SEP 23 2023

MAT520 — Func-al Analy. — HW1 Sol-ns

Q1

Claim: In $\mathbb{C}^n$, vector addition and scalar mul. are cont.

Proof: WTS $+ : (\mathbb{C}^n)^2 \to \mathbb{C}^n$ is cont.

$$(u,v) \mapsto u+v$$

Suffice to take some $B_r(z) \in \text{Open}(\mathbb{C}^n)$ and show $+^{-1}(B_r(z)) \in \text{Open}((\mathbb{C}^n)^2)$. Since open sets in the prod. top. are unions of products of open balls.

$$+^{-1}(B_r(z)) \equiv \{ (u,v) \in (\mathbb{C}^n)^2 \mid \|u+v-z\| < r \}$$

Let $(u,v) \in +^{-1}(B_r(z))$, i.e., $u+v \in B_r(z)$.

Since $B_r(z) \in \text{Open}(\mathbb{C}^n)$, $\exists\ \varepsilon > 0 : B_\varepsilon(u+v) \subseteq B_r(z)$.

Claim: $B_{\varepsilon/3}(u) \times B_{\varepsilon/3}(v) \subseteq +^{-1}(B_\varepsilon(u+v))$

Proof: If $(\tilde{u}, \tilde{v}) \in B_{\varepsilon/3}(u) \times B_{\varepsilon/3}(v)$,

$$\|\tilde{u} + \tilde{v} - u - v\| \leq 2\varepsilon/3 < \varepsilon$$

$\Longrightarrow \quad B_{\varepsilon/3}(u) \times B_{\varepsilon/3}(v) \in Nbhd(u+v)$

and $\quad B_{\varepsilon/3}(u) \times B_{\varepsilon/3}(v) \subseteq +^{-1}(B_\varepsilon(u+v))$

$\subseteq +^{-1}(B_r(z))$.

$\Longrightarrow \quad +^{-1}(B_r(z)) \in Open((\mathbb{C}^n)^2)$ and hence $+$ is cont. $\blacksquare$

Next, W.T.S. $\quad \cdot : \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is cont.

$(\alpha, u) \longmapsto \alpha u$

Again W.T.S. $\quad \cdot^{-1}(B_r(z)) \in Open(\mathbb{C} \times \mathbb{C}^n)$.

Let $\quad (\alpha, u) \in \cdot^{-1}(B_r(z)) \Leftrightarrow \|\alpha u - z\| < r$.

$\Leftrightarrow \quad \alpha u \in B_r(z)$

So $\exists \quad \varepsilon > 0 : \quad B_\varepsilon(\alpha u) \subseteq B_r(z)$.

Want $\quad \|\tilde{\alpha}\tilde{u} - \alpha u\| < \varepsilon$:

$$\|\tilde{\alpha}\tilde{u} - \alpha u\| = \|\tilde{\alpha}\tilde{u} - \tilde{\alpha}u + \tilde{\alpha}u - \alpha u\|$$

$$\leq |\tilde{\alpha}| \, \|\tilde{u} - u\| + |\tilde{\alpha} - \alpha| \, \|u\|$$

$$\leq (|\alpha - \tilde{\alpha}| + |\alpha|) \, \|\tilde{u} - u\| + |\tilde{\alpha} - \alpha| \, \|u\|$$

$$\leq (1 + |\alpha|) \, \|\tilde{u} - u\| + |\tilde{\alpha} - \alpha| \, \|u\| \overset{!}{<} \frac{2\varepsilon}{3} < \varepsilon.$$

So pick $\quad \mathcal{U} := B_{\frac{1}{\|u\|}\frac{\varepsilon}{3}}(\alpha) \times B_{\min\{1, \frac{1}{1+|\alpha|}\frac{\varepsilon}{3}\}}(u)$.

This guarantees $(a,u) \in \mathcal{U} \subseteq \cdot^{-1}(B_\varepsilon(\alpha u))$

and hence $\cdot^{-1}(B_r(z)) \in Open(\mathbb{C} \times \mathbb{C}^n)$. ✓ ∎

Q2

Claim: $\mathbb{C}$ w/ the french metro metric is NOT homeomorphic to $\mathbb{C}$ w/ Euclidean metric.

Pf.: Example 2 in lecture notes.

Since we know $\exists$ only one TVS (up to homeomorphisms) in dim $n < \infty$, the french metro metric's top. cannot make a TVS out of $\mathbb{C}$.

Q3

Claim: $\overline{A} + \overline{B} \subseteq \overline{A+B}$

Proof: Let $a \in \overline{A}$, $b \in \overline{B}$.

W.T.S. $a+b \in \overline{A+B}$.

Let $W \in Nbhd(a+b)$. So W.T.S. $W \cap (A+B) \neq \emptyset$.

Since addition is cont.,

$\exists (\mathcal{U}, \mathcal{V}) \in Nbhd(a) \times Nbhd(b) : \mathcal{U}+\mathcal{V} \subseteq W$.

Since $a \in \overline{A}$, $\exists$ $\tilde{a} \in A \cap \mathcal{U}$

$b \in \overline{B}$, $\tilde{b} \in B \cap \mathcal{V}$

and so $\tilde{a}+\tilde{b} \in A+B$ and

$\tilde{a}+\tilde{b} \in \mathcal{U}+V \subseteq \mathcal{W}$.

Q4 Claim: If $A \subseteq X$ is a v/s/sp. then so is $\bar{A}$.

Proof: By Rudin pp. 6.,

$$S \subseteq X \text{ is a v/s/sp} \Leftrightarrow \begin{cases} 0_X \in S \\ \alpha S + \beta S \subseteq S \\ \quad \forall\, \alpha, \beta \in \mathbb{C} \end{cases}$$

Clearly since $0_X \in A$ and $A \subseteq \bar{A}$, $0_X \in \bar{A}$.

WTS $\alpha \hat{A} + \beta \bar{A} \subseteq \bar{A}$ $\forall\, \alpha, \beta \in \mathbb{C}$.

Claim: $\alpha \bar{A} = \overline{\alpha A}$ $\forall$ $\alpha \in \mathbb{C}$

Proof: If $\alpha = 0$ true.

Else: Let $\bar{A} \equiv \bigcap_{\substack{F \in \mathrm{Closed}(X) \\ F \supseteq A}} F$

$f(u) := \frac{1}{\alpha} u$

$$\alpha \bar{A} \overset{\checkmark}{=} f^{-1}(\bar{A}) = f^{-1}\Big(\bigcap_{\substack{F \in \mathrm{Closed}(X) \\ F \supseteq A}} F \Big)$$

$$= \bigcap_{\substack{F \in \mathrm{Closed}(X) \\ F \supseteq A}} f^{-1}(F)$$

$G := f^{-1}(F)$

$$\overset{\circ}{=} \bigcap_{\substack{F \in \mathrm{Closed}(X) \\ F \supseteq f^{-1}(A)}} F$$

$$= \overline{\alpha A}.$$

Hence
$$\alpha \bar{A} + \beta \bar{A} = \overline{\alpha A} + \overline{\beta A}$$

v/s/sp.

$$\overset{\circ}{=} \bar{A} + \bar{A}$$
$$\overset{\circ}{\subseteq} \overline{A+A}$$
$$= \bar{A}. \quad \blacksquare$$

Q5 Claim: $2A \subseteq A+A$

Proof: Let $a \in A$. Then $2a = a + a \in A + A$. $\blacksquare$

Q6 Claim: Unions and intersections of balanced are balanced.

Proof: Let $\{B_\alpha\}_\alpha$ be balanced, i.e.,

$$zB_\alpha \subseteq B_\alpha \qquad \forall \alpha,\ |z|\leq 1.$$

Let now $z\in\mathbb{C} : |z|\leq 1$. W.T.S.

$$z\bigcup_\alpha B_\alpha \subseteq \bigcup_\alpha B_\alpha \quad \text{and}$$

$$z\bigcap_\alpha B_\alpha \subseteq \bigcap_\alpha B_\alpha.$$

If $z=0$, $zB_\alpha=\{0\}$ so $0_X\in B_\alpha\ \forall\alpha$

and $\nexists$ anything to prove.

Else, Let $v\in z\bigcap_\alpha B_\alpha$. Then

$\frac{1}{z}v\in B_\alpha\ \forall\ \alpha.$

Since B_α is balanced, $v\in B_\alpha\ \forall\ \alpha.$ ✓

Similarly for the union. ∎

Q7

<u>Claim</u>: If A,B are balanced, so is $A+B$.

<u>Proof</u>: Let $z\in\mathbb{C} : |z|\leq 1$ and $v\in z(A+B)$.
Then $v=z(a+b)$ $\exists$ $a\in A, b\in B$.
So $v=za+zb\in A+B$ as A,B are bal.
∎

Q8 **Claim:** If A, B are bdd. then $A+B$ is bdd.

Pf.: Let $N \in \mathrm{Nbhd}(0_X)$. WTS.

$$(A+B) \subseteq tN$$

for all $t > 0$ large enough.

By cont. of $+$ $\exists$ $M \in \mathrm{Nbhd}(0_X)$: $M+M \subseteq N$.

Then $A \subseteq tM$
$B \subseteq tM$

$$\Rightarrow A+B \subseteq tM + tM \subseteq t(M+M) \subseteq tN. \quad \blacksquare$$

Claim: If A, B are cpt. then $A+B$ is cpt.

Pf.: $+ : X^2 \to X$ is cont.

$A \times B \in \mathrm{Cpt}(X^2)$ by def. of prod. top.

$A+B = +(A \times B)$ and cont. image of cpt. is cpt. $\blacksquare$

Q9 **Claim:** $\exists$ $A, B \in \mathrm{Closed}(X)$: $A+B \notin \mathrm{Closed}(X)$.

Proof: Let $A \subseteq \mathbb{C}$ be given by

$$A := \mathbb{N} \in \text{Closed}(\mathbb{C})$$

$$B := \left\{ -n + \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\} \in \text{Closed}(\mathbb{C}).$$

$\frac{1}{n} \in A+B \quad \forall\, n \in \mathbb{N}$ and $0 \notin A+B$!

Q10 **Claim:** X, Y TVS w/ $\dim(Y) < \infty$.

$\Lambda : X \to Y$ lin. & surj.

Then (1) Λ is open and

(2) $\ker(\Lambda) \in \text{Closed}(X) \Rightarrow \Lambda$ is cont.

Proof: By Rudin Thm. 1.21 (a), $Y = \mathbb{C}^n$ WLOG.

Let $\{e_j\}_{j=1}^{n}$ be the std. basis.

Since Λ is surj., $\exists\, f_j \in X : \Lambda f_j = e_j$.

Define $\Gamma : \mathbb{C}^n \to X$ via

$$v \mapsto \sum_{j=1}^{n} v_j f_j$$

By def. Γ is lin.

By Rudin's Lemma 1.20, Γ is cont.

By L.N. Claim 3.31, suffice to show that if $N \in Nbhd(0_X)$ then ΛN contains some $M \in Nbhd(0_{\mathbb{C}^n})$.

Study $\Gamma^{-1}N$: Since $\Lambda \Gamma v = v \ \forall v \in \mathbb{C}^n$:

$$\Gamma^{-1}N = \Lambda \Gamma \Gamma^{-1} N \subseteq \Lambda N.$$

But Γ is cont., so $\Gamma^{-1}N \in Open(\mathbb{C}^n)$ and by linearity, $0_{\mathbb{C}^n} \in \Gamma^{-1}N$.

Hence Λ is indeed open. $\Rightarrow$ (1).

Next, assume $\ker \Lambda \in Closed(X)$ and WTS $\Lambda : X \to \mathbb{C}^n$ is cont.

Unfortunately, the easiest way to do this seems to involve quotient TVS, so no pts. will be deducted for mistakes here.

$\hat{\Lambda} : X/\ker(\Lambda) \xrightarrow{\sim} \mathbb{C}^n$ is a VS isomorphism and hence a TVS isomorphism

Note $X/\ker(\lambda)$ only makes sense if $\ker(\lambda)$ is closed, and VS $\Rightarrow$ TVS iso. bcs. of finite dimensions. ∎

$\boxed{Q11}$ $\mathcal{C} := \{ f : [0,1] \to \mathbb{C} \mid f \text{ is cont.} \}$

$$d(f,g) := \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} \, dx$$

Claim: d is a metric on $\mathcal{C}$.

Pf.: (1) If $d(f,g)=0$,

$$\int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} \, dx = 0$$

Since integrand ≥ 0 and cont., it must equal zero $\Rightarrow$ $f=g$. ✓

(2) d is symm. ✓

(3) Triangle ineq. follows from

The fact

$$\tilde{d}(a,b) := \frac{|a-b|}{1+|a-b|}$$

obeys $\Delta \neq$ on $\mathbb{C}^n$.

Note $[0,\infty) \ni \alpha \overset{r}{\longmapsto} \dfrac{\alpha}{1+\alpha}$ is

increasing, and we're trying to show

$$r(|a-b|) \leqslant r(|a-c|) + r(|c-b|)$$

Note $r(\alpha) + r(\beta) \geqslant r(\alpha+\beta)$.

Indeed, $\dfrac{\alpha}{1+\alpha} + \dfrac{\beta}{1+\beta} \geqslant \dfrac{\alpha+\beta}{1+\alpha+\beta}$

$$\frac{\alpha(1+\beta)+\beta(1+\alpha)}{(1+\alpha)(1+\beta)} = \frac{\alpha+\beta+2\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

$$\geqslant \frac{\alpha+\beta+\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

$$\geqslant \frac{\alpha+\beta}{1+\alpha+\beta}.$$

So this follows tis ordinary $\Delta \neq$ on $\mathbb{C}$.

Claim: C is a VS.

Pf.: Obvious.

Claim: (C, d) is a TVS.

Pf.: (1) All metric spaces are T1. ✓

(+) We note d is transl.-invar:

$$d(f,g) = d(f+h, g+h).$$

Hence

$$d(f+g, \tilde{f}+\tilde{g}) = d(f-\tilde{f}, \tilde{g}-g)$$
$$\leq d(f-\tilde{f}, 0) + d(0, \tilde{g}-g)$$

and the rest of of the proof follows as in $\boxed{Q1}$.

(·) We don't have homogeneity, but

$$d(\alpha f, \alpha g) \leq (1+|\alpha|)\, d(f,g).$$

Indeed,

$$\frac{|\alpha z|}{1+|\alpha z|} = |\alpha| \frac{|z|}{1+|\alpha||z|}$$

$$1+|\alpha||z| \geqslant 1+|z| \quad \text{if } |\alpha| \geqslant 1.$$

But if $|\alpha|<1$,

$$\frac{1+|\alpha||z|}{|\alpha|} \geqslant 1+|z|$$

So $$\frac{|\alpha z|}{1+|\alpha z|} \leqslant \max(\{1, |\alpha|\}) \frac{|z|}{1+|z|}$$

$$\leqslant (1+|\alpha|) \frac{|z|}{1+|z|} .$$

The rest of the proof follows similarly to $\boxed{Q1}$.

$\blacksquare$

Countable basis w/ $\{B_{1/n}(0)\}_{n\in\mathbb{N}}$.

$\boxed{Q12}$ Claim: Let $V \in \mathrm{Nbhd}(0_X)$. Then $\exists$ $f: X \to \mathbb{R}$ cont. w/ $\begin{cases} f(0)=0 \\ f(x)=1 \ \forall x \in V^c. \end{cases}$

Proof: Let $\{V_n\}_n$ be a seq. in $\mathrm{Nbhd}(0_X)$ which are all balanced and obey:

$$V_n + V_n \subseteq V_{n-1}$$

$$V_1 + V_1 \subseteq V$$

Define

$$D := \left\{ q \in \mathbb{Q} \;\middle|\; q = \sum_{n=1}^{\infty} \alpha_n 2^{-n} \text{ and } \alpha : \mathbb{N} \to \{0,1\} \text{ is s.t. } |\alpha^{-1}(\{1\})| < \infty \right\}.$$

$\forall\, q \in D$, let $\alpha(q)$ be the corresp. finite seq.

Then $q \geqslant 0$ and

$q \leqslant 1$.

Define

$$A : D \cup [1, \infty) \longrightarrow \mathcal{P}(X)$$

$$q \longmapsto \begin{cases} X & q \geq 1 \\ \sum_{j=1}^{\infty} \alpha_j(q) V_j & q \in D \end{cases}$$

$$f : X \to [0,1]$$

$$x \mapsto \inf(\{ r \in D \cup [1, \infty) \mid x \in A(r) \}).$$

Since $0_X \in V_n \;\forall\, n$, $0_X \in A(r) \;\forall\, r$.

$\Rightarrow f(0_X) = 0.$

If $x \in V^c$, want $\underset{\Uparrow}{f(x)=1}$.

$x \notin A(r)\ \forall\ r \in D$.

But if $x \in V^c$, x cannot lie in any V_n, and hence not in any of its sums.

Claim: f is cont.

Pf.: ① f is cont. @ 0_X:

$\forall\ \varepsilon > 0$, let $N: 2^{-N} < \varepsilon$.

Then $\sup|f(V_N)| \leq 2^{-N} < \varepsilon$. ✓

② $|f(x) - f(y)| \leq f(x-y)$

which follows as in the proof of Rudin 1.24.

∎

Q13 $X := \{ f: (0,1) \to \mathbb{C} \mid f \text{ cont.} \}$ VS.

$V(f,r) := \{ g \in X \mid |g(x) - f(x)| < r\ \forall x \in (0,1) \}$

Claim: $\{V(f,r)\}_{f\in X, r>0}$ is NOT a basis.

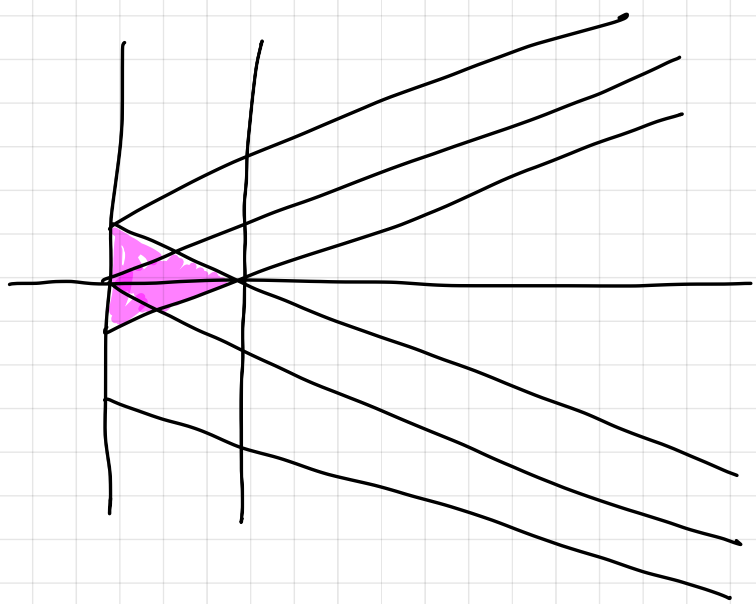
Pf.: Need $\forall$ $f, g \in X$, $r, s > 0$:

$$V(f,r) \cap V(g,s) \neq \emptyset,$$

some $V(h,t) \subseteq V(f,r) \cap V(g,s)$.

Take $V(x,1) \cap V(-x,1)$ which intersect at the zero fn.

But it is impossible to find $V(h,r)$ inside this as



area tends to zero.

So this is a sub-basis.

Claim: $+$ is cont.

Pf.: Define for $f, g \in X$: $f < g$:

$$R(f,g) := \{ h \in X \mid f < h < g \}.$$

Then $V(f,r) \equiv \{ g \in X \mid |g-f| < r \}$

$$= \{ g \in X \mid f-r < g < f+r \}$$

$$= R(f-r, f+r)$$

Actually $R(f,g)$ is open $\circledast$.

Then if $g+h \in V(f,r)$,

$$g+h \in \bigcap_{j=1}^{n} \underbrace{V(f_j, r_j)}_{R(f_j - r_j,\, f_j + r_j)} \subseteq V(f,r)$$

$$\underbrace{}_{R(\underbrace{\min_j f_j - r_j}_{L},\ \underbrace{\max_j f_j + r_j}_{H})}$$

$$L_1 := g - \tfrac{1}{2}(g+h-L)$$

$$L_2 := h - \tfrac{1}{2}(g+h-L)$$

$$H_1 := g + \tfrac{1}{2}(H - (g+h))$$
$$H_2 := h + \tfrac{1}{2}(H - (g+h))$$

Then $(g,h) \in R(L_1, H_1) \times R(L_2, H_2)$
$\subseteq +^{-1}(R(L,H))$. ✓

To see scalar mul. is NOT cont., consider $(x \mapsto \frac{1}{x}) \in X$ w/ mul. by 0, which yields the zero f^n.

However, $\nexists$ nbhd of $(0, x \mapsto \frac{1}{x})$ which will land in an arbitrarily small ball of the zero f^n.

(*) To see $R(f,g)$ are open, write $R(f,g) = \bigcup_\alpha \bigcap_{\ell=1}^{n_\alpha} V(f_\ell^\alpha, r_\ell^\alpha)$.