MAT 520 HW2

3. It is clear that it is an equivalent relation, and hence it suffices to show that any norm is equivalent to the canonical one $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. After normalization, we need to show that there are $a, b > 0$ such that $a \leq \|x\| \leq b$ for all $x \in \mathbb{C}^n$ such that $\|x\|_2 = 1$. Observe that $K = \{x \in \mathbb{C}^n : \|x\|_2 = 1\}$ is compact with respect to $\|\cdot\|_2$. If $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ is continuous with respect to $\|\cdot\|_2$, then $\|\cdot\|$ achieves maximum b and minimum a on K, which cannot be zero, since $\|x\| = 0$ implies that $x = 0 \in K$. By writing elements in $\mathbb{C}^n$ using the standard basis of $\mathbb{C}^n$, it is not hard to see that $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ is indeed continuous with respect to $\|\cdot\|_2$.

4. Let $T : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ be the identity map. Then T is bijective and continuous since $\|Tx\|_2 = \|x\|_2 \leq C\|x\|_1$. Thus we can use the inverse mapping theorem to conclude.

11. The only if part is clear. To show the other direction, let $\{x_n\}$ be Cauchy. There exists a subsequence $\{x_{n_k}\}$ so that $\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}$. Let $y_k = x_{n_{k+1}} - x_{n_k}$. We have $\sum_k \|y_k\| \leq \sum_k 2^{-k} < \infty$. Thus $\sum_k y_k$ converges. Since $\sum_{k=1}^l y_k = x_{n_{l+1}} - x_{n_1}$, it follows that x_{n_k} converges to some x. Since $\{x_n\}$ is Cauchy, it is clear that x_n also converges to x.

12. Recall the Cantor set is $C = \bigcap_{k=0}^\infty C_k$ where $C_0 = [0,1]$ and $C_1 = [0,1/3] \cup [2/3,1]$ and so on. If the interior of C is nonempty, then there is an interval $I \subset C$, which is not possible. Indeed, for any two points x, y in the Cantor set such that $|x-y| \geq 1/3^k$, then x, y belongs to two different C_k and hence there is some point not in C but lies between x and y.

13. We show that $W \cap \bigcap V_j$ is nonempty for any nonempty open set W. Since V_1 is dense, it follows that $W \cap V_1$ is nonempty. Since X is a locally compact Hausdorff space, there exists an open set U_1 such that $U_1 \subset \overline{U}_1 \subset W \cap V_1$, and that $\overline{U}_1$ is compact. Similarly, choose and open set U_2 with $\overline{U}_2$ compact such that $U_2 \subset \overline{U}_2 \subset U_1 \cap V_1$, and so on. We obtain a nested sequence of nonempty compact sets $\overline{U}_1 \supset \overline{U}_2 \supset \cdots$, and hence $\bigcap \overline{U}_j$ is nonempty.

15. Let Y be a finite-dimensional subspace of X. We know that Y is closed in a TVS. Suppose Y contains some open set U. Pick $u \in U$. Since $U - u$ is absorbing, for any $x \in X$, we have $tx + u \in U$ for sufficiently small $t > 0$. It follows that $x \in Y$ and $X \subset Y$, which is a contradiction, since X is assumed

to be infinite-dimensional. Thus Y is nowhere dense in X. In particular, X is of Baire's first category. For the second part, let X be an infinite-dimensional Banach space that has a countable Hamel basis $\{f_j\}_{j=1}^{\infty}$. Let Y_n be the span of $\{f_j\}_{j=1}^{n}$. Then $X = \bigcup_n Y_n$. However, Y_n is finite-dimensional and hence nowhere dense in X, implying that X is of first category, which contradicts the Baire's category theorem.

16. Let E_n be a Cantor-like set where at k^{th} stage we remove 2^{k-1} centrally situated open intervals each of length l_{nk} such that $\sum_{k=1}^{\infty} 2^{k-1} l_{nk} = 2^{-n}$. This can be achieved with $l_{nk} = 2^{-2k-n+1}$. Then $m(E_n) = 1 - 2^{-n}$ where m is the Lebesgue measure. We have $E_1 \subset E_2 \subset \cdots$ and let $E = \bigcup E_n$. Then $m(E) = \lim_n m(E_n) = 1$. In particular, each E_n is nowhere dense.

17. If f is twice continuously differentiable, then $\hat{f}(n) = O(1/|n|^2)$ as $|n| \to \infty$, and hence $\lim_n \Lambda_n f$ exists. This space is dense in $L^2(\mathbb{S}^1)$. For the second part, denote E to be the set of $f \in L^2(\mathbb{S}^1)$ such that $\lim_n \Lambda_n f$ exists, and let E_N be the set of $f \in L^2(\mathbb{S}^1)$ such that $|\Lambda_n f| \leq N$. It is clear that $E \subset \bigcup_N E_N$ since convergent sequence is bounded. The set E_N is closed since Λ_n is linear and bounded $|\Lambda_n f| \leq \sqrt{2n+1}\|f\|_2$. It remains to show that E_N has no interior. Suppose E_N contains a ball B around f of radius $r > 0$. Let $g \in L^2(\mathbb{S}^1)$ corresponds to the Fourier coefficients $\{1/k\}_{k=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$. Now $f + \epsilon g \notin E_N$ for all $\epsilon > 0$, since $|\sum_{k=-n}^{n} (\hat{f}(k) + \epsilon/k)| \geq \epsilon |\sum_{k=-n}^{n} 1/k| - N$ can be made arbitrarily large. However, $f + \epsilon g \in B$ for ϵ sufficiently small.

18. If Y intersects with $Y + x$ for all $x \in X$, then we are done, using the fact that Y is a subspace. If Y does not intersect $Y + x$, then $Y + x \subset Y^c$ is of first category. This cannot be true since $X = Y \cup Y^c$ will then be of first category.

19. Let $x_n \to x$. Since K is compact, then there is a subsequence x_{n_k} for which $f(x_{n_k}) \to y$ converges. Since the graph of f is closed, it follows that $y = f(x)$.