MAT520 – FA – FINAL – SAMPLE SOLUTIONS

DEC 16 – DEC 21 2023

$\boxed{Q1}$ On $\mathcal{H} := L^2([0,1] \to \mathbb{C})$, define K via

$$(K\psi)(x) := \int_{y=x}^{1} \int_{z=0}^{y} \psi(z)\, dz \qquad (x \in [0,1],\ \psi \in \mathcal{H})$$

<u>Claim</u>: $K \in \mathcal{B}(\mathcal{H} \to \mathcal{H})$

<u>Proof</u>: Define $V : \mathcal{H} \to \mathcal{H}$ via

$$(V\psi)(x) := \int_{y=0}^{x} \psi(y)\, dy \qquad (x \in [0,1],\ \psi \in \mathcal{H})$$

See HW8 Q7 (3)

<u>Claim</u>: $V \in \mathcal{B}(\mathcal{H})$

<u>Proof</u>:
$$\|V\psi\|_{L^2}^2 \equiv \int_{x=0}^{1} \left| \int_{y=0}^{x} \psi(y)\, dy \right|^2 dx$$
$$\leq \int_{x=0}^{1} \left(\int_{y=0}^{x} |\psi(y)|\, dy \right)^2 dx$$
$$\leq \int_{x=0}^{1} \left(\int_{y=0}^{1} |\psi(y)|\, dy \right)^2 dx$$
$$= \left(\|\psi\|_{L^1} \right)^2.$$

But $\|\psi\|_{L^1} \equiv \int_0^1 |\psi| = \int_0^1 |\psi| \cdot 1 \overset{\text{C.S.}}{\leq} \|\psi\|_{L^2}$

$\Rightarrow \|V\|_{\mathcal{B}(L^2)} \leq 1$. Linearity is clear. ∎

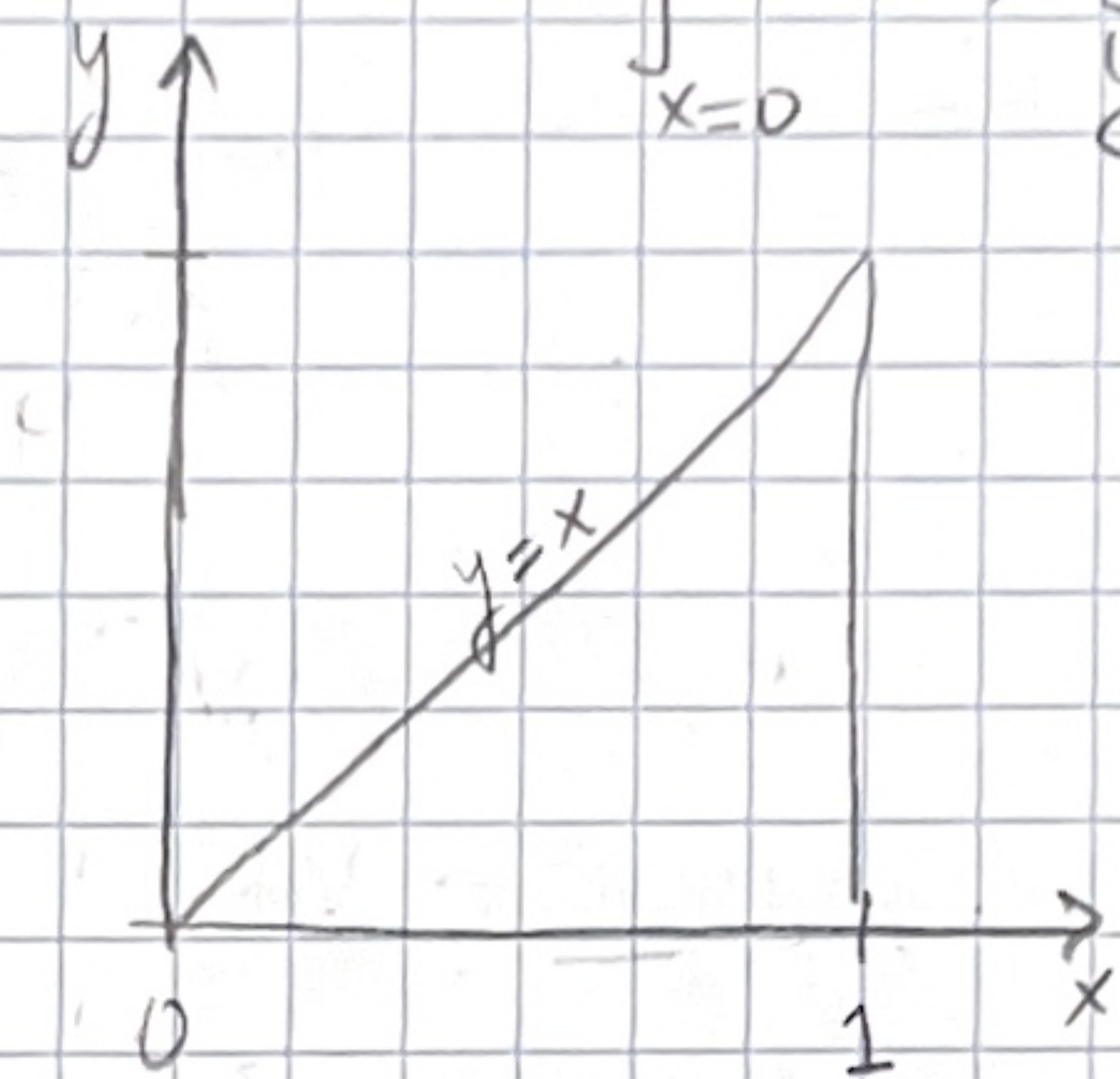
[2]

Claim: $(V^*\psi)(x) = \int_{y=x}^{1} \psi(y)\, dy \qquad (x\in[0,1],\ \psi\in\mathcal{H})$

Proof: $\langle V^*\psi, \varphi\rangle \equiv \langle \psi, V\varphi\rangle$

$$= \int_{x=0}^{1} \overline{\psi(x)}\, (V\varphi)(x)\, dx$$

$$= \int_{x=0}^{1} \overline{\psi(x)} \int_{y=0}^{x} \varphi(y)\, dy\, dx$$



Fubini

$$\overset{\text{Fubini}}{=} \int_{(x,y)\in\Delta} \overline{\psi(x)}\,\varphi(y)\, dx\, dy$$

$$= \int_{y=0}^{1} dy \int_{x=y}^{1} dx\, \overline{\psi(x)}\,\varphi(y)$$

$$\equiv \int_{y=0}^{1} dy\, \overline{(V^*\psi)(y)}\,\varphi(y).$$

■

Automatically due to $\|A^*\| = \|A\|$ we get $\|V^*\| \leq 1$.

Claim: $K = |V|^2$

Proof: $(V^* V \psi)(x) \equiv \int_{y=x}^{1} (V\psi)(y)\, dy$

$$= \int_{y=x}^{1} \left(\int_{z=0}^{y} \psi(z)\, dz \right) dy.$$

$\Rightarrow K$ is clearly linear and bdd.

(a) Since $K = |V|^2$, it is clearly self-adjoint.

(b) Claim: K is cpt.

Proof 1: We note that the integral kernel of

V is $V(x,y) = \chi_{[0,x]}(y)$

V^* is $V^*(x,y) = \chi_{[x,1]}(y)$

Hence, e.g.

$$\|K\|_1 \equiv \|V^*V\|_1$$

$$\leq \underbrace{\|V^*\|_{\mathcal{B}}}_{\leq 1} \underbrace{\|V\|_1}_{\leq \int_{x,y=0}^{1} V(x,y)\, dx dy}$$

$$\leq \int_{x,y=0}^{1} \chi_{[0,x]}(y)\, dx\, dy$$

$$= \frac{1}{2} < \infty.$$

Since K is trace-class, it is cpt.
(see e.g. Prop. 9.73 in lecture notes).

Alt. proof w/o trace: (Using Ascoli)

Let $\{\psi_n\}_n \subseteq B_M(0_{\mathcal{H}})$ be some bdd. seq. By Lemma 9.34 in LN, if we can show $\{V\psi_n\}_n \subseteq \mathcal{H}$ has a convergent subseq. then V would be cpt. But that would imply cpt.-ness of K as $K(\mathcal{H})$ is a two-sided ideal.

Claim: $\{V\psi_n\}_n$ are pointwise bdd.

Proof: $\left|\int_0^x \psi_n\right| \leq \int_0^1 |\psi_n| \leq \|\psi_n\|_{L^2} < M$

Claim: $\{V\psi_n\}_n$ is equicontinuous.

Proof: $|(V\psi_n)(x) - (V\psi_n)(y)| = \left|\int_0^x \psi_n - \int_0^y \psi_n\right|$

$$= \left|\int_x^y \psi_n\right| = |\langle \chi_{[x,y]}, \psi_n\rangle|$$

$$\leq \underbrace{\|\chi_{[x,y]}\|_{L^2}}_{\sqrt{|x-y|}} \underbrace{\|\psi_n\|_{L^2}}_{\leq M} \leq \sqrt{|x-y|}\, M$$

$\Rightarrow$ By Ascoli's thm. (Munkers Thm. 45.4) $\{V\psi_n\}_n$ has cpt. closure in $C([0,1] \to \mathbb{C})$ where the latter is taken with the uniform top.

Hence by the seq. char. of cpt. sets, $\{V\psi_n\}_n$ has a conv. subseq. ∎

Alt. proof.: (using fin. rank)

$$(V\psi)(x) \equiv \int_0^x \psi$$

$$\equiv \int_0^1 \chi_{[0,x]} \psi$$

$$= \langle \chi_{[0,x]}, \psi \rangle$$

$$= \int_0^1 \langle \chi_{[0,y]}, \psi \rangle \delta(x-y)\, dy$$

Riemann sum approx., δ-fⁿ approx.

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \frac{1}{N} \times \langle \chi_{[0,\frac{j}{N}]}, \psi \rangle \times N \chi_{[\frac{j}{N}, \frac{j+1}{N}]}(x)$$

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \langle \chi_{[0,\frac{j}{N}]}, \psi \rangle \chi_{[\frac{j}{N}, \frac{j+1}{N}]}(x)$$

But $\{\chi_{[0,j/N]}\}_{j=1}^{N}$ and $\{\chi_{[j/N, \frac{j+1}{N}]}\}_{j=1}^{N}$

are two sets in L^2, and the conv. is actually in op. norm:

$$V_N := \sum_{j=1}^{N} \chi_{[j/N, \frac{j+1}{N}]} \otimes \chi_{[0,j/N]}^{*}$$

$$\|(V-V_N)\psi\|_{L^2} \equiv \int_{x=0}^{1} |((V-V_N)\psi)(x)|^2\, dx$$

$$((V-V_N)\psi)(x) = \int_0^x \psi - \sum_{j=1}^{N} \langle \chi_{[0,j/N]}, \psi \rangle \chi_{[j/N, \frac{j+1}{N}]}(x)$$

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$$= \int_0^x \psi - \langle \chi_{[0, j_*/N]}, \psi \rangle$$

where $j_* = 1, \dots, N$ is the unique index so that $x \in [\frac{j_*}{N}, \frac{j_*+1}{N}]$.

$$= \int_{j_*/N}^{x} \psi$$

$$\Rightarrow |((V - V_N)\psi)(x)| = |\langle \chi_{[j_*/N, x]}, \psi \rangle|$$

$$\leq \sqrt{x - j_*/N}\, \|\psi\|_{L^2}$$

$$\Rightarrow \|(V - V_N)\psi\|_{L^2}^2 \leq \int_{x=0}^{1} \underbrace{(x - j_*/N)}_{\leq \frac{1}{N}} \|\psi\|_{L^2}^2 \, dx$$

$$\leq \frac{\|\psi\|^2}{N}$$

$$\Rightarrow \boxed{\|V - V_N\| \leq \frac{1}{\sqrt{N}}}.$$

∎

(c) Want to determine $\sigma(K)$. Since it is cpt., we know by Riesz-Schauder that $\sigma_{ess}(K) = \{0\}$ (Thm. 9.60 in LN) so we only need to determine fin. deg. eigenvalues which are real and non-zero.

I.e. we want to find $\psi \in L^2 \setminus \{0\}$

s.t. $K\psi = \lambda\psi$ $\exists$ $\lambda \in \mathbb{R} \setminus \{0\}$.

Moreover, $K = |V|^2$, so $K \geq 0$ and hence

we may assume $\lambda > 0$.

Intuition: $V\psi \equiv \int_0^\cdot \psi$ is like

the inverse of the momentum

op. on $L^2([0,1])$, so by

spectral mapping we expect

K to have spec. which is

the inverse of Dirichlet Laplacian

on $[0,1]$.

$$\int_{y=x}^1 dy \int_{z=0}^y dz\, \psi(z) \overset{!}{=} \lambda\psi(x) \qquad x \in [0,1]$$

$$\Leftrightarrow \quad \psi(x) = \frac{1}{\lambda}\int_{y=x}^1 dy\, (V\psi)(y)$$

$\Rightarrow$ ψ is abs. conti.

Differentiate eigeneq-n to get

$$\psi'(x) = \frac{-1}{\lambda}(V\psi)(x)$$

This shows ψ' is diff. so diff. again we

get $\psi''(x) = -\frac{1}{\lambda}\psi(x)$

hence $\psi(x) = A\exp\left(i\frac{1}{\sqrt{\lambda}}x\right) + B\exp\left(-i\frac{1}{\sqrt{\lambda}}x\right)$

for some $A, B \in \mathbb{C}$.

B.C. are determined from the eq-n itself:

$$\psi'(x) = -\frac{1}{\lambda}\int_0^x \psi \overset{x=0}{\Longrightarrow} \boxed{\psi'(0) = 0}$$

$$\psi(x) = \frac{1}{\lambda}\int_{y=x}^1 dy\,(V\psi)(y) \overset{x=1}{\Longrightarrow} \boxed{\psi(1) = 0}$$

This yields:
$$\begin{cases} Ae^{i\frac{1}{\sqrt{\lambda}}} + Be^{-i\frac{1}{\sqrt{\lambda}}} \overset{!}{=} 0 \\ i\frac{1}{\sqrt{\lambda}}A - i\frac{1}{\sqrt{\lambda}}B \overset{!}{=} 0 \end{cases}$$

$$\Longrightarrow \boxed{A = B}$$

and $\cos\left(\frac{1}{\sqrt{\lambda}}\right) = 0$

$$\Leftrightarrow \frac{1}{\sqrt{\lambda}} = \left(n + \frac{1}{2}\right)\pi \qquad (n \in \mathbb{Z})$$

But $\lambda > 0$ so $n \in \mathbb{N}$.

We find

$$\boxed{\sigma(K) = \underbrace{\{0\}}_{\text{ess. spec.}} \cup \underbrace{\left\{ \frac{1}{(n+\frac{1}{2})^2\pi^2} \,\middle|\, n \in \mathbb{N}_{\geq 0} \right\}}_{\text{point spec.}}}.$$

Q2 This is Thm. 10.27 in LN.

Q3 See Thm. 8.15 & Lemma 9.5 in LN

Claim: Let $A \in \mathcal{B}(\mathcal{H})$ be given. Then TFAE:

i. $\langle \psi, A\psi \rangle \geq 0 \quad (\psi \in \mathcal{H})$

ii. $A = A^* \wedge \sigma(A) \subseteq [0, \infty)$

iii. $A = |B|^2 \quad \exists B \in \mathcal{B}(\mathcal{H})$.

Proof: $\boxed{i \Rightarrow ii}$

Write $A_R = \mathrm{Re}\{A\} \equiv \frac{1}{2}(A + A^*)$

$A_I := \mathrm{Im}\{A\} \equiv \frac{1}{2i}(A - A^*)$.

Then $A = A_R + iA_I$, and

$\langle \psi, A\psi \rangle = \langle \psi, A_R\psi \rangle + i\langle \psi, A_I\psi \rangle$

For any $B = B^*$,

$\langle \psi, B\psi \rangle \equiv \langle B\psi, \psi \rangle = \langle \psi, B^*\psi \rangle = \langle \psi, B\psi \rangle$

$\Rightarrow \langle \psi, A_R\psi \rangle, \langle \psi, A_I\psi \rangle \in \mathbb{R}$.

But $\langle \psi, A\psi \rangle \geq 0$ for any ψ.

$\Rightarrow A_I = 0 \Leftrightarrow A = A^*$. ✓

By Weyl's criterion (Thm. 9.22), iff

$\lambda \in \sigma(A) \; \exists \{\varphi_n\}_n \subseteq \partial B_1(0_{\mathcal{H}})$:

$\lim_n \|(A - \lambda \mathbb{1})\varphi_n\| = 0$.

$\Leftrightarrow \forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N}$: if $n \geq N_\varepsilon$,

$\|(A - \lambda \mathbb{1})\varphi_n\| < \varepsilon$.

Thus
$$\begin{aligned}\lambda &= \lambda \|\varphi_n\|^2 \\ &= \langle \varphi_n, \lambda \varphi_n \rangle \\ &= -\langle \varphi_n, (A-\lambda\mathbb{1})\varphi_n \rangle + \langle \varphi_n, A\varphi_n \rangle \\ &\geq \langle \varphi_n, A\varphi_n \rangle - \|(A-\lambda\mathbb{1})\varphi_n\| \\ &\geq \underbrace{\langle \varphi_n, A\varphi_n \rangle}_{\geq 0 \text{ by hypothesis}} - \varepsilon \\ &\geq -\varepsilon,\end{aligned}$$

Since ε was arbitrary, we conclude $\lambda \geq 0$.

$\Rightarrow \quad \sigma(A) \subseteq [0, \infty)$.

$\boxed{ii \Rightarrow iii}$

Using the spectral thm.,
$$A = \int_{\lambda \in \mathbb{R}} \lambda \, \underbrace{dP(\lambda)}_{\text{proj.-valued spec. msr. of } A}$$

Since $\sigma(A) \subseteq [0, \infty)$, P is supported only on $[0, \infty)$, so we may write
$$\sqrt{A} = \int_{\lambda=0}^{\infty} \sqrt{\lambda} \, dP(\lambda)$$

Let U be any unitary and define $B := U\sqrt{A}$. Then

$$|B|^2 = \sqrt{A}\,U^*U\sqrt{A}$$
$$= (\sqrt{A})^2 = A.$$

$\boxed{iii \Rightarrow i}$

Let $\psi \in \mathcal{H}$. Then

$$\langle \psi, A\psi \rangle = \langle \psi, |B|^2 \psi \rangle$$
$$= \langle B\psi, B\psi \rangle$$
$$= \|B\psi\|^2 \geq 0.$$

$\blacksquare$

(b) This is Thm. 10.20 in LN.

Note: See the proof of the Krammers-Kronig relation in my MAT330 LN.

(c) Claim: Let $A \in \mathcal{B}(\mathcal{H})$ be normal and $\psi \in \mathcal{H}$ be cyclic for A:

$$\overline{\{A^n \psi \mid n \in \mathbb{N}_{\geq 0}\}} = \mathcal{H}.$$

Then ψ is cyclic for A^*.

Proof: Let $\varphi \in \mathcal{H}$ and $\varepsilon > 0$. Then

$$\|\varphi - \sum_{n=0}^{N} a_n A^n \psi\| < \varepsilon/2$$

$\exists\, N \in \mathbb{N}, \{a_n\}_{n=0}^{N} \subseteq \mathbb{C}$, since ψ is cyclic for A

Applying cyclicity of ψ again on the vector $\left(\sum_{n=0}^{N} a_n A^n\right)^* \psi$, we find

$$\left\| \left(\sum_{n=0}^{N} a_n A^n\right)^* \psi - \sum_{n=0}^{M} b_n A^n \psi \right\| < \frac{\varepsilon}{2}$$

for some $M \in \mathbb{N}$, $\{b_n\}_{n=0}^{M} \subseteq \mathbb{C}$.

Note $\|B\psi\| = \|B^*\psi\|$ if $|B|^2 = |B^*|^2$:

$$\|B\psi\|^2 = \langle \psi, |B|^2 \psi \rangle$$
$$= \langle \psi, |B^*|^2 \psi \rangle$$
$$= \|B^*\psi\|^2 \quad \checkmark$$

<u>Claim</u>: $B := \left(\sum_{n=0}^{N} a_n A^n\right)^* - \sum_{n=0}^{M} b_n A^n$ is normal.

<u>Proof</u>: $B = \sum_n a_n (A^*)^n - b_n A^n$

$$B^* B = \sum_{n,m} \underbrace{\left(a_n (A^*)^n - b_n A^n\right)^*}_{\overline{a_n} A^n - \overline{b_n} (A^*)^n} \left(a_m (A^*)^m - b_m A^m\right)$$

$$= \sum_{n,m} \overline{a_n} a_m A^n (A^*)^m - \dots$$

$$= \dots = BB^* \quad \text{as } [A, A^*] = 0.$$

$$\Rightarrow \left\| \sum_{n=0}^{N} a_n A^n \psi - \sum_{n=0}^{M} \overline{b_n} (A^*)^n \psi \right\| < \frac{\varepsilon}{2}.$$

$$\Rightarrow \left\| \psi - \sum_{n=0}^{M} \overline{b_n} (A^*)^n \psi \right\| < \varepsilon.$$

Since ψ may be approx. by an arbitrary poly. of A^*, ψ is a cyclic vector of A^*. ∎

Q4 (a) Let $A = A^* \in \mathcal{B}(\mathcal{H})$, $z \in \mathbb{C} : \mathrm{Im}\{z\} \neq 0$.

Claim: $U := (A + \bar{z}\mathbb{1})(A + z\mathbb{1})^{-1}$ is unitary.

Proof: Since $\mathrm{Im}\{z\} \neq 0$ and $A = A^*$, $(A + z\mathbb{1})^{-1}$ makes sense.

Let $f : \mathbb{R} \to \mathbb{C}$

$\lambda \mapsto \frac{\lambda + \bar{z}}{\lambda + z}$ holomorphic in a strip.

Then $U = f(A)$ via the holo. f^n-al calc.

$U^* = \bar{f}(A)$ by the $*$-morphism prop.

But $\bar{f} \cdot f = 1$ $*$.

$\Rightarrow$ By the morphism prop., U is unitary.

$*$ $\bar{f}(\lambda) f(\lambda) = \frac{\cancel{\lambda + z}}{\cancel{\lambda + \bar{z}}} \cdot \frac{\cancel{\lambda + \bar{z}}}{\cancel{\lambda + z}} = 1$. ∎

(b) This is Lemma 7.18 in the LN.

(c) Let $A \in \mathcal{B}(\mathcal{H})$ w/ $\dim \operatorname{im} A = 1$.

Claim: $A = \psi \otimes \varphi^*$ $\exists\, \psi, \varphi \in \mathcal{H} \setminus \{0\}$.

Proof: Let $\operatorname{im} A = \mathbb{C}\psi$ $\exists\, \psi \in \mathcal{H} \setminus \{0\}$.

Then $A\xi = \alpha_\xi \psi$ $\exists\, \alpha_\xi \in \mathbb{C}$ $(\xi \in \mathcal{H})$.

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$$\|A\xi\| = |\alpha_\xi|\,\|\psi\| \leq \|A\|$$

$$\uparrow$$

$$A \in \mathcal{B}(\mathcal{H})$$

$\Rightarrow$ The linear functional

$$\mathcal{H} \ni \xi \mapsto \alpha_\xi \in \mathbb{C}$$

is bounded, and hence by Riesz (Thm. 7.10) $\exists!\ \varphi \in \mathcal{H}\setminus\{0\}$ s.t.

$$\alpha_\xi = \langle \varphi, \xi \rangle \qquad (\xi \in \mathcal{H}).$$

I.e., $\boxed{A = \psi \otimes \varphi^*}$. ■

$$\|A\xi\|^2 = \langle \xi, |A|^2 \xi \rangle$$

$$|A|^2 \equiv A^* A = \underbrace{(\psi \otimes \varphi^*)^*}_{\varphi \otimes \psi^* \ \boxed{\square}} \psi \otimes \varphi^*$$

$$= \|\psi\|^2 \varphi \otimes \varphi^*$$

$$= \langle \xi, \|\psi\|^2 \varphi \otimes \varphi^* \xi \rangle$$

$$= \|\psi\|^2 |\langle \varphi, \xi \rangle|^2 \leq \|\psi\|^2 \|\varphi\|^2 \|\xi\|^2$$

and it may be saturated by $\xi = \varphi$.

$\Rightarrow \boxed{\|A\| = \|\psi\|\,\|\varphi\|}$.

$\boxed{\square}$ $\langle a, A^* b \rangle \equiv \langle Aa, b \rangle = \langle \psi \otimes \varphi^* a, b \rangle =$

$$= \overline{\langle \varphi, a \rangle} \langle \psi, b \rangle = \langle a, \varphi \rangle \langle \psi, b \rangle$$

$$= \langle a, \varphi \otimes \psi^* b \rangle \Rightarrow \boxed{A^* = \varphi \otimes \psi^*}.$$

Since A is rank-1, it is cpt. So $\sigma_{ess}(A) = \{0\}$ and we only need to determine the point spectrum, i.e., solve the eigenequation, for $\lambda \neq 0$,

$$A\xi = \lambda\xi \iff \psi\otimes\varphi^* \xi = \lambda \xi$$

$$\iff \langle \varphi, \xi\rangle \psi = \lambda \xi$$

So $\xi = \frac{1}{\lambda}\langle\varphi,\xi\rangle\psi$, i.e., $\xi \in \mathbb{C}\psi$.

Let Then $\xi = \alpha\psi$. Then

$$\alpha\langle\varphi,\psi\rangle\psi = \alpha\lambda\psi$$

$$\Rightarrow \boxed{\lambda = \langle\varphi,\psi\rangle}$$

$$\boxed{\sigma(\psi\otimes\varphi^*) = \{0, \langle\varphi,\psi\rangle\}}$$

(d) Let $A = A^* = A^{-1} \in \mathcal{B}(\mathcal{H})$.

Claim: $\exists$ two ortho. proj. $P, Q \in \mathcal{B}(\mathcal{H})$;

$$A = P - Q$$

Proof: Since $A = A^*$, $\sigma(A) \subseteq \mathbb{R}$.

Since $A^* = A^{-1}$, $\sigma(A) \subseteq \mathbb{S}^1$

$\Rightarrow \sigma(A) \subseteq \{\pm 1\}$.

So let

$$P := \chi_{\{+1\}}(A)$$

$$Q := \chi_{\{-1\}}(A)$$

via the msrbl. f^{n}-al calc. (or just via the Riesz proj. formula of the holo. f^{n}-al calc.).

Note these operators may happen to be the zero proj., e.g., if $A = \pm \mathbb{1}$. Anyway, then

$$A = \int_{\lambda\in\mathbb{R}} \lambda \, d\chi_{(-\infty,\lambda]}(A)$$

$$= P - Q \quad .$$

$\blacksquare$

Since we also have

$$\mathbb{1} = \int_{\lambda\in\mathbb{R}} d\chi_{(-\infty,\lambda)}(A)$$

$$= P + Q$$

then we must have $\mathcal{H} = \underbrace{(P\mathcal{H})}_{=:\mathcal{H}_+} \oplus \underbrace{(Q\mathcal{H})}_{=:\mathcal{H}_-}$.

(e) $R :=$ ~~fini.~~ right shift on $\ell^2(\mathbb{N})$.

(i) $|R|^2 = R^* R = \mathbb{1}$

$|R^*|^2 = \mathbb{1} - \delta_1 \otimes \delta_1^* \equiv (\delta_1 \otimes \delta_1^*)^\perp$

both via direct computation.

(ii) $\sum_{n=1}^{\infty} \langle \delta_n, R \delta_n \rangle = \sum_{n=1}^{\infty} \underbrace{\langle \delta_n, \delta_{n+1} \rangle}_{=0} = 0.$

$\sum_{n=1}^{\infty} \langle \delta_n, R^* \delta_n \rangle = \sum_{n=2}^{\infty} \underbrace{\langle \delta_n, \delta_{n-1} \rangle}_{=0} = 0.$

$\sum_{n=1}^{\infty} \langle \delta_n, |R|^2 \delta_n \rangle = \sum_{n=1}^{\infty} 1 = \infty.$

$\sum_{n=1}^{\infty} \langle \delta_n, |R^*|^2 \delta_n \rangle = \sum_{n=2}^{\infty} 1 = \infty.$

$\sum_{n=1}^{\infty} \langle \delta_n, \underbrace{(|R|^2 - |R^*|^2)}_{\mathbb{1} - (\mathbb{1} - \delta_1 \otimes \delta_1^*) = \delta_1 \otimes \delta_1^*} \delta_n \rangle = 1.$

If we interpret

$\sum_{n=1}^{\infty} \langle \delta_n, A \delta_n \rangle \equiv \operatorname{tr}(A)$

(since $\{\delta_n\}_n$ is an ONB)

we find: $\operatorname{tr}(R) = \operatorname{tr}(R^*) = 0$

$\operatorname{tr}(|R|^2) = \operatorname{tr}(|R^*|^2) = \infty$

$\operatorname{tr}(|R|^2 - |R^*|^2) = 1$

This shows us that it is NOT appropriate to use cyclicity of the trace naively, since then we'd have:

$$\begin{aligned} 1 &= tr(|R|^2 - |R^*|^2) \\ &= tr(|R|^2) - tr(|R^*|^2) \quad \text{linearity} \\ &= tr(R^*R) - tr(RR^*) \\ &= tr(R^*R) - tr(R^*R) \quad \text{cyclicity} \\ &= 0. \end{aligned}$$

But this is NOT allowed since $|R|^2, |R^*|^2$ are not separately "trace class" (although formally $tr(R) = tr(R^*) = 0$, but that is NOT how "trace class" is defined.) See Section 9.7 in the L.N.)

$\boxed{Q5}$ (a) Let X, Y be two normed spaces, $A : X \to Y$ linear.

Suppose whenever $\{\psi_n\}_n \subseteq X$ converges weakly to zero, $\{A\psi_n\}_n \subseteq Y$ converges weakly to zero.

Claim: A is bounded.

Proof: | Claim: X^*, Y^* are Banach spaces.

<u>Proof</u>: $X^* \equiv \{\lambda : X \to \mathbb{C} \text{ linear \& bdd}\}$

w/ norm

$\|\lambda\| \equiv \sup(\{|\lambda \psi| \mid \|\psi\| = 1\})$

W.T.S. X^* is complete.

Let $\{\lambda_n\}_n \subseteq X^*$ be Cauchy.

Then $\forall x \in X$,

$$|\lambda_n(x) - \lambda_m(x)| \leq \underbrace{\|\lambda_n - \lambda_m\|}_{\text{small}} \|x\|$$

$\Rightarrow \forall x \in X$, $\{\lambda_n(x)\}_n \subseteq \mathbb{C}$ is Cauchy, so by completeness of $\mathbb{C}$ converges to some $\lambda(x) \in \mathbb{C}$.

<u>Claim</u>: $\lambda \in X^*$

<u>Proof</u>: By linearity of limit,

$\lambda : X \to \mathbb{C}$ is linear.

It is bounded too,

$|\lambda(x)| = \lim_n |\lambda_n(x)|$

But $\{\lambda_n\}_n$ is Cauchy, so it is bdd

$\Rightarrow |\lambda(x)| \leq C$

$\Rightarrow \|\lambda\| \leq C$. ∎

Clearly $\lambda_n \to \lambda$ in op. norm ∎

<u>Claim</u>: If $\{\psi_n\}_n \to \psi$ weakly in a normed space X then $\{\psi_n\}_n$ is norm bounded.

<u>Proof</u>: By Lemma 5.11,

$\lambda(\psi_n) \to \lambda(\psi) \quad \forall \lambda \in X^*$.

The injection $J : X \to X^{**}$ yields a seq. $\{J(\psi_n)\}_n$ within $\mathcal{B}(X^* \to \mathbb{C})$. Since X^* is complete, we have by Thm 3.28 (Banach-Steinhaus) we have that

$$\sup_n \|J(\psi_n)\| < \infty.$$

But J is an isometry. ∎

Now assume A is unbounded. Then $\exists \{\psi_n\}_n \subseteq X : \|\psi_n\| \leq 1$ and $\|A\psi_n\| \to \infty$. So $\{\frac{1}{n}\psi_n\}_n$ converges in norm and hence weakly to zero while $\|A\frac{1}{n}\psi_n\| = \frac{1}{n}\|A\psi_n\|$

could be made to converge to ∞. E.g. if we pick

$$\|A\psi_n\| \geq e^n$$

then $\|A\tfrac{1}{n}\psi_n\| \geq \tfrac{1}{n}e^n \to \infty$.

But by hypothesis $\{A\tfrac{1}{n}\psi_n\}_n \to 0$ weakly and by the claim above $\{A\tfrac{1}{n}\psi_n\}_n$ is supposed to be norm bdd. $\Rightarrow \boxed{\bot}$. ∎

(b) Let X, Y, Z be Banach sp.

$$\left.\begin{array}{l} A : X \to Y \\ J : Y \to Z \end{array}\right\} \text{linear}$$

Assume J is bdd. and injective.

Assume JA is bdd.

Claim: A is bdd.

Proof: By the closed graph thm. (Thm. 3.37), suffice to show

$$\Gamma(A) \in \text{Closed}(X \times Y).$$

Let $\{\psi_n\}_n \subseteq X$: $\psi_n \to \psi \; \exists \psi \in X$

$$A\psi_n \to \varphi \; \exists \varphi \in Y.$$

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W.T.S. $\varphi = A\psi$.

$$J A\psi = JA \lim_n \psi_n$$

$$= \lim_n JA\psi_n.$$

But J is injective $\Leftrightarrow$ it has a left inverse J_L^{-1} s.t. $J_L^{-1} J = \mathbb{1}_{Y_1}$. Applying J_L^{-1} on both sides of the above eq-n we get:

$$A\psi = J_L^{-1} \lim_n JA\psi_n.$$

Now, since J is bounded, we have

$$\lim_n J A\psi_n = J \lim_n A\psi_n = J\varphi$$

and so

$$A\psi = J_L^{-1} J \varphi = \varphi. \quad \blacksquare$$