BRST Transformations

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0.1 Definitions

- $F_{\mu\nu}{}^{c} \equiv \partial_\mu A_\nu{}^{c} - \partial_\nu A_\mu{}^{c} + g A_\mu{}^{a} A_\nu{}^{b} f^{abc}$
- The covariant derivative in its adjoint representation is: $D_\mu{}^{ac} \equiv \partial_\mu \delta^{ac} + g f^{abc} A_\mu{}^{b}$.
- The covariant derivative in its fundamental representation is: $D_\mu \equiv \partial_\mu - i g A_\mu{}^{a} t^{a}$
- The full Lagrangian is given by: $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}{}^{a} F^{\mu\nu a} + \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{\xi}{2} B^a B^a + B^a \partial^\mu A_\mu{}^{a} + \overline{c}^a \left(-\partial^\mu D_\mu{}^{ac}\right) c^c}$ where ξ is a gauge fixing parameter, B^a is an auxilliary Bosonic gauge fixing field, and c^a, $\overline{c}^a$ are ghost and anti-ghost fields used in the Feddeev-Popov scheme.
  - Note that the ordinary partial derivative inside $\bar{c}\left(-\partial^\mu D_\mu{}^{ac}\right)c^c \equiv \bar{c}\partial^\mu\left(-D_\mu{}^{ac}c^c\right)$ is to act on *both* the covariant derivative and the following ghost field c^c. Thus explicitly the expression for this term is:

$$
\begin{aligned}
\overline{c}^a \left(-\partial^\mu D_\mu{}^{ac}\right) c^c &= \overline{c}^a \left(-\left(\partial^\mu \partial_\mu \delta^{ac} + g f^{abc} \partial^\mu A_\mu{}^{b}\right)\right) c^c \\
&= -\overline{c}^a \partial^2 c^a - g f^{abc} \overline{c}^a \partial^\mu A_\mu{}^{b} c^c \\
&= -\overline{c}^a \partial^2 c^a - g f^{abc} \overline{c}^a \left(\partial^\mu A_\mu{}^{b}\right) c^c - g f^{abc} \overline{c}^a A_\mu{}^{b} \partial^\mu c^c \\
&= -g f^{abc} \overline{c}^a \left(\partial^\mu A_\mu{}^{b}\right) c^c - \overline{c}^a D_\mu{}^{ac} \partial^\mu c^c
\end{aligned}
$$

0.1.1 Claim

The classical equations of motion stemming from this Lagrangian are:
$$
\begin{cases}
D_\beta{}^{dc} F^{\beta\sigma\, c} = -g\bar{\psi}\gamma^\sigma t^d \psi + \partial^\sigma B^d + g f^{dac}\left(\partial^\sigma \overline{c}^a\right) c^c \\
\sum_j \partial_\sigma \bar{\psi}_{\alpha,\, j} i \gamma^\sigma{}_{ji} - \sum_\beta \sum_j \bar{\psi}_{\beta,\, j}\left(g A_\mu{}^{a} \gamma^\mu{}_{ji} t^a{}_{\beta\alpha} - m \delta_{ji}\delta_{\beta\alpha}\right) = 0 \\
\left(i\gamma^\mu D_\mu - m\right)\psi = 0 \\
B^b = \frac{1}{\xi}\partial^\mu A_\mu{}^{b} \\
D_\sigma{}^{da} \partial^\sigma \overline{c}^a = 0 \\
\partial^\sigma D_\sigma{}^{dc} c^c = 0
\end{cases}
$$

Proof In order to use the Euler-Lagrange equations, we must have a Lagrangian that depends only on the fields on their spacetime derivatives. While that is *usually* the case, it is not the case in our Lagrangian at hand:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu}{}^{a} F^{\mu\nu a} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{\xi}{2} B^a B^a + B^a \partial^\mu A_\mu{}^{a} + \overline{c}^a \left(-\partial^\mu D_\mu{}^{ac}\right) c^c \\
&= -\frac{1}{4} F_{\mu\nu}{}^{a} F^{\mu\nu a} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{\xi}{2} B^a B^a + B^a \partial^\mu A_\mu{}^{a} - \overline{c}^a \partial^2 c^a - g f^{abc} \overline{c}^a \left(\partial^\mu A_\mu{}^{b}\right) c^c - g f^{abc} \overline{c}^a A_\mu{}^{b} \partial^\mu c^c
\end{aligned}
$$

The first term of the ghost field is differentiated twice, and if we pressed on with this Lagrangian, we would have modified Euler-Lagrange equations. Instead of following that route, we can instead add a four-divergence to the Lagrangian (as that would not change the equations of motion) so that there will be only at most first derivatives in the Lagrangian. To this end, we write:

$$\begin{aligned}
-\bar{c}^a \partial^2 c^a &= -\bar{c}^a \partial_\mu \partial^\mu c^a \\
&= -\bar{c}^a \partial_\mu \partial^\mu c^a \\
&= -\partial_\mu \left(\bar{c}^a \partial^\mu c^a\right) + \left(\partial_\mu \bar{c}^a\right) \partial^\mu c^a \\
&\sim \left(\partial_\mu \bar{c}^a\right) \partial^\mu c^a
\end{aligned}$$

and so instead, we will work with the following Lagrangian:

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu a} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{\xi}{2} B^a B^a + B^a \partial^\mu A_\mu{}^a + \left(\partial_\mu \bar{c}^a\right)\partial^\mu c^a - g f^{abc} \bar{c}^a \left(\partial^\mu A_\mu{}^b\right) c^c - g f^{abc} \bar{c}^a A_\mu{}^b \partial^\mu c^c}$$

and use the usual Euler-Lagrange equations $\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0$ for each field $\phi \in \left\{A_\mu{}^a,\, \psi,\, \bar{\psi},\, B^a,\, c^a,\, \bar{c}^a\right\}$.

1. Start with the gauge fields $A_\mu{}^a$:

   (a) First the field strength tensor:

$$\begin{aligned}
\frac{\delta}{\delta A_\sigma{}^d} F_{\mu\nu}{}^c &= \frac{\delta}{\delta A_\sigma{}^d}\left(\partial_\mu A_\nu{}^c - \partial_\nu A_\mu{}^c + g A_\mu{}^a A_\nu{}^b f^{abc}\right) \\
&= g f^{abc}\left(A_\mu{}^a \delta_\nu^\sigma \delta^{db} + A_\nu{}^b \delta_\mu^\sigma \delta^{da}\right) \\
&= g f^{adc}\left(A_\mu{}^a \delta_\nu^\sigma - A_\nu{}^a \delta_\mu^\sigma\right)
\end{aligned}$$

   (a) Now the whole Lagrangian:

$$\begin{aligned}
\frac{\delta}{\delta A_\sigma{}^d}\mathcal{L} &= \frac{\delta}{\delta A_\sigma{}^d}\left(-\frac{1}{4}\eta^{\alpha\beta}\eta^{\gamma\delta} F_{\alpha\gamma}{}^c F_{\beta\delta}{}^c + \bar{\psi}\left(i\gamma^\mu\left(-igA_\mu{}^c t^c\right)\right)\psi - g f^{abc}\bar{c}^a A_\mu{}^b \partial^\mu c^c\right) \\
&= -\frac{1}{4}\eta^{\alpha\beta}\eta^{\gamma\delta}\left(F_{\alpha\gamma}{}^c g f^{adc}\left(A_\beta{}^a \delta_\delta^\sigma - A_\delta{}^a \delta_\beta^\sigma\right) + g f^{adc}\left(A_\alpha{}^a \delta_\gamma^\sigma - A_\gamma{}^a \delta_\alpha^\sigma\right) F_{\beta\delta}{}^c\right) \\
&\quad + \bar{\psi}\left(i\gamma^\mu\left(-ig\delta_\mu^\sigma \delta^{cd} t^c\right)\right)\psi - g f^{abc}\bar{c}^a \delta_\mu^\sigma \delta^{bd}\partial^\mu c^c \\
&= -g f^{dac} A_\beta{}^a F^{\sigma\beta\, c} + g\bar{\psi}\gamma^\sigma t^d \psi + g f^{dac}\bar{c}^a \partial^\sigma c^c
\end{aligned}$$

   (b) And for the derivative:

$$\begin{aligned}
\frac{\delta}{\delta \partial_\lambda A_\sigma{}^d} F_{\mu\nu}{}^c &= \frac{\delta}{\delta \partial_\lambda A_\sigma{}^d}\left[\partial_\mu A_\nu{}^c - \partial_\nu A_\mu{}^c\right] \\
&= \delta_\mu^\lambda \delta_\nu^\sigma \delta^{dc} - \delta_\nu^\lambda \delta_\mu^\sigma \delta^{dc}
\end{aligned}$$

$$
\begin{aligned}
\frac{\delta}{\delta \partial_\lambda A_\sigma{}^d}\mathcal{L} &= \frac{\delta}{\delta \partial_\lambda A_\sigma{}^d}\left[-\frac{1}{4}\eta^{\alpha\beta}\eta^{\gamma\delta}F_{\alpha\gamma}{}^c F_{\beta\delta}{}^c + B^a\eta^{\alpha\beta}\partial_\alpha A_\beta{}^a - gf^{abc}\bar{c}^a\left(\partial^\mu A_\mu{}^b\right)c^c\right] \\
&= -\frac{1}{4}\left[\eta^{\alpha\beta}\eta^{\gamma\delta}\left(F_{\alpha\gamma}{}^c\left(\delta^\lambda_\beta\delta^\sigma_\delta\delta^{dc} - \delta^\lambda_\delta\delta^\sigma_\beta\delta^{dc}\right) + \left(\delta^\lambda_\alpha\delta^\sigma_\gamma\delta^{dc} - \delta^\lambda_\gamma\delta^\sigma_\alpha\delta^{dc}\right)F_{\beta\delta}{}^c\right)\right] \\
&\quad + B^a\eta^{\alpha\beta}\delta^\lambda_\alpha\delta^\sigma_\beta\delta^{ad} - gf^{abc}\bar{c}^a\left(\eta^{\alpha\beta}\delta^\lambda_\alpha\delta^\sigma_\beta\delta^{ab}\right)c^c \\
&= -F^{\lambda\sigma\,d} + B^d\eta^{\lambda\sigma} - g\eta^{\lambda\sigma}f^{adc}\bar{c}^a c^c
\end{aligned}
$$

(c) Thus the equation of motion is given by

$$
\begin{aligned}
\partial_\lambda\left(-F^{\lambda\sigma\,d} + B^d\eta^{\lambda\sigma} - g\eta^{\lambda\sigma}f^{adc}\bar{c}^a c^c\right) - \left(-gf^{dac}A_\beta{}^a F^{\sigma\beta\,c} + g\bar{\psi}\gamma^\sigma t^d\psi + gf^{dac}\bar{c}^a\partial^\sigma c^c\right) &= 0 \\
-\partial_\lambda F^{\lambda\sigma\,d} + \partial^\sigma B^d - gf^{adc}\partial^\sigma\bar{c}^a c^c + gf^{dac}A_\beta{}^a F^{\sigma\beta\,c} - g\bar{\psi}\gamma^\sigma t^d\psi - gf^{dac}\bar{c}^a\partial^\sigma c^c &= 0 \\
\partial_\lambda F^{\lambda\sigma\,d} + gf^{dac}A_\beta{}^a F^{\beta\sigma\,c} &= -g\bar{\psi}\gamma^\sigma t^d\psi \\
&\quad + \partial^\sigma B^d \\
&\quad + gf^{dac}\left(\partial^\sigma\bar{c}^a\right)c^c
\end{aligned}
$$

(d) After simplifying a bit and using we obtain $\boxed{D_\beta{}^{dc}F^{\beta\sigma\,c} = -g\bar{\psi}\gamma^\sigma t^d\psi + \partial^\sigma B^d + gf^{dac}\left(\partial^\sigma\bar{c}^a\right)c^c}$.

2. For the spinor field $\psi_{\alpha,\,i}$ (where α is the spin-index and i is the flavor index, both of which are usually implicit)

   (a) For the field itself:

$$
\begin{aligned}
\frac{\delta}{\delta\psi_{\alpha,\,i}}\mathcal{L} &= \frac{\delta}{\delta\psi_{\alpha,\,i}}\left[\sum_{\beta,\,\gamma}\sum_{j,\,k}\bar{\psi}_{\beta,\,j}\left(gA_\mu{}^a\gamma^\mu{}_{jk}t^a{}_{\beta\gamma} - m\delta_{jk}\delta_{\beta\gamma}\right)\psi_{k,\,\gamma}\right] \\
&= -\sum_{\beta,\,\gamma}\sum_{j,\,k}\bar{\psi}_{\beta,\,j}\left(gA_\mu{}^a\gamma^\mu{}_{jk}t^a{}_{\beta\gamma} - m\delta_{jk}\delta_{\beta\gamma}\right)\delta_{ik}\delta_{\alpha\gamma} \\
&= -\sum_{\beta}\sum_{j}\bar{\psi}_{\beta,\,j}\left(gA_\mu{}^a\gamma^\mu{}_{ji}t^a{}_{\beta\alpha} - m\delta_{ji}\delta_{\beta\alpha}\right)
\end{aligned}
$$

   (b) For its derivative:

$$
\begin{aligned}
\frac{\delta}{\delta\partial_\sigma\psi_{\alpha,\,i}}\mathcal{L} &= \frac{\delta}{\delta\partial_\sigma\psi_{\alpha,\,i}}\left[\sum_{\beta}\sum_{j}\bar{\psi}_{\beta,\,j}\left(i\gamma^\mu\partial_\mu\right)\psi_{j,\,\beta}\right] \\
&= -\bar{\psi}_{\alpha,\,i}i\gamma^\sigma
\end{aligned}
$$

   (c) Thus the equation of motion for $\psi_{\alpha,\,i}$ is: $\boxed{\partial_\sigma\bar{\psi}_{\alpha,\,i}i\gamma^\sigma - \sum_{\beta}\sum_{j}\bar{\psi}_{\beta,\,j}\left(gA_\mu{}^a\gamma^\mu{}_{ji}t^a{}_{\beta\alpha} - m\delta_{ji}\delta_{\beta\alpha}\right) = 0}$ which is the adjoint Dirac equation.

3. For the adjoint spinor field $\bar{\psi}_{\alpha,\,i}$:

(a) For the field itself:

$$\begin{aligned}
\frac{\delta}{\delta\bar{\psi}_{\alpha,\,i}}\mathcal{L} &= \frac{\delta}{\delta\bar{\psi}_{\alpha,\,i}}\left[\sum_{\beta,\,\gamma}\sum_{j,\,k}\bar{\psi}_{\beta,\,j}\left(i\gamma^{\mu}{}_{jk}D_{\mu\,\beta\gamma} - m\delta_{jk}\delta_{\beta\gamma}\right)\psi_{k,\,\gamma}\right] \\
&= \sum_{\gamma}\sum_{k}\left(i\gamma^{\mu}{}_{ik}D_{\mu\,\alpha\gamma} - m\delta_{jk}\delta_{\beta\gamma}\right)\psi_{k,\,\gamma} \\
&= \left(i\gamma^{\mu}D_{\mu} - m\right)\psi
\end{aligned}$$

(b) For its derivative:

$$\frac{\delta}{\delta\partial_{\sigma}\bar{\psi}_{\alpha,\,i}}\mathcal{L} = 0$$

(c) Thus the equation of motion for $\bar{\psi}_{\alpha,\,i}$ is: $\boxed{\left(i\gamma^{\mu}D_{\mu} - m\right)\psi = 0}$ which is the Dirac equation, but with the covariant derivative replaced with the ordinary derivative, to accomodate for gauge theory.

4. For the auxillary field B^a:

(a) For the field itself:

$$\begin{aligned}
\frac{\delta}{\delta B^b}\mathcal{L} &= \frac{\delta}{\delta B^b}\left[-\frac{\xi}{2}B^aB^a + B^a\partial^{\mu}A_{\mu}{}^{a}\right] \\
&= -\frac{\xi}{2}\left(B^a\delta^{ab} + \delta^{ab}B^a\right) + \delta^{ab}\partial^{\mu}A_{\mu}{}^{a} \\
&= -\xi B^b + \partial^{\mu}A_{\mu}{}^{b}
\end{aligned}$$

(b) For its derivative, $\frac{\delta}{\delta\partial_{\sigma}B^b}\mathcal{L} = 0$.

(c) Thus the equation of motion for B^b is $\boxed{B^b = \frac{1}{\xi}\partial^{\mu}A_{\mu}{}^{b}}$.

5. For the ghost field c^a:

(a) For the field itself:

$$\begin{aligned}
\frac{\delta}{\delta c^d}\mathcal{L} &= \frac{\delta}{\delta c^d}\left[-gf^{abc}\bar{c}^a\left(\partial^{\mu}A_{\mu}{}^{b}\right)c^c\right] \\
&= +gf^{abd}\bar{c}^a\left(\partial^{\mu}A_{\mu}{}^{b}\right)
\end{aligned}$$

(b) For its derivative:

$$\begin{aligned}
\frac{\delta}{\delta\partial_{\sigma}c^d}\mathcal{L} &= \frac{\delta}{\delta\partial_{\sigma}c^d}\left[\left(\partial^{\mu}\bar{c}^a\right)\partial_{\mu}c^a - gf^{abc}\bar{c}^aA^{\mu\,b}\partial_{\mu}c^c\right] \\
&= -\partial^{\sigma}\bar{c}^d + gf^{abd}\bar{c}^aA^{\sigma\,b}
\end{aligned}$$

(c) and so the equation of motion is:

$$\begin{aligned} \partial_\sigma \left(-\partial^\sigma \bar{c}^d + g f^{abd} \bar{c}^a A^{\sigma\ b}\right) - g f^{abd} \bar{c}^a \left(\partial^\mu A_\mu{}^b\right) &= 0 \\ -\partial_\sigma \partial^\sigma \bar{c}^d - g f^{dba} A^{\sigma\ b} \partial_\sigma \bar{c}^a &= 0 \\ D_\sigma{}^{da} \partial^\sigma \bar{c}^a &= 0 \end{aligned}$$

(d) Finally $\boxed{D_\sigma{}^{da} \partial^\sigma \bar{c}^a}$.

6. For the adjoint ghost field, $\bar{c}^d$ we have:

(a) For the field itself:

$$\begin{aligned} \frac{\delta}{\delta \bar{c}^d} \mathcal{L} &= \frac{\delta}{\delta \bar{c}^d} \left[-g f^{abc} \bar{c}^a \left(\partial^\mu A_\mu{}^b\right) c^c - g f^{abc} \bar{c}^a A_\mu{}^b \partial^\mu c^c\right] \\ &= -g f^{dbc} \left(\partial^\mu A_\mu{}^b\right) c^c - g f^{dbc} A_\mu{}^b \partial^\mu c^c \end{aligned}$$

(b) For its derivative:

$$\begin{aligned} \frac{\delta}{\delta \partial_\sigma \bar{c}^d} \mathcal{L} &= \frac{\delta}{\delta \partial_\sigma \bar{c}^d} \left[\left(\partial_\mu \bar{c}^a\right) \partial^\mu c^a\right] \\ &= \partial^\sigma c^d \end{aligned}$$

(c) and so the equation of motion is:

$$\begin{aligned} \partial_\sigma \partial^\sigma c^d + g f^{dbc} \left(\partial^\mu A_\mu{}^b\right) c^c + g f^{dbc} A_\mu{}^b \partial^\mu c^c &= 0 \\ \partial^\mu \left(\partial_\mu \delta^{dc} + g f^{dbc} A_\mu{}^b\right) c^c &= 0 \\ \partial^\sigma D_\sigma{}^{dc} c^c &= 0 \end{aligned}$$

(d) and so finally $\boxed{\partial^\sigma D_\sigma{}^{dc} c^c}$.

0.2 BRST Infinitesimal Transformations

- The infinitesimal transformation laws (where ϵ is an infinitesimal anti-commuting parameter, which means $\epsilon^2 \equiv 0$) of BRST are given by:

$$\begin{aligned} A_\mu{}^a &\mapsto A_\mu{}^a + \epsilon D_\mu{}^{ac} c^c \\ \psi &\mapsto \left(1 + i g \epsilon c^a t^a\right) \psi \\ c^a &\mapsto c^a - \frac{1}{2} g \epsilon f^{abc} c^b c^c \\ \bar{c}^a &\mapsto \bar{c}^a + \epsilon B^a \\ B^a &\mapsto B^a \end{aligned}$$

- As a result the covariant derivative in its adjoint representation transforms as:

$$\begin{aligned} D_\mu{}^{ac} &\mapsto \partial_\mu \delta^{ac} + g f^{abc} \left(A_\mu{}^b + \epsilon \left(D_\mu{}^{bd} c^d\right)\right) \\ &= \partial_\mu \delta^{ac} + g f^{abc} A_\mu{}^b + g \epsilon f^{abc} \left(D_\mu{}^{bd} c^d\right) \\ &= D_\mu{}^{ac} + g \epsilon f^{abc} \left(D_\mu{}^{bd} c^d\right) \end{aligned}$$

0.2.1 Claim

The Lagrangian is invariant under BRST transformations.

Proof

- The non-gauge-invariant part of the Lagrangian, namely, $B^a\partial^\mu A_\mu{}^a + \bar{c}^a\left(-\partial^\mu D_\mu{}^{ac}\right)c^c$, transforms as:

$$\begin{aligned}
B^a\partial^\mu A_\mu{}^a - \bar{c}^a\partial^\mu D_\mu{}^{ac}c^c \quad \mapsto \quad & B^a\partial^\mu\left(A_\mu{}^a + \epsilon\left(D_\mu{}^{ac}c^c\right)\right) \\
& -\left(\bar{c}^a + \epsilon B^a\right)\partial^\mu\left(D_\mu{}^{ac} + g\epsilon f^{abc}\left(D_\mu{}^{bd}c^d\right)\right)\left(c^c - \frac{1}{2}g\epsilon f^{cef}c^e c^f\right) \\
= \quad & \underbrace{B^a\partial^\mu A_\mu{}^a} \cancel{+B^a\epsilon\partial^\mu D_\mu{}^{ac}c^c} + \\
& \underbrace{-\bar{c}^a\partial^\mu D_\mu{}^{ac}c^c} \\
& +\frac{1}{2}g\bar{c}^a\partial^\mu D_\mu{}^{ac}\epsilon f^{cef}c^e c^f \\
& -g\bar{c}^a\partial^\mu\epsilon f^{abc}\left(D_\mu{}^{bd}c^d\right)c^c \\
& +\frac{1}{2}g\bar{c}^a\partial^\mu g\epsilon f^{abc}\left(D_\mu{}^{bd}c^d\right)\epsilon f^{cef}c^e c^f \\
& \cancel{-\epsilon B^a\partial^\mu D_\mu{}^{ac}c^c} \\
& +\frac{1}{2}g\epsilon B^a\partial^\mu D_\mu{}^{ac}\epsilon f^{cef}c^e c^f \\
& -g\epsilon B^a\partial^\mu\epsilon f^{abc}\left(D_\mu{}^{bd}c^d\right)c^c \\
& +\frac{1}{2}g^2\epsilon B^a\partial^\mu\epsilon f^{abc}\left(D_\mu{}^{bd}c^d\right)\epsilon f^{cef}c^e c^f \\
= \quad & B^a\partial^\mu A_\mu{}^a - \bar{c}^a\partial^\mu D_\mu{}^{ac}c^c \\
& +\frac{1}{2}g\bar{c}^a\partial^\mu D_\mu{}^{ac}\epsilon f^{cef}c^e c^f - g\bar{c}^a\partial^\mu\epsilon f^{abc}\left(D_\mu{}^{bd}c^d\right)c^c
\end{aligned}$$

  where we discarded all terms which contain twice ϵ, because $\epsilon^2 \equiv 0$ due to anti-commutativity of ϵ.

  - The first line is equal to the original term in the Lagrangian, so to prove invariance of the Lagrangian under BRST transformations, we have to show now that $\Upsilon := \frac{1}{2}g\bar{c}^a\partial^\mu D_\mu{}^{ac}\epsilon f^{cef}c^e c^f - g\bar{c}^a\partial^\mu\epsilon f^{abc}\left(D_\mu{}^{bd}c^d\right)c^c = 0$:

$$\begin{aligned}
\Upsilon &= \frac{1}{2}g\bar{c}^a\partial^\mu\left(\partial_\mu\delta^{ac} + gf^{abc}A_\mu{}^b\right)\epsilon f^{cef}c^e c^f - g\bar{c}^a\partial^\mu\epsilon f^{abc}\left(\left(\partial_\mu\delta^{bd} + gf^{bed}A_\mu{}^e\right)c^d\right)c^c \\
&= \frac{1}{2}g\bar{c}^a\partial^\mu\left(\epsilon f^{aef}\partial_\mu c^e c^f + gf^{abc}A_\mu{}^b\epsilon f^{cef}c^e c^f\right) - g\bar{c}^a\partial^\mu\epsilon f^{abc}\left(\left(\partial_\mu c^b\right)c^c + gf^{bed}A_\mu{}^e c^d c^c\right) \\
&= g\bar{c}^a\epsilon\partial^\mu\left[\frac{1}{2}f^{aef}\partial_\mu c^e c^f - f^{abc}\left(\partial_\mu c^b\right)c^c\right] + \\
&\quad +g^2\bar{c}^a\epsilon\partial^\mu\left[\frac{1}{2}f^{abc}f^{cef}A_\mu{}^b c^e c^f - f^{abc}f^{bed}A_\mu{}^e c^d c^c\right]
\end{aligned}$$

– First we will show that $\frac{1}{2}f^{aef}\partial_\mu c^e c^f - f^{abc}\left(\partial_\mu c^b\right)c^c = 0$:

$$\begin{aligned}
\frac{1}{2}f^{aef}\partial_\mu c^e c^f - f^{abc}\left(\partial_\mu c^b\right)c^c \quad &= \quad \frac{1}{2}f^{abc}\partial_\mu c^b c^c - f^{abc}\left(\partial_\mu c^b\right)c^c \\
&\overset{\text{Leibnitz}}{=} \quad \frac{1}{2}f^{abc}\left(\partial_\mu c^b\right)c^c + \frac{1}{2}f^{abc}c^b\partial_\mu c^c - f^{abc}\left(\partial_\mu c^b\right)c^c \\
&= \quad -\frac{1}{2}f^{abc}\left(\partial_\mu c^b\right)c^c + \frac{1}{2}f^{abc}c^b\partial_\mu c^c \\
&\overset{\left\{\partial_\mu c^b,\, c^c\right\}=0}{=} \quad +\frac{1}{2}f^{abc}c^c\partial_\mu c^b + \frac{1}{2}f^{abc}c^b\partial_\mu c^c \\
&\overset{f^{abc}=-f^{acb}}{=} \quad -\frac{1}{2}f^{acb}c^c\partial_\mu c^b + \frac{1}{2}f^{abc}c^b\partial_\mu c^c \\
&\overset{c\leftrightarrow b}{=} \quad -\frac{1}{2}f^{abc}c^b\partial_\mu c^c + \frac{1}{2}f^{abc}c^b\partial_\mu c^c \\
&= \quad 0
\end{aligned}$$

[Note that even though c and $\bar{c}$ are Grassmann variables, the ordinary derivative with respect to a spacetime coordinate commutes with those fields!]

– And now we will also show that $\frac{1}{2}f^{abc}f^{cef}A_\mu{}^b c^e c^f - f^{abc}f^{bed}A_\mu{}^e c^d c^c = 0$:

$$\begin{aligned}
\frac{1}{2}f^{abc}f^{cef}A_\mu{}^b c^e c^f - f^{abc}f^{bed}A_\mu{}^e c^d c^c \;&=\; \frac{1}{2}f^{abc}f^{cde}A_\mu{}^b c^d c^e - f^{abc}f^{bed}A_\mu{}^e c^d c^c \\
&=\; \frac{1}{2}f^{abc}f^{cde}A_\mu{}^b c^d c^e - \frac{1}{2}f^{abc}f^{cde}A_\mu{}^e c^d c^b - \frac{1}{2}f^{abc}f^{cde}A_\mu{}^e c^d c^b \\
&=\; \frac{1}{2}f^{abc}f^{cde}A_\mu{}^b c^d c^e + \frac{1}{2}f^{abc}f^{cde}A_\mu{}^e c^b c^d + \frac{1}{2}f^{abc}f^{cde}A_\mu{}^d c^e c^b \\
&=\; \frac{1}{2}f^{abc}f^{cde}\left(A_\mu{}^b c^d c^e + A_\mu{}^d c^e c^b + A_\mu{}^e c^b c^d\right) \\
&=\; \frac{1}{2}\left(f^{abc}f^{cde}A_\mu{}^b c^d c^e + f^{aec}f^{cbd}A_\mu{}^b c^d c^e + f^{adc}f^{ceb}A_\mu{}^b c^d c^e\right) \\
&=\; \frac{1}{2}\left(f^{abc}f^{cde} + f^{aec}f^{cbd} + f^{adc}f^{ceb}\right)A_\mu{}^b c^d c^e \\
&=\; -\frac{1}{2}\underbrace{\left(-f^{aec}f^{cbd} - f^{ceb}f^{adc} - f^{abc}f^{cde}\right)}_{\text{zero by the Jacobi identity}}A_\mu{}^b c^d c^e
\end{aligned}$$

where we have used the Jacobi identity $\boxed{f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0}$ in order to ascertain that:

$$\begin{aligned}
f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} &= 0 \\
f^{ace}f^{bdc} + f^{bce}f^{dac} + f^{abc}f^{dce} &= 0 \\
-f^{aec}f^{cbd} - f^{ceb}f^{adc} - f^{abc}f^{cde} &= 0
\end{aligned}$$

- [QED] Thus we have shown that the Lagrangian is invariant under the BRST transformation.

0.2.2 Claim

The BRST transformation is nilpotent on all the fields.

Proof

1.

$$\begin{aligned}
A_\mu{}^a & \overset{\text{BRST with } \epsilon_1}{\mapsto} & A_\mu{}^a + \epsilon_1 D_\mu{}^{ac} c^c \\
& \overset{\text{BRST with } \epsilon_2}{\mapsto} & A_\mu{}^a + \epsilon_2 D_\mu{}^{ac} c^c + \epsilon_1 \left(D_\mu{}^{ac} + g\epsilon_2 f^{abc} \left(D_\mu{}^{bd} c^d \right) \right) \left(c^c - \frac{1}{2} g\epsilon_2 f^{cb'c'} c^{b'} c^{c'} \right) \\
& = & A_\mu{}^a + (\epsilon_2 + \epsilon_1) D_\mu{}^{ac} c^c - \frac{1}{2} g\epsilon_1\epsilon_2 f^{cb'c'} D_\mu{}^{ac} c^{b'} c^{c'} + g\epsilon_1\epsilon_2 f^{abc} \left(D_\mu{}^{bd} c^d \right) c^c \\
& = & A_\mu{}^a + (\epsilon_2 + \epsilon_1) D_\mu{}^{ac} c^c + g\epsilon_1\epsilon_2 \underbrace{\left(-\frac{1}{2} f^{cb'c'} D_\mu{}^{ac} c^{b'} c^{c'} + f^{abc} \left(D_\mu{}^{bd} c^d \right) c^c \right)}_{\text{this was shown to be zero above}} \\
& = & A_\mu{}^a + (\epsilon_2 + \epsilon_1) D_\mu{}^{ac} c^c
\end{aligned}$$

2.

$$\begin{aligned}
\psi & \overset{\text{BRST with } \epsilon_1}{\mapsto} & \psi + ig\epsilon_1 c^a t^a \psi \\
& \overset{\text{BRST with } \epsilon_2}{\mapsto} & \psi + ig\epsilon_2 c^a t^a \psi + ig\epsilon_1 \left(c^a - \frac{1}{2} g\epsilon_2 f^{ab'c'} c^{b'} c^{c'} \right) t^a \left(\psi + ig\epsilon_2 c^a t^a \psi \right) \\
& = & \psi + ig\epsilon_2 c^a t^a \psi + ig\epsilon_1 \left(c^a - \frac{1}{2} g\epsilon_2 f^{ab'c'} c^{b'} c^{c'} \right) t^a \left(\psi + ig\epsilon_2 c^{a'} t^{a'} \psi \right) \\
& = & \psi + ig (\epsilon_1 + \epsilon_2) c^a t^a \psi - g^2 \epsilon_1 c^a t^a \epsilon_2 c^{a'} t^{a'} \psi - \frac{1}{2} ig^2 \epsilon_1\epsilon_2 f^{ab'c'} c^{b'} c^{c'} t^a \psi \\
& = & \psi + ig (\epsilon_1 + \epsilon_2) c^a t^a \psi + g^2 \epsilon_1\epsilon_2 c^a c^{a'} \left(t^a t^{a'} - \frac{1}{2} i f^{daa'} t^d \right) \psi \\
& = & \psi + ig (\epsilon_1 + \epsilon_2) c^a t^a \psi
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
c^a c^{a'} \left(t^a t^{a'} - \frac{1}{2} i f^{daa'} t^d \right) & = & \left(\frac{1}{2} c^a c^{a'} t^a t^{a'} + \frac{1}{2} c^a c^{a'} t^a t^{a'} - \frac{1}{2} i c^a c^{a'} f^{daa'} t^d \right) \\
& = & \left(\frac{1}{2} c^a c^{a'} t^a t^{a'} - \frac{1}{2} c^{a'} c^a t^a t^{a'} - \frac{1}{2} i c^a c^{a'} f^{daa'} t^d \right) \\
& = & \left(\frac{1}{2} c^a c^{a'} t^a t^{a'} - \frac{1}{2} c^a c^{a'} t^{a'} t^a - \frac{1}{2} i c^a c^{a'} f^{daa'} t^d \right) \\
& = & c^a c^{a'} \left(\frac{1}{2} \left[t^a, t^{a'} \right] - \frac{1}{2} i f^{daa'} t^d \right) \\
& = & \frac{1}{2} c^a c^{a'} \left(i f^{daa'} t^d - i f^{daa'} t^d \right) \\
& = & 0
\end{aligned}$$

3.

$$\begin{aligned}
c^a \quad & \overset{\text{BRST with } \epsilon_1}{\mapsto} \quad c^a - \frac{1}{2} g \epsilon_1 f^{abc} c^b c^c \\
& \overset{\text{BRST with } \epsilon_2}{\mapsto} \quad c^a - \frac{1}{2} g \epsilon_2 f^{abc} c^b c^c - \frac{1}{2} g \epsilon_1 f^{abc} \left(c^b - \frac{1}{2} g \epsilon_2 f^{bb'c'} c^{b'} c^{c'} \right) \left(c^c - \frac{1}{2} g \epsilon_2 f^{cb''c''} c^{b''} c^{c''} \right) \\
& = \quad c^a - \frac{1}{2} g \left(\epsilon_1 + \epsilon_2 \right) f^{abc} c^b c^c - \frac{1}{4} g^2 \epsilon_1 \epsilon_2 \left(f^{abc} f^{cb''c''} c^b c^{b''} c^{c''} - f^{abc} f^{bb'c'} c^{b'} c^{c'} c^c \right) \\
& = \quad c^a - \frac{1}{2} g \left(\epsilon_1 + \epsilon_2 \right) f^{abc} c^b c^c - \frac{1}{4} g^2 \epsilon_1 \epsilon_2 \left(f^{abc} f^{cde} - f^{ace} f^{cbd} \right) c^b c^d c^e \\
& = \quad c^a - \frac{1}{2} g \left(\epsilon_1 + \epsilon_2 \right) f^{abc} c^b c^c + \frac{1}{4} g^2 \epsilon_1 \epsilon_2 f^{ceb} f^{adc} c^b c^d c^e \\
& = \quad c^a - \frac{1}{2} g \left(\epsilon_1 + \epsilon_2 \right) f^{abc} c^b c^c
\end{aligned}$$

where we have used the fact that $-f^{ace} f^{cbd} + f^{abc} f^{cde} = -f^{ceb} f^{adc}$ by the Jacobi identity and also the fact that

$$\begin{aligned}
f^{ceb} f^{adc} c^b c^d c^e &= \frac{1}{3} \left(f^{ceb} f^{adc} c^b c^d c^e + f^{ceb} f^{adc} c^b c^d c^e + f^{ceb} f^{adc} c^b c^d c^e \right) \\
&= \frac{1}{3} \left(f^{ceb} f^{adc} c^b c^d c^e + f^{ceb} f^{adc} c^d c^e c^b + f^{ceb} f^{adc} c^e c^b c^d \right) \\
&= \frac{1}{3} \left(f^{ceb} f^{adc} + f^{cde} f^{abc} - f^{cbd} f^{ace} \right) c^b c^d c^e \\
&= 0
\end{aligned}$$

4.

$$\begin{aligned}
\bar{c}^a \quad & \overset{\text{BRST with } \epsilon_1}{\mapsto} \quad \bar{c}^a + \epsilon_1 B^a \\
& \overset{\text{BRST with } \epsilon_2}{\mapsto} \quad \bar{c}^a + \epsilon_2 B^a + \epsilon_1 B^a \\
& = \quad \bar{c}^a + \left(\epsilon_1 + \epsilon_2 \right) B^a
\end{aligned}$$

5. And the field B^a trivially obeys this:

$$\begin{aligned}
B^a \quad & \overset{\text{BRST with } \epsilon_1}{\mapsto} \quad B^a + \epsilon_1 0 \\
& \overset{\text{BRST with } \epsilon_2}{\mapsto} \quad B^a + \epsilon_2 0 + \epsilon_1 0 \\
& = \quad B^a
\end{aligned}$$

- [QED] Thus we have proven that the BRST transformation is nilpotent.

0.2.3 Claim

The conserved Noether current associated with the BRST symmetry is given by: ${j_{BRST}}^{\mu} =$

Proof Since the Lagrangian is invariant under BRST, we expect that under that transformation, $\Delta\mathcal{L} = 0$:

$$
\begin{aligned}
0 &\stackrel{!}{=} \Delta\mathcal{L} \\
&= \sum_i \left[\frac{\delta\mathcal{L}}{\delta\phi_i}\epsilon\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\epsilon\Delta\partial_\mu\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\epsilon\Delta\partial_\nu\partial_\mu\phi_i \right]
\end{aligned}
$$

The Euler-Lagrange equations are:

$$
\begin{aligned}
0 &\stackrel{!}{=} \delta S \\
&= \int d^4x \left\{ \sum_i \left[\frac{\delta\mathcal{L}}{\delta\phi_i}\delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\delta\partial_\mu\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\delta\partial_\nu\partial_\mu\phi_i \right] \right\} \\
&= \int d^4x \left\{ \sum_i \left[\frac{\delta\mathcal{L}}{\delta\phi_i}\delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\partial_\mu\delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\partial_\nu\partial_\mu\delta\phi_i \right] \right\} \\
&= \int d^4x \left\{ \sum_i \left[\frac{\delta\mathcal{L}}{\delta\phi_i}\delta\phi_i + \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\delta\phi_i \right) - \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i} \right) \delta\phi_i + \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\partial_\nu\delta\phi_i \right) - \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i} \right) \partial_\nu\delta\phi_i \right] \right\} \\
&= \int d^4x \left\{ \sum_i \left[\left(\frac{\delta\mathcal{L}}{\delta\phi_i} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i} \right) \delta\phi_i - \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i} \right) \partial_\nu\delta\phi_i \right] \right\} \\
&= \int d^4x \left\{ \sum_i \left[\left(\frac{\delta\mathcal{L}}{\delta\phi_i} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i} \right) \delta\phi_i - \partial_\nu \left(\left(\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i} \right) \delta\phi_i \right) + \left(\partial_\nu \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i} \right) \right) \delta\phi_i \right] \right\} \\
&= \int d^4x \left\{ \sum_i \left[\left(\frac{\delta\mathcal{L}}{\delta\phi_i} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i} + \partial_\nu\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i} \right) \delta\phi_i \right] \right\}
\end{aligned}
$$

Thus the Euler-Lagrange equations are $\boxed{\frac{\delta\mathcal{L}}{\delta\phi_i} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i} + \partial_\nu\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}}$ for all $\phi_i \in \left\{A_\mu{}^a,\, \psi_{\alpha,\,i},\, \bar{\psi}_{\alpha,\,i},\, B^a,\, c^a,\, \bar{c}^a\right\}$. We can use this equation in our derivation of the Noether current:

$$\begin{aligned}
0 &= \sum_i \left[\frac{\delta\mathcal{L}}{\delta\phi_i}\epsilon\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\epsilon\Delta\partial_\mu\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\epsilon\Delta\partial_\nu\partial_\mu\phi_i\right] \\
&= \sum_i \left[\frac{\delta\mathcal{L}}{\delta\phi_i}\epsilon\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\partial_\mu\epsilon\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\partial_\mu\epsilon\Delta\partial_\nu\phi_i\right] \\
&= \sum_i \left[\frac{\delta\mathcal{L}}{\delta\phi_i}\epsilon\Delta\phi_i + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\epsilon\Delta\phi_i\right) - \left(\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\right)\epsilon\Delta\phi_i + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\epsilon\Delta\partial_\nu\phi_i\right) - \left(\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\right)\epsilon\Delta\partial_\nu\phi_i\right] \\
&= \sum_i \left[\left(\frac{\delta\mathcal{L}}{\delta\phi_i} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\right)\epsilon\Delta\phi_i + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\epsilon\Delta\phi_i\right) + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\epsilon\Delta\partial_\nu\phi_i\right) - \left(\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\right)\partial_\nu\epsilon\Delta\phi_i\right] \\
&= \sum_i \left[\left(\underbrace{\frac{\delta\mathcal{L}}{\delta\phi_i} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i} + \partial_\nu\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}}_{0}\right)\epsilon\Delta\phi_i + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\epsilon\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\epsilon\Delta\partial_\nu\phi_i\right) - \partial_\nu\left(\left(\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\mu\phi_i}\right)\epsilon\Delta\phi_i\right)\right] \\
&= \epsilon\partial_\mu\sum_i \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\partial_\nu\Delta\phi_i - \left(\partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\right)\Delta\phi_i\right]
\end{aligned}$$

and so the conserved Noether current is generally given by $\boxed{j^\mu = \sum_i \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\partial_\nu\Delta\phi_i - \left(\partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\right)\Delta\phi_i\right]}$. Thus for our BRST transformation we can compute the conserved Noether current corresponding to the BRST transformation, noting that c^a

is the *only* field that has dependence on its second spacetime derivative, and so, for all other fields, the current is usual:

$$
\begin{aligned}
j_{BRST}{}^{\mu} &= \sum_i \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\Delta\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\partial_\nu\Delta\phi_i - \left(\partial_\nu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\right)\Delta\phi_i \right] \\
&= \sum_a \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\sigma{}^{a}}\delta A_\sigma{}^{a} \\
&\quad + \sum_i\sum_\alpha \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi_{\alpha,\, i}}\delta\psi_{\alpha,\, i} \\
&\quad + \sum_i\sum_\alpha \frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\psi}_{\alpha,\, i}}\bar{\psi}_{\alpha,\, i} \\
&\quad + \sum_a \frac{\delta\mathcal{L}}{\delta\partial_\mu B^{\,a}}\delta B^{\,a} \\
&\quad + \sum_a \frac{\delta\mathcal{L}}{\delta\partial_\mu \bar{c}^{\,a}}\delta \bar{c}^{\,a} \\
&\quad + \sum_a \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu c^{\,a}}\delta c^{\,a} + \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu c^{a}}\partial_\nu\delta c^{\,a} - \left(\partial_\nu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu c^{a}}\right)\delta c^{\,a} \right] \\
&= \sum_a \left(-F^{\mu\sigma\ a} + B^a\eta^{\mu\sigma} - g\eta^{\mu\sigma} f^{bac}\bar{c}^b c^c\right)\left(D_\sigma{}^{ac}c^c\right) \\
&\quad + \sum_i\sum_\alpha \left(-\bar{\psi}_{\beta,\, i} i\gamma^{\mu}{}_{\beta\alpha}\right)\left(igc^a t^a\psi_{\alpha,\, i}\right) \\
&\quad +0 \\
&\quad +0 \\
&\quad +0 \\
&\quad + \sum_a \left[\left(gf^{dba}\bar{c}^d A^{\mu\ b}\right)\left(-\frac{1}{2}gf^{abc}c^b c^c\right) + \left(\eta^{\mu\nu}\bar{c}^a\right)\partial_\nu\left(-\frac{1}{2}gf^{abc}c^b c^c\right) - \left(\partial_\nu\eta^{\mu\nu}\bar{c}^a\right)\left(-\frac{1}{2}gf^{abc}c^b c^c\right) \right] \\
&= \left(-F^{\mu\sigma\ a} + B^a\eta^{\mu\sigma} - g\eta^{\mu\sigma} f^{bac}\bar{c}^b c^c\right)\left(D_\sigma{}^{ac}c^c\right) + g\bar{\psi}\gamma^\mu c^a t^a\psi - \frac{1}{2}g^2 f^{cba}f^{ade}A^{\mu\ b}\bar{c}^c c^d c^e - \frac{1}{2}gf^{abc}\bar{c}^a\partial^\mu c^b c^c + \frac{1}{2}gf^{abc}\left(\partial^\mu\bar{c}^a\right)c^b c^c
\end{aligned}
$$

0.3 BRST Cohomology

Because classically there is a conserved current, when following the *canonical quantization* procedure, the classically conserved charge (denote it by $Q \equiv \int d^3\vec{x} j_{BRST}{}^{0}$) becomes an operator, and this operator is exactly the operator which "generates" the BRST transformation, in the sense that $\left[\hat{Q},\, \hat{\phi}_i\right]_{\pm} = \delta\phi_i$ where $\phi_i \mapsto \phi_i + \delta\phi_i$ under BRST and the commutator or anti-commutator is chosen based on the Grassmannian nature of the field at hand. Because we have proven that the BRST transformation is nilpotent, $\hat{Q}^2 = 0$ as an operator identity now. In addition, $\left[\hat{Q},\, \hat{H}\right] = 0$ because $\hat{Q}$ is a symmetry generator.

Now $\hat{Q}$ divides the eigenstate space of $\hat{H}$ into three subspaces:

1. $\mathcal{H}_1 := \left\{ |\psi\rangle \in \mathcal{H} :\ \hat{Q}\,|\psi\rangle \neq 0 \right\}$

2. The image of $\hat{Q}$: $\mathcal{H}_2 := \left\{ |\psi\rangle \in \mathcal{H} :\ \exists\, |\psi'\rangle \in \mathcal{H}_1 \text{ s.t. } |\psi\rangle = \hat{Q}\,|\psi'\rangle \right\}$.

(a) Observe that $\hat{Q}\,|\psi\rangle = 0\,\forall\,|\psi\rangle \in \mathcal{H}_2$ due to the nilpotency of $\hat{Q}$.

(b) Observe that $\langle\psi\,|\,\psi'\rangle = 0\,\forall\,(|\psi\rangle\,,\,|\psi'\rangle) \in {\mathcal{H}_2}^2$ because $\exists\,|\psi''\rangle \in \mathcal{H}_1 : |\psi'\rangle = \hat{Q}\,|\psi''\rangle$ and then $\langle\psi\,|\,\psi'\rangle = \left\langle\psi\,\middle|\,\hat{Q}\,\middle|\,\psi''\right\rangle = 0$. ($\hat{Q}$ is Hermitian).

3. The cohomology of $\hat{Q}$: $\mathcal{H}_0 := \left\{|\psi\rangle \in \mathcal{H}\backslash\mathcal{H}_2 : \hat{Q}\,|\psi\rangle = 0\right\}$.

(a) Observe that $\langle\psi\,|\,\psi'\rangle = 0\,\forall\,(|\psi\rangle\,,\,|\psi'\rangle) \in \mathcal{H}_0 \times \mathcal{H}_2$ because $\exists\,|\psi''\rangle \in \mathcal{H}_1 : |\psi'\rangle = \hat{Q}\,|\psi''\rangle$ and then $\langle\psi\,|\,\psi'\rangle = \left\langle\psi\,\middle|\,\hat{Q}\,\middle|\,\psi''\right\rangle = 0$.

It is clear that $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2$.

0.3.1 Claim

Only gauge Bosons with transverse polarization exists in $\mathcal{H}_0$ and all other particles are in $\mathcal{H}_1 \cup \mathcal{H}_2$.

0.3.2 Claim

The *restricted* S matrix (restricted to the subspace of transverse polarization gauge Bosons) is also unitary.

0.3.3 Claim

If the gauge group is Abelian, the requirement that $\hat{Q}\,|\psi\rangle = 0$ on $|\psi\rangle \in \mathcal{H}_2$ is equivalent to the Gupta-Bleuler subsidiary condition $(\partial_\mu A^\mu)^+\,|physical\rangle = 0$ (which replaces the classical Lorentz gauge condition).

0.4 Slavnov-Taylor Identities of the Effective Action

0.4.1 Claim

If $\mathcal{L}$ is invariant under the BRST then so is $\mathcal{L} + \sum_i \left[\hat{Q},\,\hat{\phi}_i\right]_\pm$.

Proof This stems from the nilpotency condition we have proven above, which means that the variation of the fields, $\left[\hat{Q},\,\hat{\phi}_i\right]_\pm$, is stable under BRST.

0.4.2 Remark

Recall that the BRST transformation is given by:

$$\begin{aligned}
{A_\mu}^a &\mapsto {A_\mu}^a + \epsilon {D_\mu}^{ac} c^c \\
\psi &\mapsto (1 + ig\epsilon c^a t^a)\,\psi \\
c^a &\mapsto c^a - \frac{1}{2} g\epsilon f^{abc} c^b c^c \\
\bar{c}^a &\mapsto \bar{c}^a + \epsilon B^a \\
B^a &\mapsto B^a
\end{aligned}$$

Thus we realize that *the most general Lagrangian with the same BRST symmetry* is in fact

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}{}^{a}F^{\mu\nu a}+\bar{\psi}\left(i\not{D}-m\right)\psi-\frac{\xi}{2}B^{a}B^{a}+B^{a}\partial^{\mu}A_{\mu}{}^{a}+\bar{c}^{a}\left(-\partial^{\mu}D_{\mu}{}^{ac}\right)c^{c}\\ &\quad +K_{A}{}^{\mu a}\epsilon D_{\mu}{}^{ac}c^{c}+igK_{\psi\ i}\epsilon c^{a}t^{a}{}_{ij}\psi_{j}-igK_{\bar{\psi}\ i}\epsilon c^{a}t^{a}{}_{ij}\bar{\psi}_{j}-\frac{1}{2}gK_{c}{}^{a}\epsilon f^{abc}c^{b}c^{c}+K_{\bar{c}}{}^{a}\epsilon B^{a}\end{aligned}$$

where $K_A{}^{\mu a}$, $K_{\psi\ i}$, $K_{\bar{\psi}\ i}$, $K_c{}^a$ and $K_{\bar{c}}{}^a$ are the corresponding arbitrary "sources" to the variations. Note that $K_{\psi\ i}$, $K_{\bar{\psi}\ i}$, $K_c{}^a$ and $K_{\bar{c}}{}^a$ are Grassmann-valued sources. Together with the usual sources, J, the Lagrangian becomes:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}{}^{a}F^{\mu\nu a}+\bar{\psi}\left(i\not{D}-m\right)\psi-\frac{\xi}{2}B^{a}B^{a}+B^{a}\partial^{\mu}A_{\mu}{}^{a}+\bar{c}^{a}\left(-\partial^{\mu}D_{\mu}{}^{ac}\right)c^{c}\\ &\quad +K_{A}{}^{\mu a}\epsilon D_{\mu}{}^{ac}c^{c}+igK_{\psi\ i}\epsilon c^{a}t^{a}{}_{ij}\psi_{j}-igK_{\bar{\psi}\ i}\epsilon c^{a}t^{a}{}_{ij}\bar{\psi}_{j}-\frac{1}{2}gK_{c}{}^{a}\epsilon f^{abc}c^{b}c^{c}+K_{\bar{c}}{}^{a}\epsilon B^{a}\\ &\quad +J_{A}{}^{\mu a}A_{\mu}{}^{a}+J_{\psi\ i}\psi_{i}+J_{\bar{\psi}\ i}\bar{\psi}_{i}+J_{c}{}^{a}c^{a}+J_{\bar{c}}{}^{a}\bar{c}^{a}+J_{B}{}^{a}B^{a}\end{aligned}$$

where $J_{\psi\ i}$, $J_{\bar{\psi}\ i}$, $J_c{}^a$ and $J_{\bar{c}}{}^a$ are Grassmann-valued sources.

0.4.3 Claim

$\left\langle 0\,\middle|\,T\left\{e^{iS}\left(i\int d^4x\left[J_A{}^{\mu a}\epsilon D_\mu{}^{ac}c^c+igJ_{\psi\ i}\epsilon c^a t^a{}_{ij}\psi_j-igJ_{\bar{\psi}\ i}\epsilon c^a t^a{}_{ij}\bar{\psi}_j-\frac{1}{2}gJ_c{}^a\epsilon f^{abc}c^b c^c+J_{\bar{c}}{}^a\epsilon B^a\right]\right)\right\}\middle|\,0\right\rangle=0.$

Proof

$$T\left(e^{iS}\right)\ \mapsto\ T\left\{e^{iS}\left(S+i\int d^4x\left[J_A{}^{\mu a}\epsilon D_\mu{}^{ac}c^c+igJ_{\psi\ i}\epsilon c^a t^a{}_{ij}\psi_j-igJ_{\bar{\psi}\ i}\epsilon c^a t^a{}_{ij}\bar{\psi}_j-\frac{1}{2}gJ_c{}^a\epsilon f^{abc}c^b c^c+J_{\bar{c}}{}^a\epsilon B^a\right]\right)\right\}$$

Now use the fact that $\langle 0\,|\,\mathcal{O}\,|\,0\rangle=0$ where $\mathcal{O}$ is a BRST variation.