{m-2q} = \sum_{j \geq 0} a'_j f_j(x)$$

for some $f_j(x) \in k[x_0, x_1, \ldots, x_{d-1}]$. Comparing the coefficients of the a_i''s, we observe that X is defined by the following system of equations on $\mathbb{A}_k^{2d}$:

$$\begin{cases} \left(\sum_{i=0}^{d-1} c_i y_i^q\right)^p = f_0(x)\left(\sum_{i=0}^{d-1} c_i x_i^q\right)^2, \\ \sum_{i=0}^{d-1} c_i x_i^q + f_1(x)\left(\sum_{i=0}^{d-1} c_i x_i^q\right)^2 = 0, \\ f_j(x)\left(\sum_{i=0}^{d-1} c_i x_i^q\right)^2 = 0 \quad (j \geq 2). \end{cases} \tag{4.2}$$

If $h \coloneqq \sum_{i=0}^{d-1} c_i x_i^q$, then the relation $h(1+f_1 h) = 0$ guarantees that the loci $\{h = 0\}$ and $\{h \text{ is a unit}\}$ define a separation of X. For $P \in X(\overline{k})$, the associated point $P' \in X(k' \otimes \overline{k})$, given by

$$\mathrm{Spec}(k' \otimes \overline{k}) \to \mathrm{Spec}(\overline{k}) \xrightarrow{P} X',$$

corresponds to a pair $(x, y) \in (k' \otimes \overline{k})^2$ satisfying the equation $y^{qp} = a'x^q + x^m$, with $x = \sum_{i=0}^{d-1} a_i' \otimes x_i$ and $y = \sum_{i=0}^{d-1} a_i' \otimes y_i$. Since $\overline{h} = \overline{x}^q$ in the residue field of the Artinian local ring $k' \otimes \overline{k}$, we see that the locus $\{h \text{ is a unit}\}$ coincides with U, as defined above.

Moreover, if $V \coloneqq X \backslash U$ denotes the open subscheme of X defined by $h = 0$, then setting $h = 0$ in the equations (4.2) shows that V is cut out by h and $\left(\sum_{i=0}^{d-1} c_i y_i^q\right)^p$; in particular, V is nowhere reduced.

Example 4.7. This is a continuation of the discussion of Example 4.6. In the absence of a smoothness hypothesis, Corollary 3.8 fails: the dimension of the non-smooth plane curve X' is 1 and the degree of the extension k'/k is $d > 1$, but the dimension of the component V of X is $2d - 1$. As $d > 1$, the dimension of X' times the degree of k'/k cannot be the dimension of X on V.

Example 4.8. The Weil restriction of a smooth, affine group scheme with positive-dimensional fibres along a non-étale, finite, flat morphism of Noetherian rings is never reductive. More precisely, by considering the geometric fibres at the non-étale points of the base, one shows the following: if k is an algebraically closed field, k' is a non-reduced, local, finite k-algebra, and G is a nontrivial, connected, smooth, affine k'-group scheme, then the smooth, connected, affine k-group scheme $\mathrm{R}_{k'/k}(G)$ is not reductive.

In the special case when $G = \mathrm{GL}_{n,k[\epsilon]}$, $\mathrm{R}_{k[\epsilon]/k}(G) = \mathrm{GL}_n \ltimes \mathfrak{gl}_n$ where the semi-direct product is taken using the adjoint action of GL_n on $\mathfrak{gl}_n$. More generally,, if G is a smooth, affine k-group scheme with Lie algebra $\mathfrak{g}$, then one sees that $\mathrm{R}_{k[\epsilon]/k}(G_{k[\epsilon]}) = G \ltimes \mathfrak{g}$, and $G \ltimes \mathfrak{g}$ is clearly not reductive. For the general case, see [Oes84, Proposition A.3.5].

Example 4.9. Proposition 2.7 (and Proposition 5.6) assert that the Weil restriction of a smooth and surjective morphism between schemes (or algebraic spaces) is again smooth and surjective; however, it is not true that the Weil restriction of a surjective morphism between quasi-compact, smooth schemes (or algebraic spaces) is necessarily surjective. More precisely, if $k \subseteq k'$ is a finite field extension and $X' \to Y'$ is a surjective morphism of smooth, quasi-projective k'-schemes, then it is not necessarily true that the induced map $\mathrm{R}_{k'/k}(X') \to \mathrm{R}_{k'/k}(Y')$ is surjective. Let k be an imperfect field of characteristic $p > 0$. Pick $a \in k \setminus k^p$ and consider the finite degree-p extension $k' = k(a^{1/p})$ of k. Consider the exact sequence

$$1 \to \mu_p \to \mathrm{SL}_p \to \mathrm{PGL}_p \to 1$$

of k'-group homomorphisms. As Weil restriction is left exact on the category of k'-groups, there is an exact sequence

$$1 \to \mathrm{R}_{k'/k}(\mu_p) \to \mathrm{R}_{k'/k}(\mathrm{SL}_p) \to \mathrm{R}_{k'/k}(\mathrm{PGL}_p)$$

of k-groups. However, $\mathrm{R}_{k'/k}(\mathrm{SL}_p)$ and $\mathrm{R}_{k'/k}(\mathrm{GL}_p)$ are smooth with the same dimension $[k' \colon k] \cdot (p^2 - 1)$, and $\mathrm{R}_{k'/k}(\mu_p)$ is positive-dimensional (see [CGP15, Example 1.3.2]), hence, $\mathrm{R}_{k'/k}(\mathrm{SL}_p) \to \mathrm{R}_{k'/k}(\mathrm{PGL}_p)$ cannot be surjective.

Example 4.10. Let $S' \to S$ be a finite étale morphism between connected schemes, and let π (resp. π') denote the étale fundamental group of S (resp. S'). Let X' be a finite étale cover of S', corresponding to the finite, discrete π'-set A'. Then, $\mathrm{R}_{S'/S}(X') \to S$ is a finite étale cover, by Proposition 2.6(6) and Proposition 2.9; it corresponds to a finite, discrete π-set A. We claim that there is a canonical identification

$$A = \mathrm{Ind}_{\pi'}^{\pi}(A'),$$

where $\mathrm{Ind}_{\pi'}^{\pi}(A')$ denotes the induced representation. The above assertion can be verified by combining the universal properties of the induced representation and of the Weil restriction: if B is a finite, discrete π-set, corresponding to the finite étale map $Y \to S$, then

$$\begin{aligned}\mathrm{Hom}_{\pi}(B, \mathrm{Ind}_{\pi'}^{\pi}(A)) &= \mathrm{Hom}_{\pi}(B, A') \\ &= \mathrm{Hom}_{S'}(Y \times_S S', X') \\ &= \mathrm{Hom}_{S}(Y, \mathrm{R}_{S'/S}(X')) \\ &= \mathrm{Hom}_{\pi}(B, A).\end{aligned}$$

Example 4.11. Let k'/k be a finite separable extension of fields. Let A' be an abelian variety over k'. Then, $A \coloneqq \mathrm{R}_{k'/k}(A')$ is an abelian variety over k by Proposition 2.9, Proposition 2.6(5), and Proposition 2.2(3). If ℓ is a prime different than the characteristic of k, then there is a canonical isomorphism

$$\mathrm{T}_{\ell}(A) = \mathrm{Ind}_{\mathrm{Gal}_{k'}}^{\mathrm{Gal}_k}(\mathrm{T}_{\ell}(A'))$$

of Galois representations of Gal_k. Here, Gal_k and $\mathrm{Gal}_{k'}$ denote the absolute Galois groups of k and k', respectively. This follows from the corresponding statements on ℓ-power torsion points, which are a consequence of Example 4.10.

5. Weil Restriction of Algebraic Spaces

Definition 5.1. Let $S' \to S$ be a morphism of schemes and let X' be an algebraic space over S'. Consider the contravariant functor $\mathrm{R}_{S'/S}(X') \colon (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets})$ given by

$$T \mapsto X'(T \times_S S').$$

If the functor $\mathrm{R}_{S'/S}(X')$ is an algebraic space X over S, then we say that X is the *Weil restriction of X' along $S' \to S$*, and we denote it by $\mathrm{R}_{S'/S}(X')$.

Thinking of the algebraic space X' as a sheaf on the big étale site of (Sch/S'), the Weil restriction $\mathrm{R}_{S'/S}(X')$ is the pushforward sheaf along $S' \to S$.

Theorem 5.2. *Let $S' \to S$ be a finite, locally free morphism of schemes and let X' algebraic space of finite presentation over S'. Then, the Weil restriction $\mathrm{R}_{S'/S}(X')$ exists as an algebraic space over S.*

Proof. Step 1 (Construction of a candidate). In order to verify that $\mathrm{R}_{S'/S}(X')$ is an algebraic space over S, we may assume that S is affine, by [Sta17, Tag 04SK]; say, $S = \mathrm{Spec}(B)$. It follows that S' is affine as well, say $S' = \mathrm{Spec}(B')$, because $S' \to S$ is finite (in particular, affine).

As $X' \to S'$ is quasi-compact, there exists an affine étale chart $U' \to X'$ (for example, the disjoint union of a finite open affine cover of some étale chart of X'). If $R' = U' \times_{X'} U'$, then $R' \rightrightarrows U'$ is an étale equivalence relation such that $X' = U'/R'$. There is a Cartesian square

$$\begin{array}{ccc} R' & \xrightarrow{\delta} & U' \times_{S'} U' \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\Delta_{X'}} & X' \times_{S'} X' \end{array} .$$

of algebraic spaces over S'. As $X' \to S'$ is finitely-presented, the diagonal $\Delta_{X'}$ is quasi-compact and hence the base change δ is quasi-compact and locally of finite type. As both R' and $U' \times_{S'} U'$ are S'-schemes, that δ is quasi-finite follows from it being a monomorphism (indeed, since it is quasi-compact, it suffices to check that the fibres are finite; as it is a monomorphism, each geometric fibre is either empty or a single point). Moreover, as δ is a monomorphism, it is separated. By [EGA, IV_4, 18.12.12], this implies that R' is quasi-affine; in particular, R' is quasi-projective, and so one may consider its Weil restriction $\mathrm{R}_{S'/S}(R')$.

By Proposition 2.6(1) and (6), the induced map

$$\mathrm{R}_{S'/S}(R') \to \mathrm{R}_{S'/S}(U' \times_{S'} U') \simeq \mathrm{R}_{S'/S}(U') \times_S \mathrm{R}_{S'/S}(U')$$

is a monomorphism such that postcomposing with either projection gives an étale S-morphism $\mathrm{R}_{S'/S}(R') \to \mathrm{R}_{S'/S}(U')$. Implicitly, we have used that $\mathrm{R}_{S'/S}(S') = S$. Thus, $\mathrm{R}_{S'/S}(R') \rightrightarrows \mathrm{R}_{S'/S}(U')$ is an étale equivalence relation and we can form the algebraic space $X \coloneqq \mathrm{R}_{S'/S}(U')/\,\mathrm{R}_{S'/S}(R')$ over S. By construction, there is a coequalizer diagram

$$\mathrm{R}_{S'/S}(R') \rightrightarrows \mathrm{R}_{S'/S}(U') \to X,$$

so the universal property of the coequalizer gives a morphism $X \to \mathrm{R}_{S'/S}(X')$ of étale sheaves on (Sch/S).

To see that $X \to \mathrm{R}_{S'/S}(X')$ is monic, it suffices to consider the T-valued points of X that lift to $\mathrm{R}_{S'/S}(U')$, by working étale-locally on T. It then suffices to show that the commutative diagram

$$\begin{array}{ccc} \mathrm{R}_{S'/S}(R') & \longrightarrow & \mathrm{R}_{S'/S}(U') \\ \downarrow & & \downarrow \\ \mathrm{R}_{S'/S}(U') & \longrightarrow & \mathrm{R}_{S'/S}(X') \end{array}$$

is Cartesian. This holds since

$$\mathrm{R}_{S'/S}(U') \times_{\mathrm{R}_{S'/S}(X')} \mathrm{R}_{S'/S}(U') = \mathrm{R}_{S'/S}(U' \times_{X'} U') = \mathrm{R}_{S'/S}(R').$$

Step 2 (Reduction to strictly henselian points). It remains to show that the map $X \hookrightarrow \mathrm{R}_{S'/S}(X')$ is a surjective morphism of étale sheaves. We claim that it suffices to show that, for any strictly henselian local B-algebra A, the induced map

$$U'(A \otimes_B B') \longrightarrow X'(A \otimes_B B') \tag{$*$}$$

is surjective. Take an S-scheme T and $\xi \in \mathrm{R}_{S'/S}(X')(T) = X'(T \times_{\operatorname{Spec} B} \operatorname{Spec} B')$. We want to lift ξ to U', étale-locally on T. We may assume T is affine, and then that T is of finite type over S because X' is assumed to be of finite presentation over B' by [Sta17, Tag 04AK].

For every $t \in T$, fix a separable closure $k(t)^{\mathrm{sep}}$ of the residue field $k(t)$ of t. If $A = \mathcal{O}_{T,t}^{\mathrm{sh}}$ is the strict henselization of $\mathcal{O}_{T,t}$, then the surjectivity of $(*)$ implies that we can find a lift $\xi_t \in U'(A \otimes_B B')$ of $\xi|_{\operatorname{Spec} A} \in X'(A \otimes_B B')$. By a standard direct limit argument, we can find an étale neighborhood V_t of $t \in T$ and a "spreading-out" $\widetilde{\xi}_t \in U'(V_t \times_{\operatorname{Spec}(B)} \operatorname{Spec}(B'))$ of ξ_t. As T is quasi-compact, the disjoint union of finitely-many of the V_t's give an étale cover V of T and a lift

$$\widetilde{\xi} \in U'(V' \times_{\operatorname{Spec}(B)} \operatorname{Spec}(B'))$$

of ξ. Thus, the morphism $X \to \mathrm{R}_{S'/S}(X')$ is surjective as étale sheaves, granting the claim.

Step 3 (Surjectivity for strictly henselian points). It remains to show that if A is a strictly henselian B-algebra, then

$$U'(A \otimes_B B') \to X'(A \otimes_B B')$$

is surjective. As $B \to B'$ is (module) finite, the base change $A \otimes_B B'$ is a (module) finite A-algebra. By [EGA, IV$_4$, Prop 18.8.10], there are finitely-many strictly henselian local rings $C_1, \ldots, C_n$ such that

$$A \otimes_B B' = C_1 \times \ldots \times C_n.$$

The map $U'(A \otimes_B B') \to X'(A \otimes_B B')$ decomposes as the product of the maps $U'(C_i) \to X'(C_i)$, hence it suffices to show that, for a strictly henselian local ring C, the map $U'(C) \to X'(C)$ is surjective. Given a point $\gamma \in X'(C)$, consider the Cartesian square

$$\begin{array}{ccc} Z & \longrightarrow & U' \\ \downarrow & & \downarrow \\ \operatorname{Spec}(C) & \xrightarrow{\gamma} & X' \end{array}$$

where Z is a scheme and $Z \to \operatorname{Spec}(C)$ is étale and surjective since $U' \to X'$ is étale and surjective. By [EGA, IV$_4$, Théorème 18.5.11(b)], the étale cover $Z \to \operatorname{Spec}(C)$ has a section $\sigma\colon \operatorname{Spec}(C) \to Z$, and hence $\operatorname{Spec}(C) \xrightarrow{\sigma} Z \to U'$ lifts γ, as required. $\square$

Remark 5.3. If $S' \to S$ is a finite, locally free morphism of schemes and X' is a finitely presented algebraic space over S', then $\mathrm{R}_{S'/S}(X')$ is finitely presented over S by construction (in particular, it is quasi-compact over S).

Proposition 5.4. *Let $S' \to S$ be a morphism of Noetherian schemes, and let X' be an algebraic space over S'.*

(1) If T is an S-scheme and $T' = T \times_S S'$, then there is an isomorphism

$$\mathrm{R}_{T'/T}(X' \times_{S'} T') \simeq \mathrm{R}_{S'/S}(X') \times_S T$$

of functors on (Sch/T).

(2) If $X' \to Z'$ and $Y' \to Z'$ are morphisms of algebraic spaces over S', then there is an isomorphism

$$\mathrm{R}_{S'/S}(X' \times_{Z'} Y') \simeq \mathrm{R}_{S'/S}(X') \times_{\mathrm{R}_{S'/S}(Z')} \mathrm{R}_{S'/S}(Y')$$

of functors on (Sch/S).

Proof. The proof is identical to Proposition 2.2(2,3). □

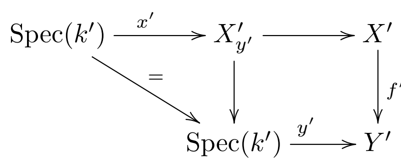
Proposition 5.5. *Let $S' \to S$ be a finite, flat morphism of Noetherian schemes. If $f' \colon X' \to Y'$ is a smooth (resp. étale) morphism between finitely presented algebraic spaces over S', then $\mathrm{R}_{S'/S}(f') \colon \mathrm{R}_{S'/S}(X') \to \mathrm{R}_{S'/S}(Y')$ is smooth (resp. étale).*

Proof. The proof is identical to the case of schemes in Proposition 2.6(5,6), since smoothness of algebraic spaces can be checked using the infinitesimal lifting criterion, by [Sta17, Tag 04AM]. □

Proposition 5.6. *Let $S' \to S$ be a finite, flat morphism of Noetherian schemes. If $f' \colon X' \to Y'$ is a smooth, surjective morphism between finitely presented algebraic spaces over S', then $\mathrm{R}_{S'/S}(f') \colon \mathrm{R}_{S'/S}(X') \to \mathrm{R}_{S'/S}(Y')$ is smooth and surjective.*

Proof. Write $X = \mathrm{R}_{S'/S}(X')$, $Y = \mathrm{R}_{S'/S}(Y')$, and $f = \mathrm{R}_{S'/S}(f')$. By Proposition 5.5, f is smooth. It suffices to show that for any geometric point $s \colon \mathrm{Spec}(k) \to S$, the k-morphism $f_s \colon X_s \to Y_s$ is surjective. Thus, we may assume that $S = \mathrm{Spec}(k)$ for an algebraically closed field k. As $S' \to S$ is finite, it follows that S' is affine, say $S' = \mathrm{Spec}(k')$; k' is a finite k-algebra, hence it decomposes as a finite product of finite local k-algebras $k' = \prod_{i=1}^m A'_i$.

The morphism $X(k) \to Y(k)$ is, by the universal property of the Weil restriction, exactly the map $X'(k') \to Y'(k')$. Given a point $y' \in Y'(k')$, it suffices to construct a point $x' \in X'(k')$ such that the following diagram commutes:



Since $X'(k') = \prod_i X'(A'_i)$ and $Y'(k') = \prod_i Y'(A'_i)$, then y' corresponds to an m-tuple of points $y'_i \in Y'(A'_i)$ and it suffices to show that each y'_i lifts to $X'(A'_i)$. So we may assume k' is a finite local k-algebra (with residue field k).

As k is algebraically closed, there is a point $z' \in X'_{y'}(k)$. The morphism $X'_{y'} \to \mathrm{Spec}(k')$ is smooth, because it is the base change of f', and so the infinitesimal lifting criterion gives a lift $x' \in X'_{y'}(k')$ of z', as required. □

Proposition 5.7. *Let $S' \to S$ be a finite, flat morphism of Noetherian schemes. Let* P *be one of the following properties of morphisms:*

(1) monomorphism;
(2) open immersion;
(3) closed immersion;
(4) separated.

If $f'\colon X' \to Y'$ is a morphism of finitely presented algebraic spaces over S' with the property P*, then the morphism* $\mathrm{R}_{S'/S}(f')\colon \mathrm{R}_{S'/S}(X') \to \mathrm{R}_{S'/S}(Y')$ *also has the property* P*.*

Proof. As in the proof of Proposition 2.6, the real content is in (2) and (3), so we focus on those assertions. Without loss of generality, we may assume that S and S' are affine. If $V' \to Y'$ is an étale cover of Y' by an affine scheme (in particular, V' is quasi-projective over S'), form the fibre product

$$\begin{array}{ccc} U' & \xrightarrow{f} & V' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f'} & Y' \end{array}$$

as algebraic spaces over S'. Then, U' is an S'-scheme and g is étale and surjective. The property P is stable under base change, so f has P. By Proposition 5.4(2), there is a Cartesian diagram

$$\begin{array}{ccc} \mathrm{R}_{S'/S}(U') & \xrightarrow{\mathrm{R}_{S'/S}(f)} & \mathrm{R}_{S'/S}(V') \\ \downarrow{\scriptstyle \mathrm{R}_{S'/S}(g)} & & \downarrow{\scriptstyle \mathrm{R}_{S'/S}(h)} \\ \mathrm{R}_{S'/S}(X) & \xrightarrow{\mathrm{R}_{S'/S}(f')} & \mathrm{R}_{S'/S}(Y') \end{array}$$

The morphism $\mathrm{R}_{S'/S}(f)$ has property P by Proposition 2.6, and the morphisms $\mathrm{R}_{S'/S}(g)$ and $\mathrm{R}_{S'/S}(h)$ are étale and surjective, by Proposition 5.6. It follows that $\mathrm{R}_{S'/S}(f')$ has property P by étale descent. □

Proposition 5.8. *Let $S' \to S$ be a finite, flat morphism of Noetherian schemes that is surjective and radicial.*

(1) If $X'_1, \ldots, X'_n$ are finitely-presented algebraic spaces over S', then

$$\mathrm{R}_{S'/S}\left(\bigsqcup_{i=1}^{n} X'_i\right) = \bigsqcup_{i=1}^{n} \mathrm{R}_{S'/S}(X'_i).$$

(2) If $\{U'_i\}_{i\in I}$ is an open (resp. étale) cover of a smooth, finitely-presented algebraic space X' over S', then $\{\mathrm{R}_{S'/S}(U'_i)\}_{i\in I}$ is an open (resp. étale) cover of $\mathrm{R}_{S'/S}(X')$*.*

Proof. The proof is identical to the case of schemes, as in Proposition 2.8. Note that, since the $\mathrm{R}_{S'/S}(X'_i)$'s and $\mathrm{R}_{S'/S}(U'_i)$'s are algebraic spaces, the need for the quasi-projectivity hypotheses (that appear in Proposition 2.8) disappear. □

Proposition 5.9. *Let k be a field and let k' a nonzero finite k-algebra. Let X' be a smooth, finitely-presented algebraic space over* $\mathrm{Spec}(k')$ *such that* $X' \to \mathrm{Spec}(k')$ *is surjective with geometrically connected fibres. Then, $X = \mathrm{R}_{S'/S}(X)$ is non-empty and geometrically connected.*

Proof. The proof is identical to that of Proposition 3.7. Note that a crucial ingredient, Lemma 3.4, is stated for algebraic spaces. □

6. Olsson's Theorem

The aim of this section is to discuss Olsson's result [Ols06, Theorem 1.5] on the Weil restriction of certain Artin stacks along a *proper* flat morphism of schemes. The notation for this section is fixed below.

Setup 6.1. Let S be a Noetherian affine scheme, let S' be a proper flat algebraic S-space, and let $\mathcal{X}' \to S'$ be one of the following:

(1) a separated Artin stack of finite type over S', with finite diagonal;
(2) a Deligne-Mumford stack of finite type over S', with finite diagonal;
(3) an algebraic space, separated of finite type over S'.

Definition 6.2. The *Weil restriction* $\mathrm{R}_{S'/S}(\mathcal{X}')$ *of* $\mathcal{X}'$ *along* $S' \to S$ is the fibered category over S which, to any S-scheme T, associates the groupoid $\mathcal{X}'(T \times_S S')$.

As in the case of schemes (c.f. Proposition 2.1), if the stack/space $\mathcal{X}'$ arises via base change from a stack/space over S, then the Weil restriction recaptures the $\underline{\mathrm{Hom}}$-stack/space of $\mathcal{X}'$:

Proposition 6.3. *If* $\mathcal{X}$ *is as in Setup 6.1,* $S' \to S$ *is a morphism of schemes, and* $\mathcal{X}' = \mathcal{X} \times_S S'$*, then there is an isomorphism*

$$\mathrm{R}_{S'/S}(\mathcal{X}') = \underline{\mathrm{Hom}}_S(S', \mathcal{X})$$

of functors on (Sch/S)*.*

If S is a scheme and $\mathcal{Y}$, $\mathcal{Z}$ are Artin stacks over S, recall that the $\underline{\mathrm{Hom}}$-stack $\underline{\mathrm{Hom}}_S(\mathcal{Y}, \mathcal{Z})$ is the fibered category over (Sch/S) that, to an S-scheme T, assigns the groupoid of T-morphisms $\mathcal{Y} \times_S T \to \mathcal{Z} \times_S T$.

Proof. This is immediate from the universal property of Weil restriction. □

The following theorem is due to Olsson; see [Ols06, Theorem 1.5].

Theorem 6.4. *With notation as in Setup 6.1, the Weil restriction* $\mathrm{R}_{S'/S}(\mathcal{X}')$ *is respectively the following:*

(1) an Artin stack, locally of finite type over S*, with quasi-compact and separated diagonal;*
(2) a Deligne–Mumford stack, locally of finite type over S*, with quasi-compact and separated diagonal;*
(3) an algebraic space, locally of finite type over S*, with quasi-compact diagonal.*

Olsson applied Theorem 6.4 to show [Ols06, Theorem 1.1], which states that if S is a Noetherian scheme, $\mathcal{X} \to S$ is a proper, flat, algebraic space of finite type, and $\mathcal{Y} \to S$ is as in Setup 6.1(1), then the associated $\underline{\mathrm{Hom}}$-stack $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack, locally of finite type over S, with quasi-compact and separated diagonal (and moreover, if $\mathcal{Y}$ is a Deligne-Mumford stack or an algebraic space, then so too is $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$).

Example 6.5. Although the Weil restriction of $\mathcal{X}' \to S'$ through a proper, flat morphism $S' \to S$ preserves the property of being locally of finite presentation over the base, it may break quasi-compactness. In fact, it is possible that there are connected components of $\mathrm{R}_{S'/S}(\mathcal{X}')$ that are *not* quasi-compact!

To explain this, let $\mathcal{X} \to S$ be as in Setup 6.1, and let $\mathcal{X}' = \mathcal{X} \times_S S'$. Then, we have

$$\mathrm{R}_{S'/S}(\mathcal{X}') = \underline{\mathrm{Sect}}(\mathcal{X}'/S') = \underline{\mathrm{Hom}}_S(S', \mathcal{X}).$$

When $\mathcal{X}$ is a proper algebraic space over S, the Weil restriction $\mathrm{R}_{S'/S}(\mathcal{X}')$ (as a $\underline{\mathrm{Hom}}$-space) will be an algebraic space *locally* of finite type. We will show that quasi-compactness fails *very* badly, in the sense that *most* connected components are not quasi-compact.

Let k be an algebraically-closed field, let $S = \operatorname{Spec} k$, and let $T = \mathbb{P}^1_k$. Consider a line L and a conic C in $\mathbb{P}^3$ that do not intersect one another. Define the scheme X by gluing two copies of $\mathbb{P}^3$, identifying L with C; X is a scheme by [Sch05, Corollary 3.7]. We claim that all components of $\mathrm{R}_{T/S}(X_T) = \underline{\mathrm{Hom}}_k(\mathbb{P}^1, X)$ are not quasi-compact, except for the components corresponding to constant maps.

Take a k-point of $\mathrm{R}_{T/S}(X_T)$ belonging to a connected component consisting of non-constant maps; it corresponds to a non-constant map $\varphi\colon \mathbb{P}^1 \to X$. First, degenerate φ into a degree-n covering of a line. Then, we can move the image to the image of L, which is the same as the image of C. Finally, we may move it outside of C and degenerate again into a double line, hence we eventually double the mapping degree. Therefore, we can see that there are infinitely-many connected components of $\underline{\mathrm{Hom}}_S(\mathbb{P}^1, \mathbb{P}^3 \backslash (L \cup C))$, which sits (as an open subscheme) in the connected component U of $\underline{\mathrm{Hom}}_k(\mathbb{P}^1, X)$ containing φ. In particular, U is not quasi-compact.

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