Functional Analysis
Princeton University MAT520
December 2023 Take-home Final Exam

December 15, 2023

Instructions: This is a take-home exam, meaning you are expected to do it "at home" by yourself in a quiet environment with no other person aiding you (this is not a group assignment). You may very well use any source material which is "not interactive" such as books / notes / lecture notes / etc. But you may *not* communicate with other people, nor post any questions online. Pretend you were in a classroom and could bring with you whichever *non-electronic offline* reference material you needed. Obviously I have no way of policing that, so we shall rely on Princeton's honor code.

- The exam has five questions, each worth 25 points, to yield a maximum of 125 points (which will then be truncated to a maximal 100).
- You are "expected" to complete the exam in four hours, starting from whenever you open it on Gradescope. You may take breaks as needed but not communicate with others about the exam during that time. Be that as it may you will have a window of 48 hours to submit it from the time it is revealed, just to avoid making it time-stressful. Hence take the four hour window with a grain of salt: my idea is simply that you plan that it should take you four hours, but please don't put a timer.
- If you somehow can't submit the assignment through Gradescope just send it to my email (shapiro@math.princeton.edu).
- During your exam it is possible to email us (shapiro@math.princeton.edu or juihui@princeton.edu) if there is any doubt about the phrasing of the questions (but don't expect to get help or hints, in the interest of fairness).

1. Define the operator K on $L^2([0,1])$ via

$$(K\psi)(x) \quad := \quad \int_{y=x}^{1}\left(\int_0^y \psi(z)\,\mathrm{d}z\right)\mathrm{d}y \qquad \left(x\in[0,1]\,,\psi\in L^2([0,1])\right)\,.$$

   Show that:

   (a) [10 pts] K is self-adjoint.
   (b) [10 pts] K is compact.
   (c) [5 pts] Find the spectrum of K.

2. [25 pts] Prove Kuiper's theorem: Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and let $A\in\mathcal{B}(\mathcal{H})$ be invertible. Show that there is an operator-norm-continuous map $\gamma:[0,1]\to\mathcal{B}(\mathcal{H})$ such that: (1) $\gamma(0)=A$, (2) $\gamma(1)=\mathbb{1}$ and (3) $\gamma(t)$ is invertible for all $t\in[0,1]$.

3. This question is divided into three independent parts.

   (a) [12 pts] Prove (with the spectral theorem) that if $A\in\mathcal{B}(\mathcal{H})$ then the following are equivalent:
      i. For any $\psi\in\mathcal{H}$, $\langle\psi,A\psi\rangle\geq 0$.
      ii. $A=A^*$ and $\sigma(A)\subseteq[0,\infty)$.
      iii. There exists some $B\in\mathcal{B}(\mathcal{H})$ such that $A=|B|^2$.

   (b) [12 pts] Prove Stone's formula: if $A\in\mathcal{B}(\mathcal{H})$ is self-adjoint and

$$\tilde{\chi}_{[a,b]}(\lambda) := \begin{cases} 1 & \lambda\in(a,b) \\ 0 & \lambda\notin[a,b] \\ \frac{1}{2} & \lambda\in\{\,a,b\,\} \end{cases} \qquad (\lambda\in\mathbb{R})$$

then

$$\tilde{\chi}_{[a,b]}(A) = \operatorname*{s-lim}_{\varepsilon\to 0^+} \frac{1}{2\pi\mathrm{i}} \int_{\lambda=a}^{b} \left[(A-(\lambda+\mathrm{i}\varepsilon)\mathbb{1})^{-1} - (A-(\lambda-\mathrm{i}\varepsilon)\mathbb{1})^{-1} \right] \mathrm{d}\lambda\,.$$

(c) [1 pts] Show that if $A \in \mathcal{B}(\mathcal{H})$ is normal and $\psi \in \mathcal{H}$ is a cyclic vector for A then it is also cyclic for A^*.

4. This problem is divided into multiple parts which are independent of each other.

(a) [5 pts] Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $z \in \mathbb{C}$ with $\mathbb{Im}\{z\} \neq 0$. Show that

$$U := (A+\overline{z}\mathbb{1})(A+z\mathbb{1})^{-1}$$

is unitary.

(b) [5 pts] Show that for any $A \in \mathcal{B}(\mathcal{H})$, $\ker\left(|A|^2\right) = \ker(A)$.

(c) [5 pts] Show that for any $A \in \mathcal{B}(\mathcal{H})$, if $\dim \operatorname{im}(A) = 1$ then there are $\varphi, \psi \in \mathcal{H} \setminus \{0\}$ such that

$$A\xi = \langle \varphi, \xi \rangle \psi \qquad (\xi \in \mathcal{H})\,.$$

Proceed to calculate: (i) $\|A\|$, (ii) A^*, (iii) $\sigma(A)$.

(d) [8 pts] Show that if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and unitary, then there are two orthogonal projections P, Q such that

$$A = P - Q$$

and that this yields a $\mathbb{Z}_2$ grading of the Hilbert space as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

(e) [2 pts] Let $R \in \mathcal{B}\left(\ell^2(\mathbb{N})\right)$ be the unilateral right shift operator

$$(R\psi)(n) \equiv \begin{cases} \psi(n-1) & n \geq 2 \\ 0 & n = 1 \end{cases} \qquad \left(\psi \in \ell^2(\mathbb{N})\right)\,.$$

Then: (i) Calculate $|R|^2$ and $|R^*|^2$, (ii) if $\{\delta_n\}_{n\in\mathbb{N}}$ is the standard basis (position basis) of $\ell^2(\mathbb{N})$, calculate the following expressions:

$$\sum_{n=1}^{\infty} \langle \delta_n, R\delta_n \rangle\,, \sum_{n=1}^{\infty} \langle \delta_n, R^*\delta_n \rangle\,, \sum_{n=1}^{\infty} \left\langle \delta_n, |R|^2 \delta_n \right\rangle\,, \sum_{n=1}^{\infty} \left\langle \delta_n, |R^*|^2 \delta_n \right\rangle\,, \sum_{n=1}^{\infty} \left\langle \delta_n, \left(|R|^2 - |R^*|^2\right) \delta_n \right\rangle\,.$$

Interpreting these expressions naively as traces, what can you conclude about cyclicity in this infinite setting?

5. The following question has two independent parts:

(a) [15 pts] Let X, Y be normed spaces and $A : X \to Y$ linear. Suppose that whenever $\{\psi_n\}_n \subseteq X$ converges weakly to zero, $\{A\psi_n\} \subseteq Y$ converges weakly to zero. Show that A is bounded.

(b) [10 pts] Let X, Y, Z be Banach spaces. Let $A : X \to Y$ and $J : Y \to Z$ be linear. Suppose that J is bounded and injective and JA is bounded. Show that A is bounded.