Functional Analysis
Princeton University MAT520
HW12, not to be submitted

December 10, 2023

1. Prove Kuiper's theorem: given any $A \in \mathcal{B}(\mathcal{H})$ invertible, there is an operator-norm-continuous path
$$\gamma : [0,1] \to \{ B \in \mathcal{B}(\mathcal{H}) \mid B \text{ is invertible} \}$$
such that $\gamma(0) = A$ and $\gamma(1) = \mathbb{1}$.

2. Show that the Fourier transform defined on the Schwarz space
$$\mathcal{F} : \mathcal{S}\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$$
given by
$$(\mathcal{F}\psi)(p) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x}\psi(x)\,\mathrm{d}x$$
is well-defined, and that actually
$$\mathcal{F}\left(\mathcal{S}\left(\mathbb{R}^d\right)\right) \subseteq \mathcal{S}\left(\mathbb{R}^d\right)\,.$$
Proceed to show also that:
   (a) $\mathcal{F}$ may be extended to the whole of L^2.
   (b) As an operator on L^2, $\mathcal{F}$ is unitary.
   (c) Show that $\sigma(\mathcal{F}) = \{ \pm 1, \pm \mathrm{i} \}$ and find the eigenvectors of $\mathcal{F}$.

3. Find an example for a self-adjoint operator $A \in \mathcal{B}\left(\ell^2(\mathbb{Z})\right)$ such that there exist $C, \mu \in (0,\infty)$ for which
$$|\langle \delta_x, A\delta_y\rangle| \leq C\mathrm{e}^{-\mu|x-y|} \qquad (x, y \in \mathbb{Z})$$
with $\{ \delta_x \}_x$ the Kronecker delta basis of $\ell^2(\mathbb{Z})$, and yet, for some $a, b \in \mathbb{R}$ with $a < b$, the operator defined via the measurable functional calculus
$$\chi_{[a,b]}(A)$$
fails to have the above locality estimate.

4. Define *the Dirichlet Laplacian* $-\Delta$ on the Hilbert space
$$\mathcal{H} := \left\{ \psi \in L^2((0,1)) \mid \psi(0) = \psi(1) = 0 \right\}$$
via
$$\mathcal{D}(-\Delta) := H^1_0((0,1)) \cap H^2((0,1))$$
where the Sobolev space $H^r((0,1))$ has been defined in the lecture notes and the subscript zero means those functions which vanish on the boundary points. Show that as such, $-\Delta$ is a self-adjoint operator which is unbounded, but with a discrete set of eigenvalues of finite multiplicities. Calculate the eigenvalues and eigenvectors.

5. On $L^2(\mathbb{R})$, define the harmonic oscillator with parameter $\omega > 0$ as
$$H = -\Delta + \frac{1}{2}\omega^2 X^2$$
and domain
$$\mathcal{D}(H) := \operatorname{span}\left(\left\{ x \mapsto x^\alpha \mathrm{e}^{-\frac{1}{2}x^2} \,\middle|\, \alpha \in \mathbb{N}_{\geq 0} \right\}\right)\,.$$
Show that H is self-adjoint, calculate the spectrum and the eigenvalue / eigenvector pairs.