Functional Analysis
Princeton University MAT520
HW10, Due Dec 1st 2023 (auto extension until Dec 3rd 2023)

December 2, 2023

1. Provide an example for a non-normal operator $A \in \mathcal{B}(\mathcal{H})$ and a point in the resolvent set $z \in \rho(A)$ where

$$\left\| (A - z\mathbb{1})^{-1} \right\| \leq \frac{1}{\operatorname{dist}(z, \sigma(A))}$$

does *not* hold.

2. Let $A = U|A|$ be the polar decomposition of an operator $A \in \mathcal{B}(\mathcal{H})$. Let

$$f_n(x) := \begin{cases} \frac{1}{x} & x \geq \frac{1}{n} \\ n & x \leq \frac{1}{n} \end{cases} \qquad (x \geq 0) \ .$$

Prove that

$$U = \operatorname*{s-lim}_{n \to \infty} A f_n(|A|) \ .$$

[extra] Do the same with

$$f_n(x) \quad := \quad \frac{1}{x + \frac{1}{n}} \qquad (x \geq 0) \ .$$

3. Prove that if $A \in \mathcal{B}(\mathcal{H})$ is normal then

$$\|A\| = r(A)$$

where $r(A)$ is the spectral radius of an operator.

4. Let $A \in \mathcal{B}(\mathcal{H})$ be normal. Show there exists some finite measure space (M, μ) and a unitary

$$U : \mathcal{H} \to L^2(M, \mu)$$

such that there exists a bounded Borel function $f : M \to \mathbb{C}$ such that

$$(UAU^* \psi)(m) \quad = \quad f(m) \psi(m) \qquad \left(m \in M; \psi \in L^2(M, \mu)\right) \ .$$

5. Show that if $A, B \in \mathcal{B}(\mathcal{H})$ are two self-adjoint operators such that $[A, B] = 0$ then there exists a finite measure space (M, μ) and a unitary

$$U : \mathcal{H} \quad \to \quad L^2(M, \mu)$$

such that there are two bounded Borel functions $f, g : M \to \mathbb{R}$ which obey

$$\begin{aligned} (UAU^* \psi)(m) \quad &= \quad f(m) \psi(m) \\ (UBU^* \psi)(m) \quad &= \quad g(m) \psi(m) \end{aligned}$$

for all $m \in M$ and $\psi \in L^2(M, \mu)$.

6. Prove that for $A \in \mathcal{B}(\mathcal{H})$ self-adjoint and $\chi_{\cdot}(A)$ the projection-valued measure of A, we have

$$\lambda \in \sigma(A) \iff \left[\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) \neq 0 \qquad (\varepsilon > 0)\right] \qquad (\lambda \in \mathbb{R}) .$$

7. Let $\mathcal{H}$ be a separable Hilbert space. Prove that the *only* operator-norm-closed star-ideals in $\mathcal{B}(\mathcal{H})$ are $\{0\}$, $\mathcal{K}(\mathcal{H})$ (the compact operators) and $\mathcal{B}(\mathcal{H})$ itself.

8. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and on it define the discrete Laplacian

$$-\Delta = 2\mathbb{1} - R - R^*$$

where R is the bilateral right shift operator.

(a) Recall the definition of a cyclic vector from the lecture notes. For $x \in \mathbb{Z}$, is δ_x a cyclic vector for $-\Delta$?

(b) Define $f : \mathbb{C}^+ \to \mathbb{C}$ via

$$f(z) := \left\langle \delta_0, (-\Delta - z\mathbb{1})^{-1} \delta_0 \right\rangle .$$

Find an explicit expression for f using the Fourier series which was presented in the previous homework.

(c) Calculate the limit

$$\lim_{\varepsilon \to 0^+} \mathbb{Im}\{f(E + \mathrm{i}\varepsilon)\}$$

for the two cases $E \in (0,4)$ and $E \in \mathbb{R} \setminus (0,4)$.

(d) Calculate the spectral measure of $(-\Delta, \delta_0)$ and determine its type (w.r.t. the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

9. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and on it define the multiplication operator

$$V(X)$$

via

$$(V(X)\psi)(x) := V(x)\psi(x) \qquad \left(x \in \mathbb{Z}, \psi \in \ell^2(\mathbb{Z})\right)$$

where $V : \mathbb{Z} \to \mathbb{R}$ is some bounded sequence.

(a) For $x \in \mathbb{Z}$, is δ_x is a cyclic vector for $V(X)$?

(b) For any $x \in \mathbb{Z}$, define $f_x : \mathbb{C}^+ \to \mathbb{C}$ via

$$f_x(z) := \left\langle \delta_x, (V(X) - z\mathbb{1})^{-1} \delta_x \right\rangle .$$

(c) Calculate both

$$\lim_{\varepsilon \to 0^+} \mathbb{Im}\{f_x(E + \mathrm{i}\varepsilon)\}$$

and

$$\lim_{\varepsilon \to 0^+} \varepsilon\, \mathbb{Im}\{f_x(E + \mathrm{i}\varepsilon)\}$$

for all $E \in \mathbb{R}$ (separate into cases).

(d) Calculate the spectral measure of $(V(X), \delta_0)$ and determine its type (w.r.t. the Lebesgue decomposition theorem where the reference measure is the Lebesgue measure, i.e., ac, sc, or pp).

10. On $\ell^2(\mathbb{N})$, let $\hat{R}$ be the *unilateral* right shift operator. Calculate $\ker \hat{R}, \ker \hat{R}^*$ and $im\hat{R}$ and show that $\hat{R}$ is a Fredholm operator. Calculate its Fredholm index.

11. Show that on $\ell^2(\mathbb{N})$, $\frac{1}{X}$ where X is the position operator, is *not* a Fredholm operator by calculating $\mathrm{im}\frac{1}{X}$ and showing that it is not closed.