DEC 17 2023

MAT520-FA — HW10 Sample Sol-ns

$\boxed{Q1}$ $A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathbb{C}^2)$.

$$A^*A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\not=$ NOT equal So A not normal.

$$AA^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\sigma(A) \equiv \{ \lambda \in \mathbb{C} \mid \underbrace{A - \lambda \mathbb{1}}_{\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}} \text{ NOT invertible} \}$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2 \overset{!}{=} 0 \implies \lambda = 0$$

$$\implies \sigma(A) = \{0\}.$$

Take $\lambda = 1 \in \rho(A)$.

$A - \mathbb{1} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ has inverse $-A - \mathbb{1}$

as $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $(A - \mathbb{1})(-A - \mathbb{1}) =$

$= -A + A + \mathbb{1} = \mathbb{1}$.

so $(A-\mathbb{1})^{-1} = -A-\mathbb{1} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^* \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\sigma\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) = \frac{1}{2}(3 \pm \sqrt{5}) \approx \{0.38, 2.61\}.$$

In particular, $\operatorname{dist}(1, \sigma(A)) = 1$

$$\|(A-\mathbb{1})^{-1}\| = \frac{1}{2}(3+\sqrt{5}) > 1.$$

$\boxed{Q2}$ Let $A = U|A|$ be the polar decomp.

$$f_n(x) := \begin{cases} \frac{1}{x} & x \geq \frac{1}{n} \\ n & x \leq \frac{1}{n} \end{cases} \qquad (x \geq 0)$$

<u>Claim</u>: $U = \operatorname*{s-lim}_{n\to\infty} A f_n(|A|)$

<u>Proof</u>: $\iff U - \operatorname*{s-lim}_{n\to\infty} U|A| f_n(|A|) = 0$

$\iff U \operatorname*{s-lim}_{n\to\infty} (\mathbb{1} - |A| f_n(|A|)) = 0$

$\iff \operatorname*{s-lim}_{n\to\infty} g_n(|A|) = 0$

w/ $g_n(x) := 1 - x f_n(x) \quad \forall \; x \geq 0.$

I.e., $$g_n(x) = \begin{cases} 0 & x \geq \frac{1}{n} \\ 1 - nx & 0 \leq x \leq \frac{1}{n} \end{cases}$$

g_n is Borel msrble. & bdd. w/ $\|g_n\|_\infty = 1$. Moreover, $g_n \to \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

the limit being L^2-equiv. to the zero f^n.

Hence by Thm. 10.16 in L.N.,

$$g_n(|A|) \to 0 \quad \text{strongly.}$$

$\blacksquare$

$\boxed{Q3}$ Claim: If $A \in \mathcal{B}(\mathcal{H})$ is normal then

$$r(A) = \|A\|.$$

Proof: By the functional calculus,

$$\|A\| = \left\| \int_{\lambda \in \mathbb{C}} \lambda \; dP_A(\lambda) \right\|$$

proj.-val. msr. of A

$$\leq \int_{\lambda \in \mathbb{C}} |\lambda| \, dP_A(\lambda) \leq r(A).$$

But $r(A) \leq \|A\|$ always (see e.g. Thm. 6.23).

Alt. proof by Gelfand's formula:

$$(A^* A)^n = (A^*)^n A^n = (A^n)^* A^n$$

$\boxed{Q4}$ will appear after $\boxed{Q5}$

$\boxed{Q5}$ Let $A, B \in \mathcal{B}(\mathcal{H})$ be s.A. ; $[A,B]=0$.

Then $[R_A(z), R_B(w)] = 0$ for

$$R_A(z) \equiv (A - z\mathbb{1})^{-1} \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

Via Stone's thm. we may recover the projection-valued measures dP_A as

$$\tfrac{1}{2}\left(\chi_{[a,b]}(A) + \chi_{(a,b)}(A)\right) = \operatorname*{s-lim}_{\varepsilon \to 0^+} \frac{1}{\pi} \int_a^b \operatorname{Im}\{R_A(E+i\varepsilon)\} dE.$$

Moreover, this formula shows

$$[dP_A, dP_B] = 0.$$

This allows us to define a msr.

$$Q_{AB}(S_1 \times S_2) := P_A(S_1)\, P_B(S_2) \qquad (S_1, S_2 \subseteq \mathbb{R})$$

on "cylinder" sets from which we may extend to msrbl. sets of $\mathbb{R}^2$.

Thus we now define, $\forall$ Borel bdd.

$$f : \mathbb{R}^2 \to \mathbb{C}$$

the operator

$$f(A,B) := \int_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} f(\lambda_1, \lambda_2)\, dQ_{AB}(\lambda_1, \lambda_2).$$

In particular, to get the unitary define, $\forall\ \psi \in \mathcal{H}$

$$\mathcal{H}_\psi := \overline{\{ f(A,B)\psi \mid f : \mathbb{R}^2 \to \mathbb{C} \text{ msrbl. bdd.} \}}$$

and

$$U : \mathcal{H}_\psi \longrightarrow L^2(dQ_{AB}\psi)$$

$$\begin{aligned} \psi &\longmapsto \underline{1} \\ A\psi &\longmapsto \lambda \mapsto \lambda_1 \\ B\psi &\longmapsto \lambda \mapsto \lambda_2 \end{aligned}$$

and if $\mathcal{H}_\psi \neq \mathcal{H}_1$ continue in this way.

For more details, see Feldman e.g.

(his notes are attached here, slightly

different approach ...)

https://personal.math.ubc.ca/~feldman/m511/
commutingNormalSpThm.pdf

Spectral Theorem for Commuting Normal Operators

Throughout these notes $\mathcal{H}$ is a Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of all bounded linear operators with domain $\mathcal{H}$ and taking values in $\mathcal{H}$. First recall

Definition 1 (Normal Operator) An operator $A \in \mathcal{L}(\mathcal{H})$ is called *normal* if $A^*A = AA^*$. That is, if A commutes with its adjoint.

Remark 2 (Normal Operators)

(a) A self–adjoint operator $A \in \mathcal{L}(\mathcal{H})$ obeys $A = A^*$ and hence is normal.

(b) A unitary operator $U \in \mathcal{L}(\mathcal{H})$ obeys $UU^* = U^*U = \mathbb{1}$ and hence is normal.

(c) Any operator $A \in \mathcal{L}(\mathcal{H})$ can be written in the form $A = \operatorname{Re} A + i \operatorname{Im} A$ with, by definition, $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A = \frac{1}{2i}(A - A^*)$. Both $\operatorname{Re} A$ and $\operatorname{Im} A$ are self–adjoint. The operator A is normal if and only if $\operatorname{Re} A$ and $\operatorname{Im} A$ commute.

In these notes we prove

Theorem 3 (Spectral Theorem for Commuting Bounded Normal Operators) *Let $n \in \mathbb{N}$ and let $\{A_1, A_2, \cdots, A_n\} \subset \mathcal{L}(\mathcal{H})$ be a finite set of commuting, normal, bounded operators. Then there exist*

- *a measure space $\langle \mathcal{M}, \Sigma, \mu \rangle$ and*
- *n bounded measurable functions $a_i : \mathcal{M} \to \mathbb{C}$, $1 \le i \le n$ and*
- *a unitary operator $U : \mathcal{H} \to L^2(\mathcal{M}, \Sigma, \mu)$*

such that

$$\big(UA_iU^{-1}\varphi\big)(m) = a_i(m)\,\varphi(m)$$

for all $\varphi \in L^2(M, \Sigma, \mu)$ and all $1 \le i \le n$. If $\mathcal{H}$ is separable, μ can be chosen to be a finite measure.

Proof: *Step 0 (Reduction to self–adjoint operators):*
By Fuglede's theorem (proven below), if the normal operators $\{A_1, A_2, \cdots, A_n\}$ commute, then so do all of the operators $\{A_1, A_2, \cdots, A_n, A_1^*, A_2^*, \cdots, A_n^*\}$. Consequently we may restrict our attention to commuting, self–adjoint, bounded operators simply by replacing $\{A_1, A_2, \cdots, A_n\}$ with $\{\operatorname{Re} A_1, \operatorname{Im} A_1, \operatorname{Re} A_2, \operatorname{Im} A_2, \cdots, \operatorname{Re} A_n, \operatorname{Im} A_n\}$. So from now on assume that $\{A_1, A_2, \cdots, A_n\} \subset \mathcal{L}(\mathcal{H})$ is a finite set of commuting, self–adjoint, bounded operators.

Step 1 ($f(A_1, \cdots, A_n)$ for some simple functions f):
Set, for $1 \le i \le n$, $I_i = [-\|A_i\|, \|A_i\|]$ and then set $I = I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$. Define the set of "rectangles" in I to be

$$\mathcal{R} = \left\{ \ B_1 \times B_2 \times \cdots \times B_n \subset I \ \middle| \ B_i \subset I_i, \text{ Borel, for each } 1 \le i \le n \ \right\}$$

There are quotation marks around "rectangles" because the sides of the "rectangles" are Borel sets rather than intervals. We are about to define $f(A_1, \cdots, A_n)$ for all simple functions $f : I \to \mathbb{C}$ that have the special form specified in

$$\mathcal{S} = \left\{ \ f(x) = \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(x) \ \middle| \ \alpha_j \in \mathbb{C}, \ R_j \in \mathcal{R}, \ 1 \le j \le m \right\}$$

We have already defined, in the functional calculus version of the spectral theorem (Theorem 27 in the notes [spectralReview.pdf]), $\chi_{B_i}(A_i)$ for each Borel $B_i \subset I_i$ and $1 \le i \le n$. We also already know the following.

- ∘ $\chi_{B_i}(A_i)$ is an orthogonal projection. (This is an immediate consequence of [spectralReview.pdf, Theorem 27.a].)
- ∘ $\chi_{B_i}(A_i)$ and $\chi_{B_j}(A_j)$ commute for all measurable $B_i \subset I_i$, $B_j \subset I_j$, $1 \le i, j \le n$. (This is an immediate consequence of [spectralReview.pdf, Theorem 27.g].)
- ∘ If the measurable sets $B_i, B_i' \subset I_i$ are disjoint, then $\chi_{B_i}(A_i)\chi_{B_i'}(A_i) = 0$. (This is an immediate consequence of [spectralReview.pdf, Theorem 27.a,b].)

We define, for each $R = B_1 \times B_2 \times \cdots \times B_n \in \mathcal{R}$

$$\chi_R(A_1, \cdots, A_n) = \prod_{j=1}^{n} \chi_{B_i}(A_i)$$

and for each $f = \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(x) \in \mathcal{S}$

$$f(A_1, \cdots, A_n) = \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(A_1, \cdots, A_n)$$

From the above bullets

- ∘ $\chi_R(A_1, \cdots, A_n)$ is an orthogonal projection for each rectangle $R \in \mathcal{R}$.
- ∘ If the rectangles $R, R' \in \mathcal{R}$ are disjoint, then $\chi_R(A_1, \cdots, A_n) \, \chi_{R'}(A_1, \cdots, A_n) = 0$.

Here is the main property that we need of the operators $f(A_1, \cdots, A_n)$, $f \in \mathcal{S}$.

Lemma 4 *If $f \in \mathcal{S}$ then*

$$\|f(A_1, \cdots, A_n)\| \le \sup_{x \in I} |f(x)|$$

Proof. Let $f \in \mathcal{S}$. We may always write f in the form $f = \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(x)$ with all of the R_j's disjoint (by possibly subdividing some of the R_j's) and with $\bigcup_{j=1}^{n} R_j = I$ (by possibly having some of the α_j's zero). Then every $x \in I$ is an element of exactly one R_j and the range of f is exactly $\left\{ \alpha_j \;\middle|\; 1 \le j \le m \right\}$. So

$$\sup_{x \in I} |f(x)| = \max\{|\alpha_j| \mid 1 \le j \le m\}$$

Now the $\chi_{R_j}(A_1, \cdots, A_n)$'s project onto mutually orthogonal subspaces of $\mathcal{H}$ and, since $\bigcup_{j=1}^{n} R_j = I$, we have $\sum_{j=1}^{m} \chi_{R_j}(A_1, \cdots, A_n) = \mathbb{1}$. So, for every $\mathbf{v} \in \mathcal{H}$,

$$\begin{aligned} \mathbf{v} &= \sum_{j=1}^{m} \chi_{R_j}(A_1, \cdots, A_n)\mathbf{v} \\ \implies \|\mathbf{v}\|^2 &= \sum_{j=1}^{m} \|\chi_{R_j}(A_1, \cdots, A_n)\mathbf{v}\|^2 \end{aligned}$$

and

$$\begin{aligned} f(A_1, \cdots, A_n)\mathbf{v} &= \sum_{j=1}^{m} \alpha_j \, \chi_{R_j}(A_1, \cdots, A_n)\mathbf{v} \\ \implies \|f(A_1, \cdots, A_n)\mathbf{v}\|^2 &= \sum_{j=1}^{m} |\alpha_j|^2 \, \|\chi_{R_j}(A_1, \cdots, A_n)\mathbf{v}\|^2 \\ &\le \max\{|\alpha_j| \mid 1 \le j \le m\}^2 \sum_{j=1}^{m} \|\chi_{R_j}(A_1, \cdots, A_n)\mathbf{v}\|^2 \\ &= \max\{|\alpha_j| \mid 1 \le j \le m\}^2 \|\mathbf{v}\|^2 \end{aligned}$$

■

The rest of the proof is identical to the corresponding parts of the proof of the multiplication operator version of the spectral theorem. Here is a very coarse outline of the remaining steps in the proof.

Step 2 ($f(A_1, \cdots, A_n)$ for continuous functions f):
By the Stone–Weierstrass Theorem, every continuous function $f : I \to \mathbb{C}$, is a uniform limit of a sequence $\{f_\ell\}_{\ell \in \mathbb{N}}$ of simple functions in $\mathcal{S}$. So we can define

$$f(A_1, \cdots, A_n) = \lim_{\ell \to \infty} f_\ell(A_1, \cdots, A_n) \in \mathcal{L}(\mathcal{H})$$

By Lemma 4 in Step 1, the right hand side converges in norm. Consequently the map $f \in C(I) \mapsto f(A_1, \cdots, A_n) \in \mathcal{L}(\mathcal{H})$ is

- ◦ continuous and
- ◦ linear and obeys
- ◦ $(fg)(A_1, \cdots, A_n) = f(A_1, \cdots, A_n)\, g(A_1, \cdots, A_n)$ and
- ◦ $f(A_1, \cdots, A_n)^* = (\bar{f})(A_1, \cdots, A_n)$.

Step 3 (Construction of $\mu_{\mathbf{v}}$):
Let $\mathbf{0} \neq \mathbf{v} \in \mathcal{H}$. Then

$$\ell_{\mathbf{v}}(f) = \langle \mathbf{v}\,,\, f(A_1, \cdots, A_n)\, \mathbf{v} \rangle_{\mathcal{H}}$$

is a positive linear functional on $C(I)$. So, by the Riesz–Markov Theorem, there is a unique, fnite, regular Borel measure $\mu_{\mathbf{v}}$ on I such that

$$\langle \mathbf{v}\,,\, f(A_1, \cdots, A_n)\, \mathbf{v} \rangle_{\mathcal{H}} = \int_I f(x)\, d\mu_{\mathbf{v}}(x)$$

for all $f \in C(I)$.

Step 4 (Construction of $\mathcal{H}_{\mathbf{v}}$ and $U_{\mathbf{v}}$):
Let $\mathbf{0} \neq \mathbf{v} \in \mathcal{H}$ and set

$$\mathcal{H}_{\mathbf{v}} = \overline{\left\{\ f(A_1, \cdots, A_n)\, \mathbf{v}\ \middle|\ f \in C(I)\ \right\}}$$

Lemma 5 *There is a unique unitary operator $U_{\mathbf{v}} : \mathcal{H}_{\mathbf{v}} \to L^2(\mu_{\mathbf{v}})$ such that*

$$U_{\mathbf{v}}\mathbf{v} = 1$$
$$(U_{\mathbf{v}} A_i U_{\mathbf{v}}^{-1}) f(x) = x_i\, f(x) \qquad 1 \leq i \leq n$$

Proof. Set

$$\mathcal{D}_{\mathbf{v}} = \left\{\ f(A_1, \cdots, A_n)\, \mathbf{v}\ \middle|\ f \in C(I)\ \right\}$$

and define $\tilde{U}_{\mathbf{v}} : \mathcal{D}_{\mathbf{v}} \to L^2(\mu_{\mathbf{v}})$ by

$$\big(\tilde{U}_{\mathbf{v}} f(A_1, \cdots, A_n)\, \mathbf{v}\big)(x) = f(x)$$

This operator is

- ◦ well–defined
- ◦ linear
- ◦ inner product preserving

As $\mathcal{D}_{\mathbf{v}}$ is dense in $\mathcal{H}_{\mathbf{v}}$, we can use the BLT theorem to define $U_{\mathbf{v}}$ as the continuous extension of $\tilde{U}_{\mathbf{v}}$ to $\mathcal{H}_{\mathbf{v}}$. Then $U_{\mathbf{v}}$ has the required properties and is indeed uniquely determined by those properties.

Step 5 (Completion of the proof by Zornification):
If $\mathcal{H}_{\mathbf{v}} = \mathcal{H}$, we are done. If not Zornify. ■

Theorem 6 *Let $A, T \in \mathcal{L}(\mathcal{H})$. If A is normal and T commutes with A, then T commutes with A^*.*

Proof: By induction $A^n T = T A^n$ for all $0 \le n \in \mathbb{Z}$. As the exponential series $e^{\bar{\lambda} A} = \sum_{n=0}^{\infty} \frac{1}{n!}(\bar{\lambda} A)^n$ converges in norm, we have

$$e^{\bar{\lambda} A} T = T e^{\bar{\lambda} A} \implies e^{\bar{\lambda} A} T e^{-\bar{\lambda} A} = T \implies e^{-\lambda A^*} e^{\bar{\lambda} A} T e^{-\bar{\lambda} A} e^{\lambda A^*} = e^{-\lambda A^*} T e^{\lambda A^*}$$

for all $\lambda \in \mathbb{C}$. As A is normal, we have that $e^{-\lambda A^*} e^{\bar{\lambda} A} = e^{-\lambda A^* + \bar{\lambda} A}$ and furthermore that $U(\lambda) = e^{-\lambda A^* + \bar{\lambda} A}$ obeys $U(\lambda)^* = U(-\lambda) = U(\lambda)^{-1}$. Thus $U(\lambda)$ is unitary and is hence of norm 1. So

$$\| e^{-\lambda A^*} T e^{\lambda A^*} \| = \| U(\lambda)\, T\, U(-\lambda) \| \le \| T \|$$

This shows that the analytic operator valued function $e^{-\lambda A^*} T e^{\lambda A^*}$ is bounded uniformly on all of $\mathbb{C}$. So $e^{-\lambda A^*} T e^{\lambda A^*}$ has to be independent of λ and

$$e^{-\lambda A^*} T e^{\lambda A^*} = e^{-\lambda A^*} T e^{\lambda A^*} \Big|_{\lambda=0} = T$$

for all λ. Differentiating with respect to λ and then setting $\lambda = 0$ gives

$$-A^* T + T A^* = 0$$

as desired. ■

$\boxed{Q4}$ Apply the above on

$$A = \mathrm{Re}\{A\} + i\,\mathrm{Im}\{A\}$$

$\boxed{Q6}$ Let $A = A^* \in \mathcal{B}(\mathcal{H})$ and $\chi_{\cdot}(A)$ the proj.-val. msr. of A.

<u>Claim</u>: $\sigma(A) = \left\{ \lambda \in \mathbb{R} \;\middle|\; \forall \varepsilon > 0,\ \chi_{B_\varepsilon(\lambda)}(A) \neq 0 \right\}$

<u>Proof</u>: We will show $\rho(A) = \{\dots\}^c$.

$\boxed{\subseteq}$ For $\mu_{A,\psi}$ the spec. msr. of (A,ψ), we know $\mathrm{Supp}(\mu_{A,\psi}) \subseteq \sigma(A)$.

So if $\lambda \in \rho(A)$, $\mu_{A,\psi}(B_\varepsilon(\lambda)) = 0 \ \exists \varepsilon > 0$.

But ψ is arbit and

$$\mu_{A,\psi}(B_\varepsilon(\lambda)) = \langle \psi, \chi_{B_\varepsilon(\lambda)}(A)\psi \rangle = 0.$$

Hence $\chi_{B_\varepsilon(\lambda)}(A) = 0$ as this is a S.A. proj.

$\boxed{\supseteq}$ Let $\lambda \in \{\dots\}^c$. Then $\exists\ \varepsilon > 0$:
$\forall\, \psi, \varphi \in \mathcal{H}$, $\langle \psi, \chi_{B_\varepsilon(\lambda)}(A)\varphi \rangle = 0$.

I.e.,

$$\langle \psi, \varphi \rangle = \langle \psi, \left(\chi_{B_\varepsilon(\lambda)}(A) + [\chi_{B_\varepsilon(\lambda)}(A)]^\perp \right) \varphi \rangle$$

by hypo.

$$\overset{\text{by hypo.}}{=} \langle \psi, \chi_{B_\varepsilon(\lambda)}(A)^\perp \varphi \rangle$$

$$\equiv \langle \psi, \chi_{B_\varepsilon(\lambda)^c}(A) \varphi \rangle .$$

Now, if $f(x) := \begin{cases} \frac{1}{x-\lambda} & x \in B_\varepsilon(\lambda)^c \\ 0 & \text{else} \end{cases}$

and $g(x) := x - \lambda$ we get

$$\langle \psi, f(A) g(A) \varphi \rangle = \langle \psi, (fg)(A) \varphi \rangle$$

$$= \int_{\lambda \in \mathbb{R}} (fg)(\lambda) \, d\mu_{A,\psi,\varphi}(\lambda)$$

$$= \int_{\lambda \in \mathbb{R}} f(\lambda) g(\lambda) \, d\mu_{A,\psi,\varphi}(\lambda)$$

$\operatorname{supp}(f) \subseteq B_\varepsilon(\lambda)^c$

$$\overset{\operatorname{supp}(f) \subseteq B_\varepsilon(\lambda)^c}{=} \int_{\lambda \in B_\varepsilon(\lambda)^c} f(\lambda) g(\lambda) \, d\mu_{A,\psi,\varphi}(\lambda)$$

$$\overset{fg=1 \text{ on } B_\varepsilon(\lambda)^c}{\geq} \int_{\lambda \in B_\varepsilon(\lambda)^c} d\mu_{A,\psi,\varphi}(\lambda)$$

$$= \langle \psi, \chi_{B_\varepsilon(\lambda)^c}(A)\, \varphi \rangle$$

$$= \langle \psi, \varphi \rangle$$

Since ψ, φ were arbitrary,

$g(A) \equiv A - \lambda \mathbb{1}$ has an inverse.

$\Leftrightarrow \lambda \in \rho(A)$.

$\blacksquare$

$\boxed{Q7}$ Let $\mathcal{H}$ be a sep. Hil. sp.

<u>Claim</u>: The only op-norm-closed $*$-ideals in $\mathcal{B}(\mathcal{H})$ are $\{0\}$, $\mathcal{K}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$.

<u>Proof</u>: Let $\mathcal{I} \subseteq \mathcal{B}(\mathcal{H})$ be some non-triv. $*$-closed ideal.

<u>Claim</u>: $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{I}$.

Proof: Let P be a rank-1 proj.
Then $\forall A \in \mathcal{I} \setminus \{0\}$, $PA \in \mathcal{I}$ is a rank-1 op.

$$PA = \varphi \otimes \psi^* \quad \exists \; \varphi, \psi \in \mathcal{H}.$$

By star-closedness, $\psi \otimes \varphi^* \in \mathcal{I}$ too.
From there by composing w/

$\varphi \mapsto \tilde{\varphi}$
$\psi \mapsto \tilde{\psi}$

we get to any other rank-1 op., and by lin. comb. to any fin. rank.
Norm closed $\Rightarrow$ cpt. op. $\blacksquare$

Now, if $A \in \mathcal{I} \setminus \mathcal{K}(\mathcal{H})$, W.T.S. $\mathbb{1} \in \mathcal{I}$.

Since A is NOT cpt., it is impossible that both $\mathrm{Re}\{A\}, \mathrm{Im}\{A\}$ are cpt., so by Thm. 9.60, $\sigma_{ess}(B) \neq \{0\}$

for some $B = B^* \in \mathcal{I}$.

This implies that B is a Fredholm op., so by Atkinson's thm.

(Thm. 9.51) that $\exists\ G \in \mathcal{F}$

s.t. $\begin{matrix} \mathbb{1} - BG \\ \mathbb{1} - GB \end{matrix} \in \mathcal{K}(\mathcal{H})$

But since $\mathcal{I}$ is an ideal,

that means $BG, GB \in \mathcal{I}$, i.e.,

$$\mathbb{1} - K_1,\ \mathbb{1} - K_2 \in \mathcal{I}$$

for some cpt. K_1, K_2. But $\mathcal{K} \subseteq \mathcal{I}$,

so $\mathbb{1} \in \mathcal{I} \Leftrightarrow \mathcal{I} = \mathcal{B}(\mathcal{H})$.

Q8, Q9 will appear after Q10, Q11:

$\boxed{\text{Q10}}$ R is the uni. shift on $\ell^2(\mathbb{N})$:

$$R\delta_n \equiv \delta_{n+1} \qquad n \geq 1.$$

$$\ker(R) \equiv \{ \psi \in \ell^2(\mathbb{N}) \mid R\psi = 0 \}$$

$$(R\psi)(n) \equiv \begin{cases} \psi(n-1) & n \geq 2 \\ 0 & n=1 \end{cases}$$

$$\psi(n-1) = 0 \qquad \forall \quad n \geq 2$$

$$\Rightarrow \quad \psi = 0. \quad \Rightarrow \quad \boxed{\ker(R) = \{0\}.}$$

$$R^* = ?$$

$$\langle \varphi, R^*\psi \rangle \equiv \langle R\varphi, \psi \rangle \equiv \sum_{n=2}^{\infty} \overline{\varphi(n-1)}\, \psi(n)$$

$$= \sum_{n=1}^{\infty} \overline{\varphi(n)}\, \psi(n+1)$$

$$\Rightarrow \quad (R^*\psi)(n) \equiv \psi(n+1) \qquad (n \geq 1).$$

$\psi(n+1) = 0 \quad \forall \quad n \geq 1$ does NOT imply $\psi = 0$ since $\psi(1)$ is free!

$$\Rightarrow \boxed{\ker(R^*) = \mathbb{C}\,\delta_1}.$$

$$\operatorname{im}(R) \equiv \{ R\psi \mid \psi \in \ell^2 \}$$
$$= \{ (0, \psi(1), \psi(2), \dots) \mid \psi \in \ell^2 \}$$
$$= \operatorname{span}(\{\delta_n\}_{n=2}^{\infty}) .$$

Claim: $R \in \mathcal{F}$

Proof: $\operatorname{coker}(R) \equiv \mathcal{H} / \operatorname{im}(R)$

$\cong \mathbb{C}\delta_1$ finite dim. ✓

$\ker(R) = \{0\}$ finite dim. ✓

$\Longleftrightarrow R \in \mathcal{F}.$

$$\operatorname{index}(R) = \dim\ker R - \\ - \dim\operatorname{coker} R$$
$$= -1.$$

Note: Can also show $\operatorname{im}(R)$ is closed and use $\operatorname{index}(R) = \dim\ker R - \\ - \dim\ker R^*$.

Q11 <u>Claim</u>: On $\ell^2(\mathbb{N})$, $\frac{1}{X}$ is NOT Fredholm even though it has index$(\frac{1}{X})$ formally equal to zero (though not well-def.).

<u>Proof</u>: We need to calculate $\text{im}(\frac{1}{X})$.

(Note we could have used Lemma 7.20 w/ $\|X\| = \infty$ so $\|\frac{1}{X}\varphi\|$ is NOT bdd. from below $\rightarrow$ $\text{im}(\frac{1}{X})$ NOT closed.

$$\text{im}(\tfrac{1}{X}) \equiv \left\{ \tfrac{1}{X}\psi \mid \psi \in \ell^2 \right\}$$

$$= \left\{ (\psi(1), \tfrac{1}{2}\psi(2), \tfrac{1}{3}\psi(3), \ldots) \mid \psi \in \ell^2 \right\}$$

$\varphi(n) := \frac{1}{n}\psi(n)$

$\Leftrightarrow \psi(n) = n\varphi(n)$

$$= \left\{ \varphi : \mathbb{N} \to \mathbb{C} \,\middle|\, \sum_{n=1}^{\infty} n^2 |\varphi(n)|^2 < \infty \right\}$$

So $n \mapsto \frac{1}{n^2}$ is NOT in this set, but all its finite supp. approx. are. Hence it is not closed.

As such $\frac{1}{x}$ cannot be Fredholm. $\blacksquare$

Q8 $-\Delta$ on $\ell^2(\mathbb{Z})$ given by

$$-\Delta \equiv 2\mathbb{1} - R - R^*$$

R: bilateral shift

(a) Claim: $\forall\ x \in \mathbb{Z}$, δ_x is NOT a cyclic vector for $-\Delta$.

Proof: $(-\Delta)^k \delta_x$ is even about x. So it cannot span δ_{x+1} e.g.

Claim: $f(z) := \langle \delta_0, (-\Delta - z\mathbb{1})^{-1} \delta_0 \rangle$ is given by

$$f(z) = \frac{-1}{\sqrt{z(z-4)}} \quad \left(z \in \mathbb{C} \setminus [0,4]\right).$$

Proof: By the Fourier series within f, we get

$$f(z) = \langle \mathcal{F}\delta_0, \mathcal{F}(-\Delta - z\mathbb{1})^{-1}\mathcal{F}^*\mathcal{F}\delta_0 \rangle_{L^2(S^1)}$$

$$= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{1}{2 - 2\cos(\theta) - z}\, d\theta$$

$\lambda := e^{i\theta}$

$d\lambda = e^{i\theta} i\, d\theta$

$= i\lambda\, d\theta$

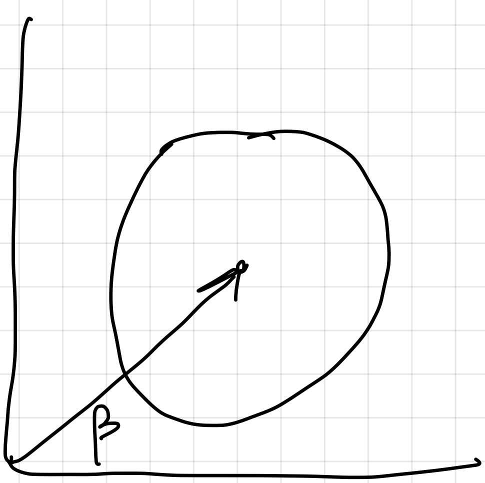
$d\theta = \frac{1}{i\lambda} d\lambda$

$$= \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{1}{2 - \lambda - \frac{1}{\lambda} - z} \frac{d\lambda}{\lambda}$$

$$= \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{1}{(2-z)\lambda - \lambda^2 - 1}\, d\lambda$$

$\beta := \frac{1}{2}(2-z)$

$$= \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{d\lambda}{(\lambda - \beta - \sqrt{\beta^2-1})(\lambda - \beta + \sqrt{\beta^2-1})}$$



$$= \frac{1}{2\pi i} \oint_{\partial B_1(\beta)} \frac{d\lambda}{(\lambda - \sqrt{\beta^2-1})(\lambda + \sqrt{\beta^2-1})}$$

Claim: One of these roots is inside the circle and the other outside, if $\beta \in \mathbb{C} \setminus [-1, 1]$.

Proof: TODO...

$$\Rightarrow \quad f(z) = \frac{-1}{2\sqrt{\beta^2 - 1}} = \frac{-1}{\sqrt{z(z-4)}}$$

$\blacksquare$

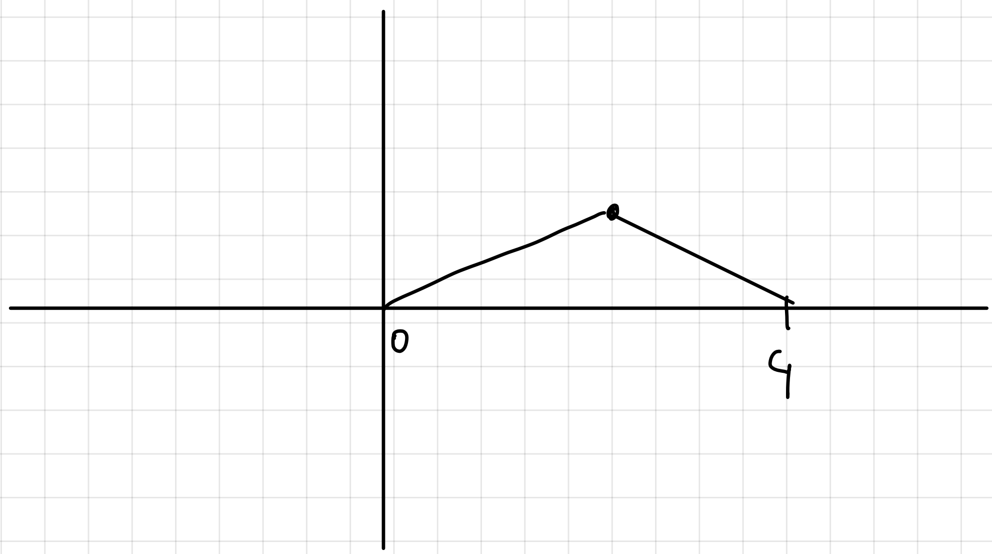
$$\operatorname{Im}\left\{-\frac{1}{z}\right\} = \operatorname{Im}\left\{-\frac{\bar{z}}{|z|^2}\right\} = \operatorname{Im}\left\{\frac{-x+iy}{x^2+y^2}\right\}$$

$$= \frac{y}{x^2+y^2} = \frac{r\sin(\theta)}{r^2} = \frac{\sin(\theta)}{r}$$

$$= \frac{\sin(\operatorname{Arg}(z))}{|z|}$$

As $\varepsilon \to 0^+$

$$\sin\Big(\underbrace{\mathrm{Arg}\big(\sqrt{(E+i\varepsilon)(E+i\varepsilon-4)}\big)}_{\frac{1}{2}\left(\mathrm{Arg}(E+i\varepsilon)+\mathrm{Arg}(E-4+i\varepsilon)\right)}\Big)$$



$$\xrightarrow{\varepsilon\to 0^+}\begin{cases} 1 & E\in(0,4) \\ 0 & E\in\mathbb{R}\setminus[0,4] \\ \frac{\sqrt{2}}{2} & E\in\{0,4\}\end{cases}$$

$$\Longrightarrow\ \boxed{\lim_{\varepsilon\to 0^+}\mathrm{Im}\{f(E+i\varepsilon)\} = \begin{cases} (E(E-4))^{-1/2} & E\in(0,4) \\ 0 & E\in\mathbb{R}\setminus[0,4] \\ \infty & E\in\{0,4\}\end{cases}}$$

By Lemma 10.11, we find

$$\boxed{d\mu_{-\Delta,\delta_0}(\lambda) = \frac{1}{\pi}(\lambda(\lambda-4))^{-1/2}\,d\lambda}$$

i.e., it is a.c. w.r.t. Lebesgue.

More precisely,

$$d\mu_{-\Delta,\delta_0}(\lambda) = \frac{1}{\pi}(\lambda(\lambda-4))^{-1/2}\chi_{(0,4)}(\lambda)\,d\lambda +$$

$$+ \, d\vartheta_{\text{singular}}(\lambda)$$

But since the first measure already integrates to 1 $(=\|\delta_0\|)$,

we must have $\vartheta_{\text{sing.}} = 0$.

$\boxed{Q9}$ (a) δ_x cannot be a cyclic vector for $V(X)$ since any power lies in $\mathbb{C}\,\delta_x$.

(b) $$f_x(z) := \langle \delta_x , (V(X) - z\mathbb{1})^{-1} \delta_x \rangle$$

$$= \langle \delta_x, (V(x) - z)^{-1} \delta_x \rangle$$

$$= (V(x) - z)^{-1}$$

(c) $$\mathrm{Im}\{f_x(z)\} = \frac{\mathrm{Im}\{z\}}{(V(x) - \mathrm{Re}\{z\})^2 + \mathrm{Im}\{z\}^2}$$

$$\lim_{\varepsilon \to 0^+} \mathrm{Im}\{f(E + i\varepsilon)\} = 0 \quad \text{if} \quad E \neq V(x).$$

$$\lim_{\varepsilon \to 0^+} \mathrm{Im}\{f(V(x) + i\varepsilon)\} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon^2} = \infty.$$

BUT $$\lim_{\varepsilon \to 0} \varepsilon\, \mathrm{Im}\{f(V(x) + i\varepsilon)\} = \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\varepsilon^2} = 1.$$

(d) By Lemma 10.11, $\mu_{V(x), \delta_x}$ is pure point supported exactly on $V(x)$:

$$d\mu_{V(x),\delta_x}(\lambda) = d\delta_{V(x)}(\lambda).$$