# Functional Analysis
# Princeton University MAT520
# HW11, Due Dec 8th 2023 (auto extension until Dec 10th 2023)

December 14, 2023

1. Let $X$ be the position operator on $L^2(\mathbb{R})$. Show that

$$\mathcal{D}(X) := \left\{ \psi \in L^2(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} x^2 |\psi(x)|^2 \mathrm{d}x < \infty \right\}$$

   is the largest vector space $V$ such that for each $\psi \in V$, $X\psi \in L^2$.

2. Let

$$\mathcal{A} := \{ \psi : [0,1] \to \mathbb{C} \mid \psi \text{ is ac and } \psi' \in L^2([0,1]) \} .$$

   Let $A_1, A_2$ both be defined as

$$\psi \mapsto -\mathrm{i}\psi'$$

   on the respective domains

$$\begin{aligned} \mathcal{D}(A_1) &:= \mathcal{A} . \\ \mathcal{D}(A_2) &:= \{ \psi \in \mathcal{A} \mid \psi(0) = 0 \} . \end{aligned}$$

   Show that both domains are dense in $L^2([0,1])$ and that $A_1, A_2$ are closed. Finally show that

$$\begin{aligned} \sigma(A_1) &= \mathbb{C} \\ \sigma(A_2) &= \varnothing . \end{aligned}$$

3. Show that if $A$ is a symmetric operator on a Hilbert space $\mathcal{H}$ then the following are equivalent:
   (a) $A$ is essentially self-adjoint.
   (b) $\ker(A^* \pm \mathrm{i}\mathbb{1}) = \{ 0 \}$.
   (c) $\overline{\operatorname{im}(A \pm \mathrm{i}\mathbb{1})} = \mathcal{H}$.

4. Let $A := -\mathrm{i}\partial$ on

$$\mathcal{D}(A) := \{ \psi \in \mathcal{A} \mid \psi(0) = \psi(1) = 0 \}$$

   with $\mathcal{A}$ as above.
   (a) Show that $A$ is symmetric as an operator $A : \mathcal{D}(A) \to L^2([0,1])$.
   (b) Calculate $A^*$ (along with $\mathcal{D}(A^*)$) and conclude $A$ is closed, symmetric but *not* self-adjoint.
   (c) For any $\alpha \in \mathbb{C}$, $|\alpha| = 1$, define $A_\alpha := -\mathrm{i}\partial$ on the domain

$$\mathcal{D}(A_\alpha) := \{ \psi \in \mathcal{A} \mid \psi(0) = \alpha\psi(1) \} .$$

   Show that $A_\alpha$ is self-adjoint, and that it is an extension of $A$, and is extended by $A^*$:

$$A \subseteq A_\alpha \subseteq A^* .$$

   Conclude that $A$ has uncountably many self-adjoint extensions.

5. Show that $A$ is closable iff $\overline{\Gamma(A)} = \Gamma(B)$ for some operator $B$. Show that this operator $B$ is the closure $\overline{A}$ of $A$.

6. Let $\{\varphi_n\}_n$ be an ONB for $\mathcal{H}$ and $\psi \in \mathcal{H}$ any vector which is *not* a finite linear combination of $\{\varphi_n\}_n$. Let $\mathcal{D}$ be the set of vectors which *are* finite linear combinations of $\{\varphi_n\}_n$ and of $\psi$. Define $A : \mathcal{D} \to \mathcal{H}$ via

$$A\left(b\psi + \sum_{i=1}^{N} a_i \varphi_i\right) := b\psi\,.$$

Calculate $\Gamma(A)$ and show that $\overline{\Gamma(A)}$ is *not* the graph of a linear operator.

7. [R&S VIII. 2] Let $A : \mathcal{D}(A) \to \mathcal{H}$ be injective.

   (a) Show that if $A$ is closed and has a closed range then $\exists C \in (0, \infty)$ such that

   $$\|A\psi\| \geq C\|\psi\| \qquad (\psi \in \mathcal{D}(A))\,. \tag{0.1}$$

   (b) Show that if $A$ has dense closed range and obeys (0.1) then $A$ is closed.

   (c) Show that if $A$ is closed and obeys (0.1) then it has a closed range.

8. Calculate the adjoint of $-\partial^2 : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R})$. Determine if $-\partial^2$ is essentially self-adjoint. Here $C_0^\infty(\mathbb{R})$ is the set of functions $f : \mathbb{R} \to \mathbb{C}$ smooth of compact support.

9. Let $-\mathrm{i}\partial : C_0^\infty([0, \infty)) \to L^2([0, \infty))$ where the domain is the set of smooth functions with compact support away from the origin. Is it essentially self-adjoint?