Functional Analysis
Princeton University MAT520
HW1, Due Sep 15th 2023 (auto extension until Sep 17th 2023)

September 8, 2023

1 Topological vector spaces

1. Prove that $\mathbb{C}^n$ with its Euclidean topology is a topological vector space, i.e., show that vector addition and scalar multiplication are continuous with respect to the Euclidean topology.

2. Prove that $\mathbb{C}$ with the French metro metric is *not* homeomorphic (=topologically isomorphic) to $\mathbb{C}$ with the Euclidean metric. Conclude (why?) that $\mathbb{C}$ with the French metro metric is not a TVS.

3. Prove that if X is a TVS and $A, B \subseteq X$ then $\overline{A} + \overline{B} \subseteq \overline{A+B}$.

4. Prove that if X is a TVS and $A \subseteq X$ is a vector subspace then so is $\overline{A}$.

5. Prove that if X is a TVS and $A \subseteq X$ then $2A \subseteq A + A$.

6. Prove that any union and any intersection of balanced sets is balanced.

7. Prove that if A, B are balanced then so is $A + B$.

8. Prove that if A, B are bounded (resp. compact) then $A + B$ is bounded (resp. compact).

9. Find two closed sets A, B whose sum $A + B$ is not closed.

10. If X, Y are TVS with $\dim(Y) < \infty$, and $\Lambda : X \to Y$ is linear with $\Lambda(X) = Y$. Show that Λ is an *open mapping*. Show further that if $\ker(\Lambda)$ is closed then Λ is continuous.

11. Let $C := \{ f : [0,1] \to \mathbb{C} \mid f \text{ is continuous} \}$ and define

$$d(f,g) := \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \mathrm{d}x \,.$$

   Show that d is a metric on C, show that C is a vector space (with pointwise addition and scalar multiplication), and show that the topology which d induces on C makes it into a TVS. Show that that TVS has a countable local base.

12. Let V be a neighborhood of zero in a TVS X. Prove that $\exists f : X \to \mathbb{R}$ continuous such that $f(0) = 0$ and $f(x) = 1$ for all $x \in X \setminus V$.

13. Let X be the VS of all continuous functions $f : (0,1) \to \mathbb{C}$. For any $f \in X$ and $r > 0$, set

$$V(f,r) := \{ g \in X \mid |g(x) - f(x)| < r \forall x \in (0,1) \}$$

   and set $\mathrm{Open}(X)$ as the topology generated by $\{ V(f,r) \}_{f \in X, r>0}$ (is this collection a basis or a sub-basis for a topology?). Show that w.r.t. $\mathrm{Open}(X)$, vector addition is continuous but scalar multiplication is *not*.