DEC 14 2023

MAT520 — HW11 — Sample Sol-ns

Q1 $\mathcal{D}(X) := \left\{ \psi \in L^2 \,\middle|\, \int_{x \in \mathbb{R}} x^2 |\psi(x)|^2 \, dx < \infty \right\}.$

Claim: $\mathcal{D}(X)$ is a v.sp.

Proof: - $0 \in \mathcal{D}(X)$ ✓

- $\psi \in \mathcal{D}(X) \implies \lambda \psi \in \mathcal{D}(X) \quad \forall \lambda \in \mathbb{C}$ ✓

- $\varphi, \psi \in \mathcal{D}(X)$ implies

$$\int x^2 \underbrace{|(\varphi + \psi)(x)|^2}_{|\varphi(x)|^2 + |\psi(x)|^2 + 2\mathrm{Re}\{\overline{\varphi(x)}\,\psi(x)\}} dx$$

only "bad" term

$$\left| \int x^2 \, 2\mathrm{Re}\{\overline{\varphi(x)}\,\psi(x)\} \, dx \right| \leq$$

$$\leq \int x^2 |\varphi(x)| |\psi(x)| \, dx \leq \left(\int x^2 |\varphi(x)|^2 dx \right)^{1/2} \left(\int x^2 |\psi(x)|^2 dx \right)^{\frac{1}{2}}$$

Cauchy-Schwarz

$< \infty.$ ∎

Claim: $\mathcal{D}(X)$ is the largest v/sp. V s.t.

if $\psi \in V$ then $X\psi \in L^2$

Proof: $\mathcal{D}(X)$ is def. as the set of all $\psi \in L^2$ s.t. $X\psi \in L^2$. As it turns out to be itself a v/sp., it is the largest such

Q2 $\mathcal{A} := \{ \psi : [0,1] \to \mathbb{C} \mid \psi$ is ac $\wedge$ $\psi' \in L^2([0,1]) \}$.

$A_j \psi := -i\psi' \quad \forall j = 1,2$ w/

$$\mathcal{D}(A_1) := \mathcal{A}$$
$$\mathcal{D}(A_2) := \{ \psi \in \mathcal{A} \mid \psi(0) = 0 \}.$$

Claim: $\overline{\mathcal{D}(A_j)} = L^2([0,1])$ for $j = 1,2$.

Proof: $\mathcal{A}$ is dense in L^2 since

$$C^\infty([0,1]) \subseteq \mathcal{A}$$

and $\overline{C^\infty([0,1])} = L^2([0,1])$.

For $\mathcal{D}(A_2)$ we multiply w/ a seq. of bump f^n's at the origin.

Claim: A_1, A_2 are both closed.

Proof: We show that $\Gamma(A_j) \in \text{Closed}(\mathcal{H}^2)$.

Let $\{\psi_n\}_n \subseteq \mathcal{D}(A_j)$. Assume

$$(\psi_n, A_j \psi_n) \xrightarrow{n\to\infty} (\psi, \varphi) \in \mathcal{H}^2.$$

W.T.S. $(\psi,\varphi) \in \Gamma(A_j)$, i.e., $\psi \in \mathcal{D}(A_j)$ and $\varphi = A_j \psi$.

Start w/ $j=1$.

Since ψ_n is ac, we may write

$$\psi_n(x) = \int_0^x \psi_n' + \psi_n(0) \quad (x \in [0,1]).$$

W.T.S. $\psi_n \to \psi$ in L^2 now implies $\psi \in \mathcal{A}$

too, and $\varphi = -i\psi' \iff \int_0^x \varphi = -i(\psi(x) - \psi(0))$.

This last eq-n actually implies ψ is a.c.

<u>Claim</u>: $\psi_n \to \psi$ in L^∞ $(\dots)$.

Hence $\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$: if $n \geq N_\varepsilon$ then

$$|\psi_n(0) - \psi(0)|,\ |\psi_n(x) - \psi(x)|,\ \|\psi_n' + i\varphi\|_{L^2}$$

are all $\leq \frac{1}{3}\varepsilon$. Thus

$$\Big|\psi(x) - \psi(0) - \int_0^x i\varphi\Big| \underset{\substack{\uparrow \\ L^2 \text{ dom. } L^1}}{\leq} \varepsilon$$

The $j=2$ statement is easier. ∎

<u>Claim</u>: $\sigma(A_1) = \mathbb{C}$

<u>Proof</u>: Let $\lambda \in \mathbb{C}$. W.T.S.

$$(A_1 - \lambda \mathbb{1}) : \mathcal{A} \to L^2$$

is NOT a bijection.

Consider $f_\lambda(x) := e^{i\lambda x}$ $\quad x \in [0,1]$.

It is certainly in $\mathcal{A}$. Moreover,

$$f_\lambda' = i\lambda f_\lambda \quad \text{so}$$

$$A_1 f_\lambda = \lambda f_\lambda \Rightarrow \ker(A_1 - \lambda \mathbb{1}) \neq \{0\}.$$

$\blacksquare$

<u>Claim</u>: $\sigma(A_2) = \emptyset$.

<u>Proof</u>: Let $\lambda \in \mathbb{C}$. W.T.S.

$$(A_2 - \lambda \mathbb{1}) : \mathcal{D}(A_2) \to L^2$$

is a bijection.

<u>Claim</u>: $\left[(A_2 - \lambda \mathbb{1})^{-1} \psi\right](x) = -i \int_0^x e^{i\lambda(x-y)} \psi(y)\, dy$

<u>Proof</u>: First we show $(A_2 - \lambda \mathbb{1})^{-1}$ is bdd.

$$\int_0^1 |f|^2 \leq \|f\|_\infty^2 \quad \text{so}$$

$$\|(A_2-\lambda \mathbb{1})^{-1}\psi\|_{L^2} \leqslant \sup_{x\in[0,1]} \left| \int_0^x e^{i\lambda(x-y)} \psi(y)\, dy \right|$$

$$\leqslant \|\psi\|_{L^2} .$$

Next,

$$(A_2-\lambda\mathbb{1})^{-1}(A_2-\lambda\mathbb{1})\psi =$$

$$= -i\int_0^x e^{i\lambda(x-y)} [(A_2-\lambda\mathbb{1})\psi](y)\, dy$$

$$= -i\int_0^x e^{i\lambda(x-y)} (-i\psi'(y)-\lambda\psi(y))\, dy$$

$$= i\lambda e^{i\lambda x}\int_0^x e^{-i\lambda y}\psi(y)\,dy - e^{i\lambda x}\underbrace{\int_0^x e^{-i\lambda y}\psi'(y)\,dy}_{e^{-i\lambda y}\psi(y)\Big|_0^x - \int_0^x (-i\lambda)e^{-i\lambda y}\psi(y)\,dy}$$

$$= \psi(x)$$

and similarly:

Leibniz int. rule

$$(A_2-\lambda\mathbb{1})(i)\int_0^{\cdot} e^{i\lambda(\cdot-y)}\psi(y)\,dy \overset{*}{=}$$

$$= \psi(x) + \lambda e^{i\lambda x} \int_0^x e^{-i\lambda y} \psi(y)\, dy$$

$$- \lambda \int_0^x e^{i\lambda(x-y)} \psi(y)\, dy$$

$$= \psi(x). \quad \checkmark$$

$\blacksquare$

Hence $A_2 - \lambda \mathbb{1}$ is indeed invertible $\forall \lambda \in \mathbb{C}$.

$\blacksquare$

$\boxed{Q3}$ This is Corollary 11.27 in the lecture notes (now fixed since DEC 17 '23).

$\boxed{Q4}$ $A := -i\partial$

$$\mathcal{D}(A) := \{ \psi \in \mathcal{A} \mid \psi(0) = \psi(1) = 0 \}$$

as in $\boxed{Q2}$

(a) **Claim:** A is densely def.

Proof: As in $\boxed{Q2}$.

Claim: A is symm.

Proof: W.T.S. $\langle A\varphi, \psi \rangle_{L^2} = \langle \varphi, A\psi \rangle_{L^2}$ $\forall$ $\varphi, \psi \in \mathcal{D}(A)$.

$$\langle \varphi, A\psi \rangle_{L^2} \equiv \int_{x\in[0,1]} \overline{\varphi(x)} \, (-i) \, \psi'(x) dx$$

$$\overset{\text{IBP}}{=} \underbrace{\varphi(x)(-i)\psi(x)\Big|_{x=0}^{1}}_{=0} - \int_{x\in[0,1]} (-i) \, \overline{\varphi'(x)} \, \psi(x) dx$$

$$= \int_{x\in[0,1]} \overline{-i\varphi'(x)} \, \psi(x) \, dx$$

$$\approx \langle -i\partial\varphi, \psi\rangle_{L^2} \approx \langle A\varphi, \psi\rangle_{L^2}. \quad \blacksquare$$

(b) $\mathcal{D}(A^*) \equiv \left\{ \varphi \in L^2 \,\middle|\, \exists\, \xi \in \mathcal{H} : \forall \psi \in \mathcal{D}(A),\ \underbrace{\langle \varphi, A\psi\rangle = \langle \xi, \psi\rangle}_{\int_{x=0}^{1} \overline{\varphi(x)}(-i)\psi'(x)\,dx = \int_0^1 \overline{\xi}\psi.} \right\}$

Claim: $\mathcal{D}(A^*) = \mathcal{A}$ w/ $A^* = -i\partial$.

Proof: $\boxed{\supseteq}$ Let $\varphi \in \mathcal{A}$. Then

$$\int_0^1 \overline{\varphi}(-i)\psi' = \int_0^1 \overline{-i\varphi'}\,\psi \qquad \forall \psi \in \mathcal{D}(A).$$

$\boxed{\subseteq}$ Let $\varphi \in \mathcal{D}(A^*)$. Then by Claim 4.15,

$\exists\, C_\varphi < \infty$:

$$|\langle \varphi, A\psi\rangle| \leq C_\varphi \|\psi\| \qquad (\psi \in \mathcal{D}(A))$$

W.T.S. $\varphi \in \mathcal{A}$, i.e., φ is a.c.:

$$\varphi(x) = \varphi(0) + \int_0^x \varphi' \qquad \text{L.a.e. } x \in [0,1].$$

The idea is to pick $\psi \in \mathcal{D}(A)$ which approx. $\chi_{[0,x]}$ within $\mathcal{A}$.

If we had that, then:

$$\langle \varphi, A\chi_{[0,x]}\rangle \approx -i\left(\overline{\varphi(x)} - \overline{\varphi(0)}\right)$$

but also, since $\varphi \in \mathcal{D}(A^*)$,

$$\langle \varphi, A\chi_{[0,x]} \rangle = \langle A^*\varphi, \chi_{[0,x]} \rangle$$

$$= \int_0^x \overline{-i\varphi'} = i\int_0^x \overline{\varphi'}$$

$$= i\overline{\int_0^x \varphi'}.$$

Hence φ is indeed a.c.

(We omit the argument for $\chi_{[0,x]}$ being approx. within $\mathcal{D}(A)$.

$\blacksquare$

$\Rightarrow$ As $\mathcal{D}(A^*) = \mathcal{A} \neq \mathcal{D}(A)$, A is NOT S.A.

But A is closed since it is A_1 of $\boxed{Q2}$. It is also symm.

(c) Let $\alpha \in \mathbb{C} : |\alpha| = 1$. Let $A_\alpha := -i\partial$ w/

$$\mathcal{D}(A_\alpha) := \{ \psi \in \mathcal{A} \mid \psi(0) = \alpha\psi(1) \}.$$

Claim: A_α is S.A.

Proof: First, by similar arguments as before, $\overline{\mathcal{D}(A_\alpha)} = L^2$, so A_α is densely def. It is indeed symm., since

$$\langle \varphi, A_\alpha \psi \rangle = \int_0^1 \overline{\varphi}(-i)\psi' \overset{\text{IBP}}{=} -i\,\overline{\varphi}\psi\Big|_0^1 + i\int_0^1 \overline{\varphi'}\psi$$

But $\overline{\varphi}(1)\psi(1) \underset{\varphi,\psi\in\mathcal{D}(A_\alpha)}{=} \frac{1}{\overline{\alpha}}\overline{\varphi(0)}\frac{1}{\alpha}\psi(0) \underset{|\alpha|^2=1}{=} \overline{\varphi(0)}\,\psi(0)$.

So we'd have $A_\alpha = A_\alpha^*$ if we could show $\mathcal{D}(A_\alpha^*) \subseteq \mathcal{D}(A_\alpha)$.

To that end, let $\psi\in\mathcal{D}(A^*)$ and $\varphi\in\mathcal{D}(A)$. Then

$$\langle \psi, A_\alpha\varphi\rangle = \int_0^1 \overline{\psi}(-i)\varphi' = -i\,\overline{\psi}\varphi\Big|_0^1 + i\int_0^1 \overline{\psi'}\varphi$$

$A_\alpha^* = -i\partial$ too

$$= -i\,\overline{\psi}\varphi\Big|_0^1 + \langle A_\alpha^*\psi, \varphi\rangle$$

$$= -i\,\overline{\psi}\varphi\Big|_0^1 + \langle \psi, A_\alpha\varphi\rangle$$

$\Longrightarrow \overline{\psi}\varphi\Big|_0^1 = 0$. But $\varphi(0) = \alpha\varphi(1)$

so $\overline{\varphi(1)}\left(\psi(1) - \overline{\alpha}\psi(0)\right) = 0$.

Pick $\varphi \in \mathcal{D}(A_\alpha)$ as

$$\varphi(x) := \alpha(1-t) + t$$

to get $\psi \in \mathcal{D}(A_\alpha)$ too.

$\Longrightarrow \mathcal{D}(A_\alpha^*) \subseteq \mathcal{D}(A_\alpha)$.

$\blacksquare$

$$\Longrightarrow \quad A \subsetneq A_\alpha = A_\alpha^* \subsetneq A^* .$$

A has uncountably many S.A. extensions.

Q5 Claim: A is closable $\Leftrightarrow$ $\overline{\Gamma(A)} = \Gamma(B)$ $\exists\, B$. Then $B = \overline{A}$.

Proof: This is Claim 11.11 in the lecture notes (now fixed since DEC 17 '23).

Q6 This is Example 11.12 in the lecture notes (now fixed since DEC 17 '23).

$\boxed{Q7}$ Let $A : \mathcal{D}(A) \to \mathcal{H}$ be injective.

Claim: If $\Gamma(A), \operatorname{im}(A)$ are both closed, then

$$\exists\ C < \infty : \qquad \|A\psi\| \overset{\circledast}{\geq} C\|\psi\| \qquad (\psi \in \mathcal{D}(A))$$

Proof:
$$\pi_2 : \mathcal{H}^2 \to \mathcal{H}$$
$$(x,y) \mapsto y$$

$$\pi_2 : \Gamma(A) \to \operatorname{im}(A)$$
$$(\psi, A\psi) \mapsto A\psi$$

is a cont. bij. on two Banach sp.

closed graph $\Rightarrow$ Hence it has a bdd. inverse, i.e.,

$\exists\ \alpha \in (0,\infty)$:

$$\underbrace{\|\pi_2(\psi, A\psi)\|}_{\equiv \|A\psi\|} \geq \alpha \underbrace{\|(\psi, A\psi)\|}_{\equiv \sqrt{\|\psi\|^2 + \|A\psi\|^2}}$$

$$\Leftrightarrow \quad (1-\alpha^2)\|A\psi\|^2 \geq \|\psi\|^2$$

$\blacksquare$

Claim: If A has dense closed range and obeys $\circledast$ then $\Gamma(A)$ is closed.

Proof: $A : \mathcal{D}(A) \to \mathcal{H}$ is a bijection w/ $\circledast$ guarantees $\|A^{-1}\| < \infty$. Hence by closed graph thm., $\Gamma(A^{-1})$ is closed.

But $\Gamma(A^{-1}) = V\,\Gamma(A)$ w/ $V(x,y) \equiv (y,x)$.

$\blacksquare$

<u>Claim</u>: If $\Gamma(A)$ is closed and obeys $\circledast$ then $\operatorname{im}(A)$ is closed.

<u>Proof</u>: Let $\{\psi_n\}_n \subseteq \mathcal{H}$: $A\psi_n \longrightarrow \eta$ $\exists\, \eta \in \mathcal{H}$.

W.T.S. $\eta \in \operatorname{im}(A)$

$\|\psi_n - \psi_m\| \leq C^{-1} \|A(\psi_n - \psi_m)\|$ small

$\Longrightarrow \{\psi_n\}_n$ is Cauchy

$\Longrightarrow \{(\psi_n, A\psi_n)\}_n$ is Cauchy and so since $\Gamma(A)$ is closed, $(\psi_n, A\psi_n) \to (\psi, A\psi)$

$\exists\, \psi \in \mathcal{D}(A)$. So $A\psi_n \to A\psi$, $A\psi_n \to \eta$ $\Rightarrow A\psi = \eta$.

$\blacksquare$

$\boxed{Q8}$ $C_0^\infty(\mathbb{R}) \equiv \{ f : \mathbb{R} \to \mathbb{C} \mid \operatorname{supp}(f)$ is cpt. and f is smooth $\}$.

$\mathcal{D}(-\partial^2) := C_0^\infty(\mathbb{R})$.

$(-\partial^2)^* = ?$

Since $-\partial^2 = (-i\partial)^2$ and $-i\partial$ is symm.,

so is $-\partial^2$. For $\psi \in \mathcal{D}((-\partial^2)^*)$, $\varphi \in C_0^\infty(\mathbb{R})$,

$$\langle \psi, -\partial^2 \varphi \rangle \equiv -\int_{\mathbb{R}} \overline{\psi} \varphi'' \equiv \int_{\mathbb{R}} \overline{A^* \psi} \varphi$$

and via IBP and cpt. supp.,

$$A^* \psi = -\psi''$$

so we merely need to calculate $\mathcal{D}(A^*)$.

Similarly to $\boxed{Q4}$ one may show

that $\mathcal{D}((-\partial^2)^*) = \{ \psi \in L^2 \mid \psi, \psi' \text{ ac and } \psi'' \in L^2 \}$

In particular, $A \neq A^*$.

Claim: $-\partial^2$ is ess. S.A.

Proof: By Corollary 11.27, WTS.

$$\ker((-\partial^2)^* \pm i\mathbb{1}) = \{0\}.$$

But $((-\partial^2)^* \pm i\mathbb{1})\psi = 0$ for $\psi \in \mathcal{D}((-\partial^2)^*)$

implies $-\psi'' = \mp i\psi$

$$\Downarrow$$

$$\psi = Ae^{\alpha\cdot} + Be^{-\alpha\cdot}$$

for $\alpha^2 = \mp i$

But $\psi \in L^2 \Rightarrow \psi$ vanishes @ $\pm\infty$

so $A = B = 0 \Rightarrow \psi = 0$.

$\blacksquare$

$\boxed{Q9}$ $-i\partial : C_0^\infty([0,\infty)) \longrightarrow L^2([0,\infty))$

By 11.27 again, $-i\partial$ is NOT ess. S.A.

Indeed, the same argument as above shows

$[0,\infty) \ni x \mapsto e^{-x} \in \mathbb{C}$

is an element of $\ker(-i\partial + i\mathbb{1})$.